

Development of Morphing for Studies of HEFT with ATLAS

Adam Kaluza
on behalf of the ATLAS Collaboration

Johannes Gutenberg-Universität Mainz,
Institut für Physik

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Bundesministerium
für Bildung
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- Theoretical framework and analysis prospects
- Morphing method
- Example: VBF $H \rightarrow WW$
- VBF vertex study

- Signal strength and kinematics depend on **many parameters**
- Parameters do not factorize trivially into individual observables
- Necessary to build a signal model taking **all parameters** into account **simultaneously** & modelling all interference effects → **Morphing**
- Use effective Lagrangian as an example, but can apply same techniques to many other BSM model with large sets of parameters

- Effective Lagrangian for the interaction of scalar and pseudo-scalar states with vector bosons

$$\mathcal{L}_0^V = \left\{ \begin{array}{l} c_\alpha \kappa_{SM} \left[\frac{1}{2} \tilde{g}_{HZZ} Z_\mu Z^\mu + \tilde{g}_{HWW} W_\mu^+ W^{-\mu} \right] \\ - \frac{1}{4} \left[c_\alpha \kappa_{H\gamma\gamma} \tilde{g}_{H\gamma\gamma} A_{\mu\nu} A^{\mu\nu} + s_\alpha \kappa_{A\gamma\gamma} \tilde{g}_{A\gamma\gamma} A_{\mu\nu} \tilde{A}^{\mu\nu} \right] \\ - \frac{1}{2} \left[c_\alpha \kappa_{HZ\gamma} \tilde{g}_{HZ\gamma} Z_{\mu\nu} A^{\mu\nu} + s_\alpha \kappa_{AZ\gamma} \tilde{g}_{AZ\gamma} Z_{\mu\nu} \tilde{A}^{\mu\nu} \right] \\ - \frac{1}{4} \left[c_\alpha \kappa_{Hgg} \tilde{g}_{Hgg} G_{\mu\nu}^a G^{a,\mu\nu} + s_\alpha \kappa_{Agg} \tilde{g}_{Agg} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} \right] \\ - \frac{1}{4} \frac{1}{\Lambda} \left[c_\alpha \kappa_{HZZ} Z_{\mu\nu} Z^{\mu\nu} + s_\alpha \kappa_{AZZ} Z_{\mu\nu} \tilde{Z}^{\mu\nu} \right] \\ - \frac{1}{2} \frac{1}{\Lambda} \left[c_\alpha \kappa_{HWW} W_{\mu\nu}^+ W^{-\mu\nu} + s_\alpha \kappa_{AWW} W_{\mu\nu}^+ \tilde{W}^{-\mu\nu} \right] \\ - \frac{1}{\Lambda} c_\alpha \left[\kappa_{H\partial\gamma} Z_\nu \partial_\mu A^{\mu\nu} + \kappa_{H\partial Z} Z_\nu \partial_\mu Z^{\mu\nu} + \kappa_{H\partial W} (W_\nu^+ \partial_\mu W^{-\mu\nu} + h.c.) \right] \end{array} \right\} \chi_0$$

Used in Run 1

Plan Run 2

- Implemented in MADGRAPH5_AMC@NLO
- $\Lambda = 1 \text{ TeV}$, $\cos \alpha = \frac{1}{\sqrt{2}}$ fixed
- Define full coupling parameter as g_x (e.g. $g_{AWW} = s_\alpha \kappa_{AWW} / \Lambda$)

Effective Lagrangian analyses in Run 1

- Only **small subset** of parameters considered
- Influences of parameters in the matrix element studied **in isolation**
- Cross-section (rate) and kinematics (shape) studied **separately**
- Couplings measurement: [arXiv:1507.04548](https://arxiv.org/abs/1507.04548)
Spin & parity measurement: [arXiv:1506.05669](https://arxiv.org/abs/1506.05669)

Plans for Run 2

- Perform combined studies of **many (all) parameters** in the matrix element
- Take **all correlations** between different operators into account
- Use constraining power from **rate & shape information**
- Combine results from different channels

- Challenge: **large parameter space** (e.g. VBF $H \rightarrow VV$ 13 free parameters)
- New method to construct predictions for signal cross section and distributions

Morphing

- **Needed:** MC samples covering wide range of values for coupling parameters
- Run 1 HWW and HZZ analyses: **Matrix Element Reweighting**
(Event by event matrix element reweighting of one source MC sample with large statistics)

$$w(\vec{g}_{target}) = w(\vec{g}_i) \frac{|\mathcal{M}(\vec{g}_{target})|^2}{|\mathcal{M}(\vec{g}_{source})|^2}$$

ME Reweighting

For every configuration point

- rerun analysis
 - write event weights to disk
 - additional interpolation
- **Morphing function:** Instead of “matrix element reweighting” use morphing to obtain a distribution with arbitrary coupling parameters
 - Can be applied directly and without change to
 - Cross sections
 - Distributions (before or after detector simulation)
 - MC events
 - **Exact continuous analytical description of rates and shapes**
 - Even possible to **fit** coupling parameters to data & derive limits

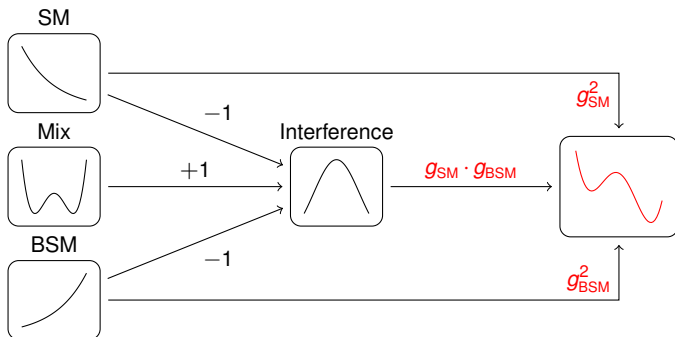
Morphing

- only calculates linear sums of coefficients
- all other inputs are pre-computed once
- computationally fast & convenient tool

- Morphing function** for an observable T_{out} at any coupling point \vec{g}_{target} constructed from weighted sum of input samples T_{in} at fixed coupling points \vec{g}_i

$$T_{out}(\vec{g}_{target}) = \sum_{i=1}^{N_{input}} w_i(\vec{g}_{target}; \vec{g}_i) \cdot T_{in}(\vec{g}_i)$$

e.g. $T = \Delta\phi_{jj}$



Example for 2 free parameters in one vertex

- Process with **two parameters** applied in **one vertex**: g_{SM} and g_{BSM}
- Matrix element can be **factorized**:

$$\begin{aligned}\mathcal{M}(g_{\text{SM}}, g_{\text{BSM}}) &= g_{\text{SM}} \mathcal{O}_{\text{SM}} + g_{\text{BSM}} \mathcal{O}_{\text{BSM}} \\ |\mathcal{M}(g_{\text{SM}}, g_{\text{BSM}})|^2 &= g_{\text{SM}}^2 |\mathcal{O}_{\text{SM}}|^2 + g_{\text{BSM}}^2 |\mathcal{O}_{\text{BSM}}|^2 + 2g_{\text{SM}}g_{\text{BSM}}\mathcal{R}(\mathcal{O}_{\text{SM}}^* \mathcal{O}_{\text{BSM}})\end{aligned}$$

- **Distribution** of a kinematic observable **proportional to the matrix element squared**

$$T(g_{\text{SM}}, g_{\text{BSM}}) \propto |\mathcal{M}(g_{\text{SM}}, g_{\text{BSM}})|^2$$

- **3 generated distributions** needed to obtain distribution with arbitrary parameters
- E.g. generate MC events for $T(1, 0)$, $T(0, 1)$, $T(1, 1)$

$$\begin{aligned}T_{in}(1, 0) &\propto |\mathcal{O}_{\text{SM}}|^2 \\ T_{in}(0, 1) &\propto |\mathcal{O}_{\text{BSM}}|^2 \\ T_{in}(1, 1) &\propto |\mathcal{O}_{\text{SM}}|^2 + |\mathcal{O}_{\text{BSM}}|^2 + 2\mathcal{R}(\mathcal{O}_{\text{SM}}^* \mathcal{O}_{\text{BSM}})\end{aligned}$$

- Distribution with **arbitrary parameters** ($g_{\text{SM}}, g_{\text{BSM}}$)

$$T_{out}(g_{\text{SM}}, g_{\text{BSM}}) = \underbrace{(g_{\text{SM}}^2 - g_{\text{SM}}g_{\text{BSM}})}_{=w_1} T_{in}(1, 0) + \underbrace{(g_{\text{BSM}}^2 - g_{\text{SM}}g_{\text{BSM}})}_{=w_2} T_{in}(0, 1) + \underbrace{g_{\text{SM}}g_{\text{BSM}}}_{=w_3} T_{in}(1, 1)$$

- Morphing function for a bin in distribution

$$T_{out}^{bin}(\vec{g}_{target}) = \sum_i w_i(\vec{g}_{target}; \vec{g}_i) T_{in}^{bin}(\vec{g}_i)$$

- For one input distribution, the bin content is calculated as follows

$$T_{in}^{bin}(\vec{g}_i) = N_{MC,in}^{bin}(\vec{g}_i) \cdot \sigma_{in}(\vec{g}_i) \mathcal{L} / N_{MC,in}$$

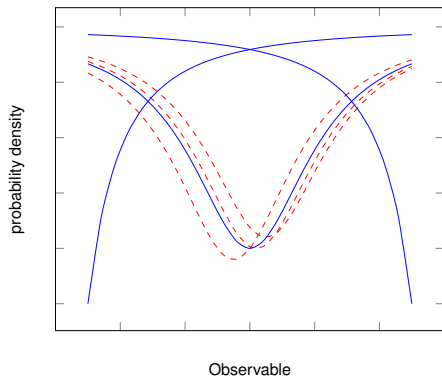
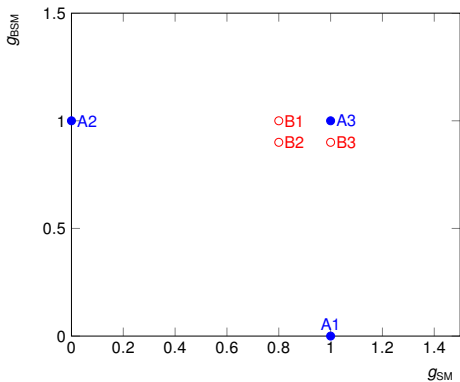
- The uncertainty on that bin is $\sqrt{N_{MC,in}^{bin}(\vec{g}_i)}$
- The propagated statistical uncertainty is

$$\Delta T_{out}^{bin} = \sqrt{\sum_i w_i^2(\vec{g}_{target}; \vec{g}_i) N_{MC,in}^{bin}(\vec{g}_i) \cdot (\sigma_{in}(\vec{g}_i) \mathcal{L} / N_{MC,in})^2}$$

- Highly **dependent** on
 - **input parameters** \vec{g}_i
 - desired **target parameters** \vec{g}_{target}

Choice of input parameters

- So far fixed parameters for input distributions: $T_{in}(1,0)$, $T_{in}(0,1)$, $T_{in}(1,1)$
- Aim to **generalize morphing** to have arbitrary g_i
- Can be chosen to **reduce statistical uncertainty**



Example for 2 free parameters in one vertex: generalization of input parameter

- Generalize to **arbitrary input parameters** \vec{g}_i used to generate input distributions $T_{in}(\vec{g}_i)$

$$T_{in}(g_{SM,i}, g_{BSM,i}) \propto g_{SM,i}^2 |O_{SM}|^2 + g_{BSM,i}^2 |O_{BSM}|^2 + 2g_{SM,i}g_{BSM,i} \mathcal{R}(O_{SM}^* O_{BSM}),$$

$$i = 1, \dots, 3$$

- Ansatz for **output distribution**

$$\begin{aligned} T_{out}(g_{SM}, g_{BSM}) = & \underbrace{(a_{11}g_{SM}^2 + a_{12}g_{BSM}^2 + a_{13}g_{SM}g_{BSM})}_{w_1} T_{in}(g_{SM,1}, g_{BSM,1}) \\ & + \underbrace{(a_{21}g_{SM}^2 + a_{22}g_{BSM}^2 + a_{23}g_{SM}g_{BSM})}_{w_2} T_{in}(g_{SM,2}, g_{BSM,2}) \\ & + \underbrace{(a_{31}g_{SM}^2 + a_{32}g_{BSM}^2 + a_{33}g_{SM}g_{BSM})}_{w_3} T_{in}(g_{SM,3}, g_{BSM,3}) \end{aligned}$$

Example for 2 Operators in one vertex

- T_{out} should be equal to T_{in} for $\vec{g}_{target} = \vec{g}_i$

$$1 = a_{11}g_{SM,1}^2 + a_{12}g_{BSM,1}^2 + a_{13}g_{SM,1}g_{BSM,1}$$

$$0 = a_{21}g_{SM,1}^2 + a_{22}g_{BSM,1}^2 + a_{23}g_{SM,1}g_{BSM,1}$$

...

- Constraints in **matrix form**

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} g_{SM,1}^2 & g_{SM,2}^2 & g_{SM,3}^2 \\ g_{BSM,1}^2 & g_{BSM,2}^2 & g_{BSM,3}^2 \\ g_{SM,1}g_{BSM,1} & g_{SM,2}g_{BSM,2} & g_{SM,3}g_{BSM,3} \end{pmatrix} = \mathbb{1}$$

$$\Leftrightarrow A \cdot G = \mathbb{1}$$

- **Definite solution** $A = G^{-1}$ requires the samples to have parameters such that $\det(G) \neq 0$
 - Very flexible in choosing the parameters for the input distributions
- Can be chosen to **reduce statistical uncertainty** in considered parameter space

General morphing and number of input distributions

- More complicated when processes share parameters between **production and decay**, for example VBF $H \rightarrow VV$
- General matrix element squared at **LO** & assuming **narrow-width-approximation** (ignoring the effect on the total width)
⇒ **polynomials** of 2nd order in production and 2nd order in decay

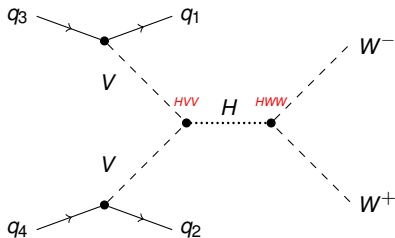
$$T(\vec{g}) \propto |\mathcal{M}(\vec{g})|^2 = \underbrace{\left(\sum_{i=1}^{n_p+n_s} g_i \mathcal{O}_i \right)^2}_{\text{production vertex}} \cdot \underbrace{\left(\sum_{j=1}^{n_d+n_s} g_j \mathcal{O}_j \right)^2}_{\text{decay vertex}}$$

with number of parameters in **production vertex** (n_p), **decay vertex** (n_d) and **shared in vertices** (n_s)

- **Number of required input distributions** equal to number of different terms in expanded matrix element squared
→ dependent on process and considered parameters
→ N_{input} function of n_p , n_d and n_s
- Example: 13 free parameters for VBF $H \rightarrow ZZ$ process:
 - $n_p = 4$ parameters in production: $g_{HWW}, g_{AWW}, g_{H\partial WR}, g_{H\partial WI}$
 - $n_s = 9$ parameters in both vertices: $g_{SM}, g_{HZZ}, g_{AZZ}, g_{H\partial Z}, g_{H\gamma\gamma}, g_{A\gamma\gamma}, g_{HZ\gamma}, g_{AZ\gamma}, g_{H\partial\gamma}$
 - $n_d = 0$, no parameters only in decay

→ **1605 samples** needed!

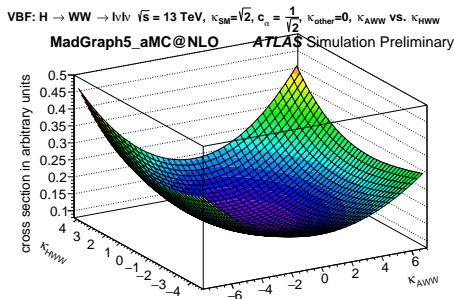
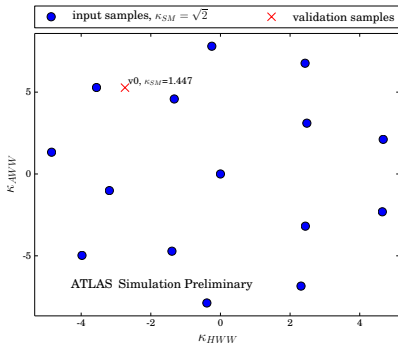
→ Reduction of considered operators favourable → see VBF study



- VBF H \rightarrow WW process with **SM** (g_{SM}) and **2 BSM** operators (g_{HWW} , g_{Aww})
- **15 samples** with different parameters needed
- 50k events generated for each sample
- Only signal considered
- Kinematic observable used: $\Delta\phi_{jj}$

VBF H \rightarrow WW example: Samples

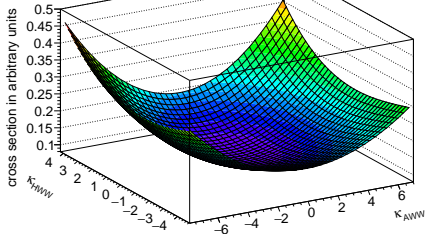
- Expect only **small deviations from SM**
- $g_{SM} = 1$ for all input samples ($\Lambda = 1 \text{ TeV}$, $\cos \alpha = \frac{1}{\sqrt{2}}$)
- BSM parameter limits chosen such that $\sigma_{\text{pure BSM}} \sim \sigma_{SM}$
- all other BSM parameters set to 0
- Scatter plot shows **blue** points in $(g_{A\text{WW}}, g_{H\text{WW}})$ space used to generate **input samples**
- A **validation sample** is produced at the **red** cross for cross-check
 - **morphing** can reproduce the distribution there
 - **fit** can reproduce the parameters from the validation sample



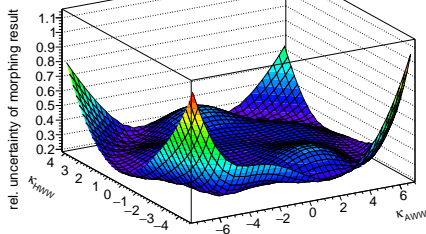
VBF $H \rightarrow WW$ example: Rel. uncertainty on number of expected events

- Dependence of **stat. uncertainty** propagated in morphing function on generated input parameter grid
- Distribution of samples in parameter space **reduces** stat. uncertainty

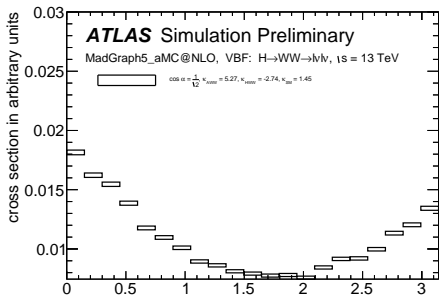
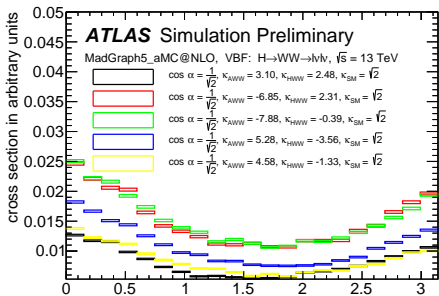
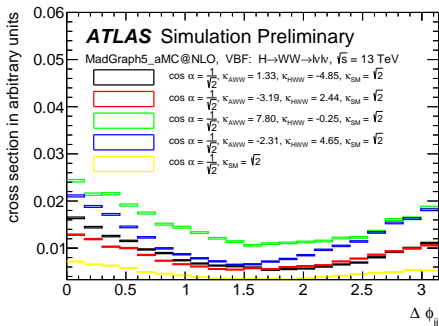
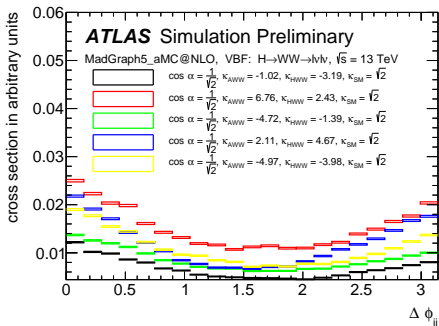
VBF: $H \rightarrow WW \rightarrow l\nu l\nu$ $\sqrt{s} = 13$ TeV, $\kappa_{SM} = \sqrt{2}$, $c_\alpha = \frac{1}{\sqrt{2}}$, $\kappa_{other} = 0$, κ_{AWW} vs. κ_{HWW}
MadGraph5_aMC@NLO ATLAS Simulation Preliminary



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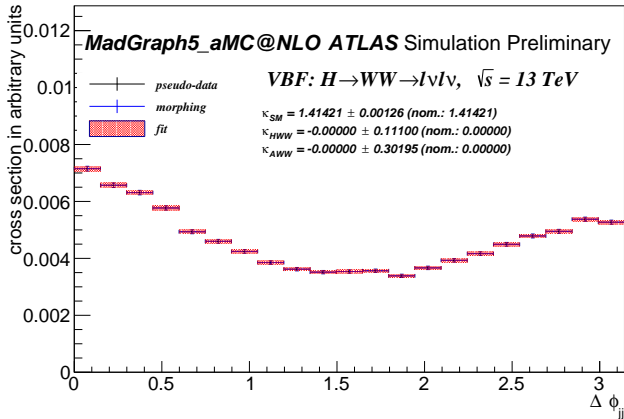
VBF H \rightarrow WW example: Input and validation distributions



VBF $H \rightarrow WW$ example: Morphing and fit to SM input sample

- **Morphing** and **fit** to SM input distribution (pseudo-data)
- MC stat. uncertainty used
- Input and morphed distribution stat. dependent
 - perfect agreement in morphing
 - Post-fit parameters match exact nominal values
- **Correlations** at SM point in table

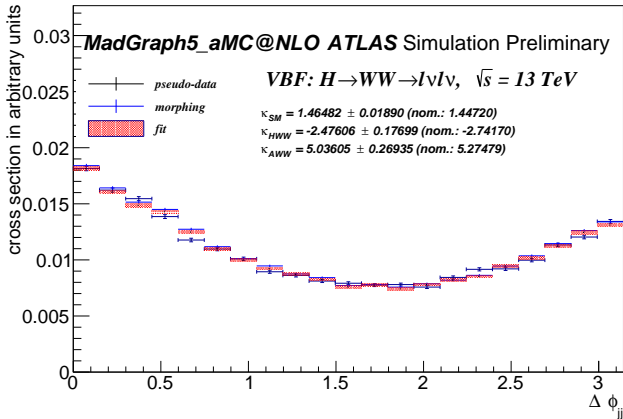
	κ_{SM}	κ_{HWW}	κ_{AWW}
κ_{SM}	1.00	0.15	-0.23
κ_{HWW}	0.15	1.00	0.36
κ_{AWW}	-0.23	0.36	1.00



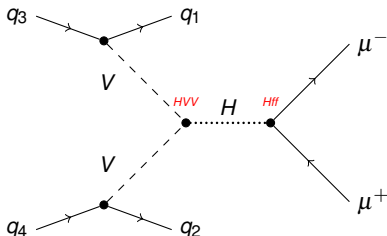
VBF $H \rightarrow WW$ example: Morphing and fit to validation sample

- **Morphing** and **fit** to validation distr. (pseudo-data)
- Validation and morphed distribution stat. independent
 - Agreement in morphing within MC stat. uncertainty
 - Fit results match nominal values within fit uncertainties
- **Correlations** vary at different parameter point

	κ_{SM}	κ_{HWW}	κ_{AWW}
κ_{SM}	1.00	0.20	-0.95
κ_{HWW}	0.20	1.00	0.09
κ_{AWW}	-0.95	0.09	1.00



- Many operators enter VBF
- Goal: **neglect operators** without losing generality to **minimize number of needed samples**

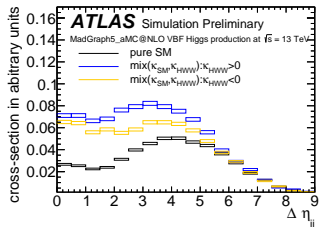
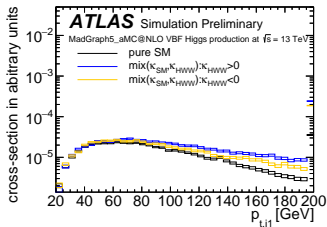
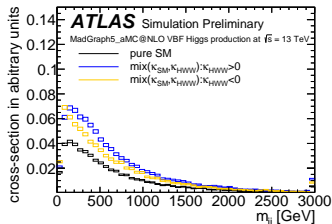
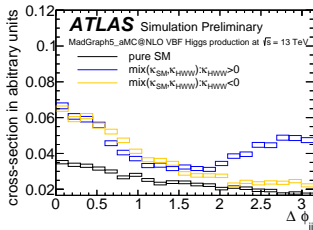


- Technical choice of $H \rightarrow \mu\mu$ decay: no crossover between production and decay
- Full set of 13 VBF prod. op. leads to **91 samples** to produce: still manageable
- Generator level, signal only samples used with 30k events each
- Setup **fit to SM input sample**
 - Learn **correlations** between operators
 - Explore **sensitivity**
 - uses observables: $\Delta\phi_{jj}$, p_T^{j1} , m_{jj} , $\Delta\eta_{jj}$
 - Understand which operators have negligible influence on VBF

Cross sections and shapes depending on VBF couplings

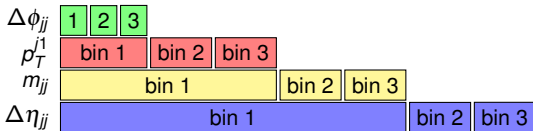
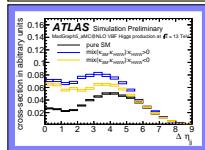
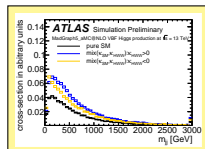
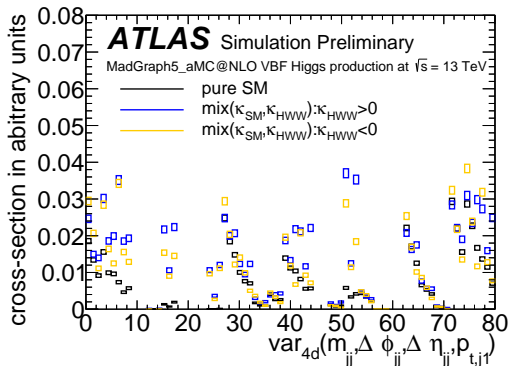
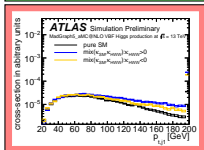
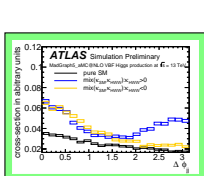
- BSM coupling parameters chosen such that one-operator pure BSM samples correspond to SM VBF cross section

κ_{SM}	1
$\kappa_{H\gamma\gamma}$	203.22
$\kappa_{A\gamma\gamma}$	408.62
$\kappa_{HZ\gamma}$	109.13
$\kappa_{AZ\gamma}$	986.88
κ_{HZZ}	5.75
κ_{AZZ}	6.96
κ_{HWW}	3.36
κ_{AWW}	3.92
$\kappa_{H\partial WR}$	0.76
$\kappa_{H\partial WI}$	0.84
$\kappa_{H\partial A}$	1.77
$\kappa_{H\partial Z}$	1.37



- Production of 91 samples ($\Lambda = 1000 \text{ GeV}$, $\cos \alpha = 1/\sqrt{2}$, $\kappa_{SM} = \sqrt{2}$)
 - 1 pure SM sample
 - 24 samples with SM + 1 BSM ($\pm \kappa_{BSM}$)
 - 66 samples with SM + 2 BSM ($+\kappa_{BSM}$)

Combined distribution of four observables



Fit result sensitivity on VBF couplings

parameter	post-fit value	+	-
Λ	1000.		
$\cos \alpha$	0.71		
$\kappa_{H\ell\ell}$	1.41		
$\kappa_{A\gamma\gamma}$	0	+219	-441
$\kappa_{A\omega\omega}$	0	+3	-2.6
$\kappa_{AZ\gamma}$	0	+441	-398
κ_{AZZ}	0	+2.7	-1.3
$\kappa_{H\gamma\gamma}$	0	+236	-91
$\kappa_{H\partial\gamma}$	0	+0.3	-0.6
$\kappa_{H\partial\omega}$	0	+1.6	-0
$\kappa_{H\partial\omega R}$	0	+0.5	-0.3
$\kappa_{H\partial Z}$	0	+1.2	-0.5
$\kappa_{H\omega\omega}$	0	+1.5	-3
$\kappa_{HZ\gamma}$	0	+38	-49
κ_{HZZ}	0	+8	-2.5
κ_{SM}	1.41	+0.22	-0.11

- Simultaneously to all parameters to SM distribution
- Assuming 8% VBF cross section uncertainty
- **Fit uncertainties** give information on **sensitivity**
- Sensitivity on $\gamma\gamma$ and $Z\gamma$ couplings small
- Close to the SM, 4 couplings ($H\gamma\gamma$, $A\gamma\gamma$, $HZ\gamma$, $AZ\gamma$) can be ignored for VBF without loss of generality
- Other measurements will limit this operator to far smaller values
- To be tested that the $\gamma\gamma$ and $Z\gamma$ operators don't influence any other VBF observable

Fit result correlations of VBF couplings

	$\kappa_{A\gamma\gamma}$	$\kappa_{A\text{WW}}$	$\kappa_{AZ\gamma}$	κ_{AZZ}	$\kappa_{H\gamma\gamma}$	$\kappa_{H\partial\gamma}$	$\kappa_{H\partial\text{WI}}$	$\kappa_{H\partial\text{WR}}$	$\kappa_{H\partial Z}$	κ_{HWW}	$\kappa_{\text{HZ}\gamma}$	κ_{HZZ}	κ_{SM}
$\kappa_{A\gamma\gamma}$	1.000	0.101	-0.093	0.045	0.113	-0.348	-0.046	0.132	-0.118	-0.167	0.058	0.062	-0.171
$\kappa_{A\text{WW}}$	0.101	1.000	0.306	-0.377	-0.355	0.220	-0.151	-0.573	0.747	0.282	0.143	-0.245	-0.130
$\kappa_{AZ\gamma}$	-0.093	0.306	1.000	-0.089	0.170	-0.044	-0.056	0.106	0.008	-0.026	0.208	-0.092	-0.056
κ_{AZZ}	0.045	-0.377	-0.089	1.000	-0.222	0.165	0.415	0.240	-0.326	0.316	0.147	-0.449	0.093
$\kappa_{H\gamma\gamma}$	0.113	-0.355	0.170	-0.222	1.000	-0.122	-0.201	0.174	-0.262	-0.457	-0.018	0.249	0.004
$\kappa_{H\partial\gamma}$	-0.348	0.220	-0.044	0.165	-0.122	1.000	-0.151	-0.380	0.369	0.231	0.299	-0.314	0.199
$\kappa_{H\partial\text{WI}}$	-0.046	-0.151	-0.056	0.415	-0.201	-0.151	1.000	0.090	-0.101	0.247	-0.156	-0.180	-0.002
$\kappa_{H\partial\text{WR}}$	0.132	-0.573	0.106	0.240	0.174	-0.380	0.090	1.000	-0.944	-0.116	-0.331	0.208	0.019
$\kappa_{H\partial Z}$	-0.118	0.747	0.008	-0.326	-0.262	0.369	-0.101	-0.944	1.000	0.193	0.294	-0.191	0.081
κ_{HWW}	-0.167	0.282	-0.026	0.316	-0.457	0.231	0.247	-0.116	0.193	1.000	-0.136	-0.747	-0.065
$\kappa_{\text{HZ}\gamma}$	0.058	0.143	0.208	0.147	-0.018	0.299	-0.156	-0.331	0.294	-0.136	1.000	-0.399	0.230
κ_{HZZ}	0.062	-0.245	-0.092	-0.449	0.249	-0.314	-0.180	0.208	-0.191	-0.747	-0.399	1.000	0.029
κ_{SM}	-0.171	-0.130	-0.056	0.093	0.004	0.199	-0.002	0.019	0.081	-0.065	0.230	0.029	1.000

- Some **large correlations** present **close to the standard model**
- Can likely neglect 1-2 more parameters after rotating into a parameter basis diagonal in this correlation matrix by principle component analysis
- E. g. anti-correlated component between $\kappa_{H\partial Z}$ and $\kappa_{H\partial\text{WR}}$
- Correlations may change away from SM

- Morphing only requires that any differential cross section can be expressed as **polynomial in BSM couplings**
- Method can be used on **any generator** that allows one to vary input couplings
- Works on **generator** and **reco-level** distributions
- **Independent of physics process**
- Works on distributions and cross sections

- Plan for Run 2: **Higgs coupling measurements**
- Combine **rate and shape information** within effective Lagrangian framework
- New method for modelling BSM effects
 - continuous
 - analytical
 - fast
- First application: detailed and complete study of **VBF production**
- Next steps
 - Similar study of WH , ZH and $H \rightarrow VV$
 - First look into most complex cases of VBF, VH or $H \rightarrow VV$
- Pub note: ATL-PHYS-PUB-2015-047

Backup

$$\begin{aligned} N_{input} = & \frac{n_p(n_p+1)}{2} \cdot \frac{n_d(n_d+1)}{2} + \binom{4+n_s-1}{4} \\ & + \left(n_p \cdot n_s + \frac{n_s(n_s+1)}{2} \right) \cdot \frac{n_d(n_d+1)}{2} \\ & + \left(n_d \cdot n_s + \frac{n_s(n_s+1)}{2} \right) \cdot \frac{n_p(n_p+1)}{2} \\ & + \frac{n_s(n_s+1)}{2} \cdot n_p \cdot n_d + (n_p + n_d) \binom{3+n_s-1}{3} \end{aligned}$$

with number of parameters in **production vertex** (n_p), **decay vertex** (n_d) and **shared in vertices** (n_s)

