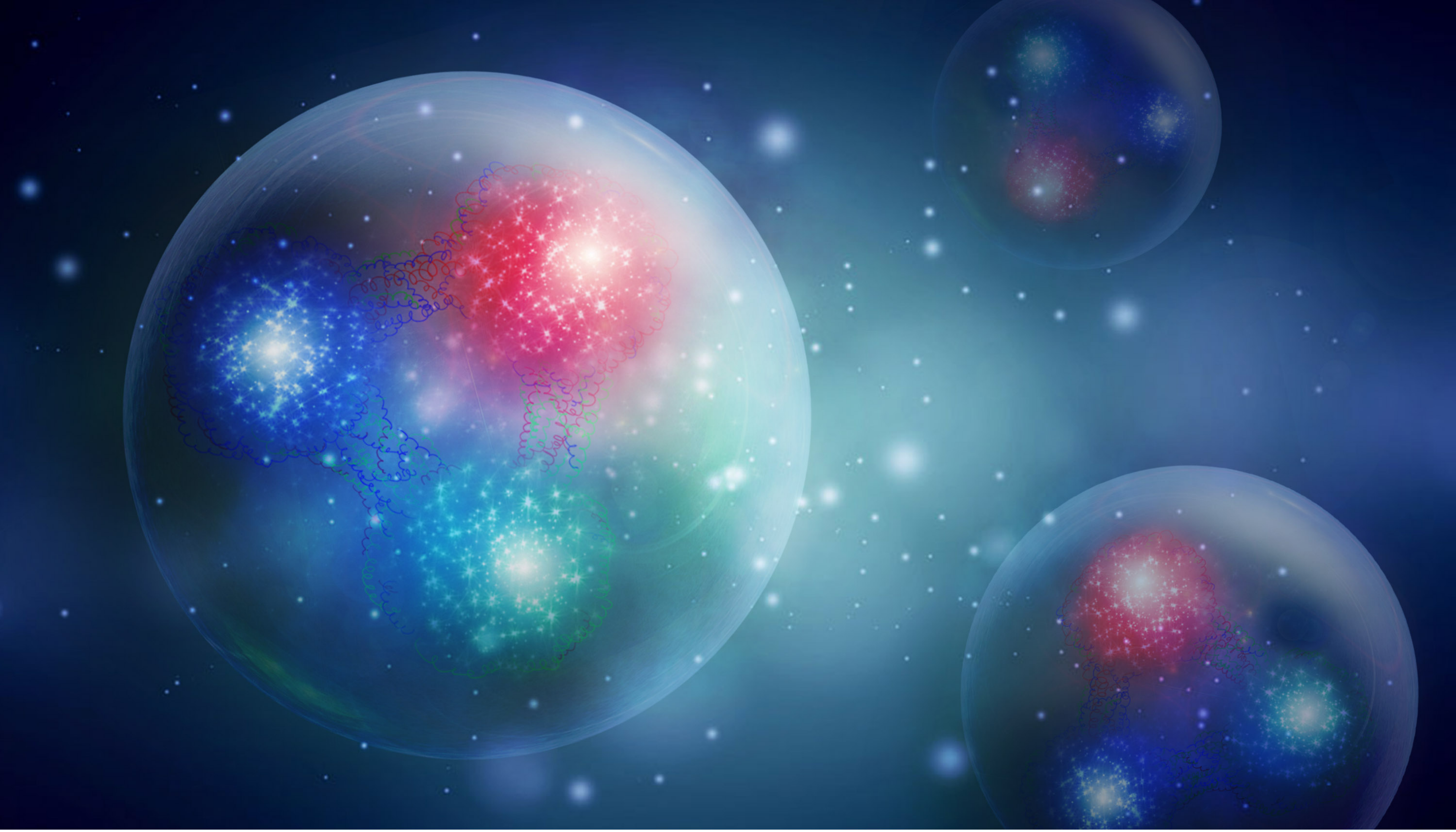


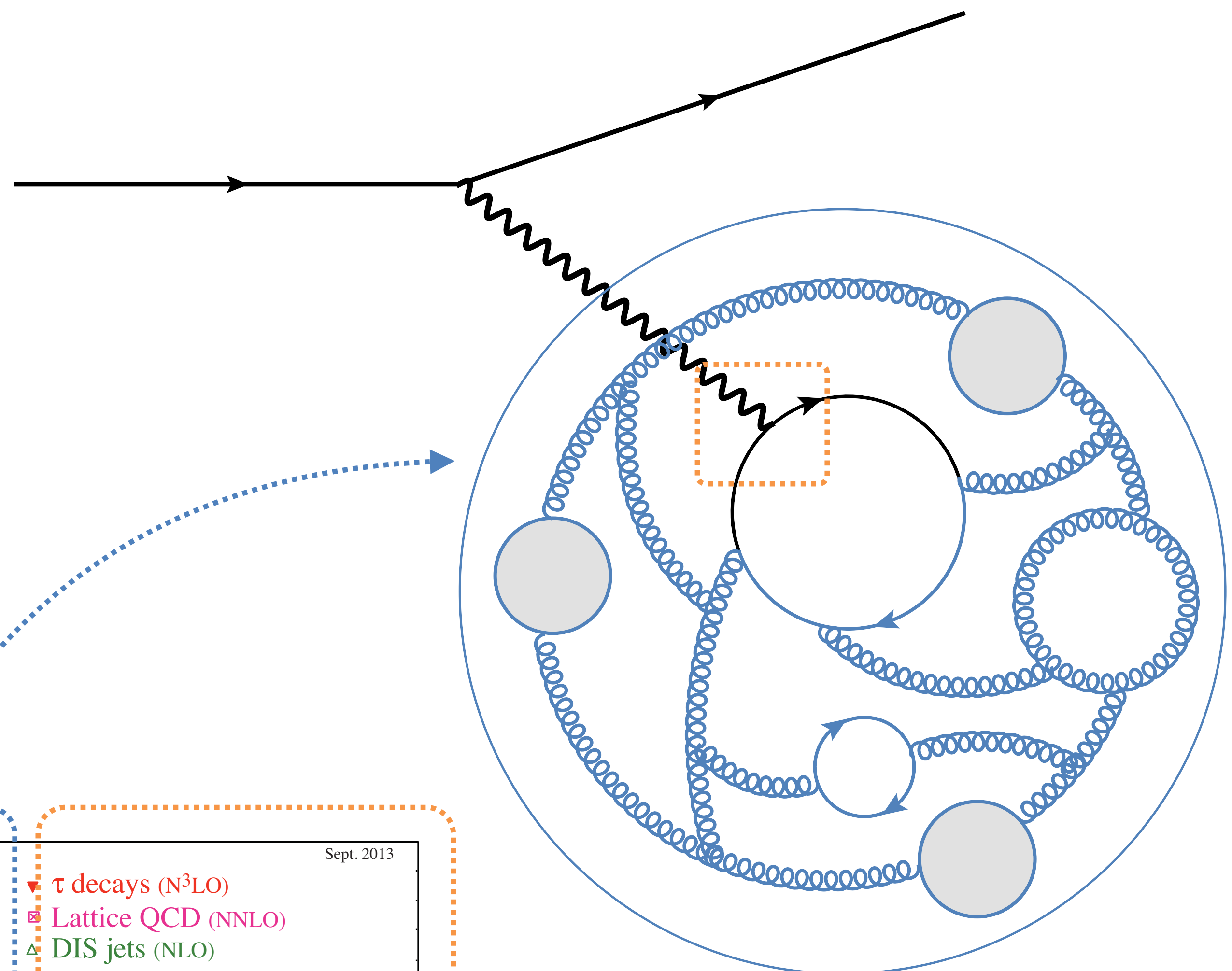
Rapidity evolution of gluon TMDs from small to moderate x

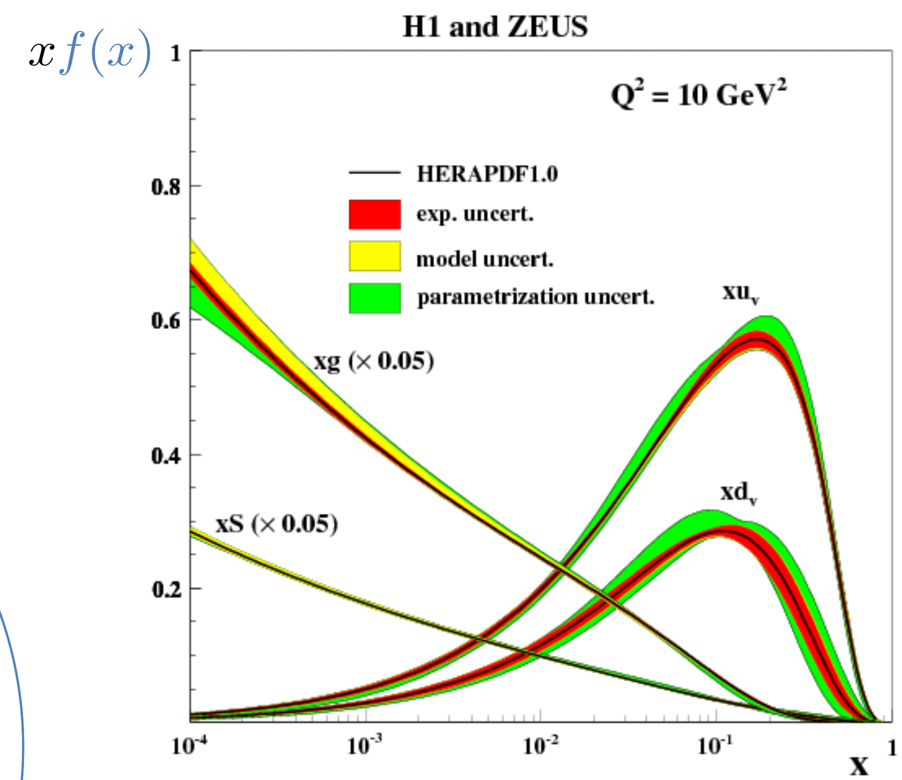
Andrey Tarasov

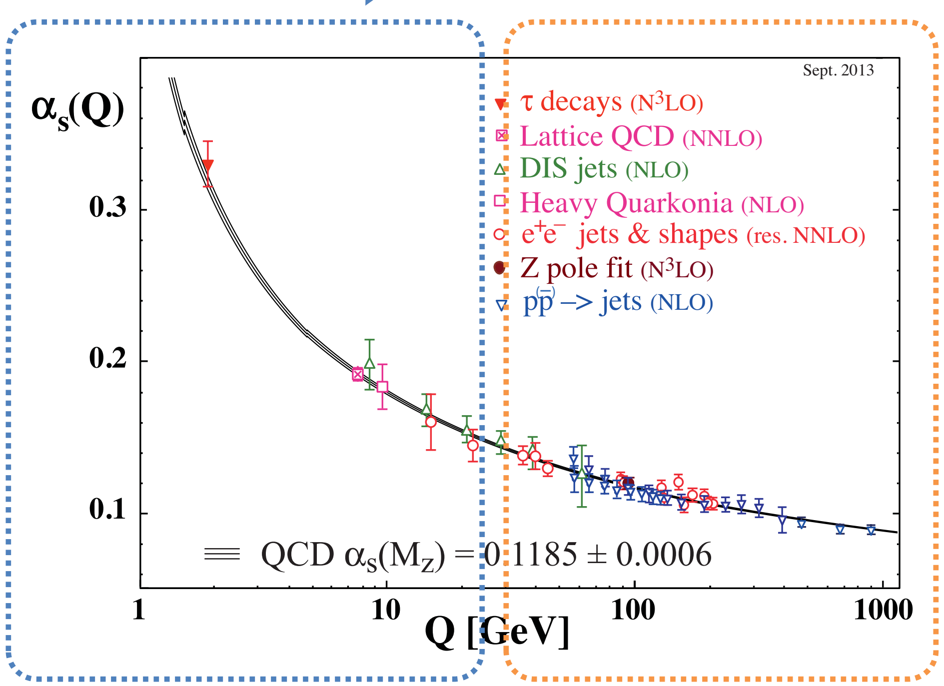
Jets and Heavy Flavor workshop, Santa Fe, January 13, 2016

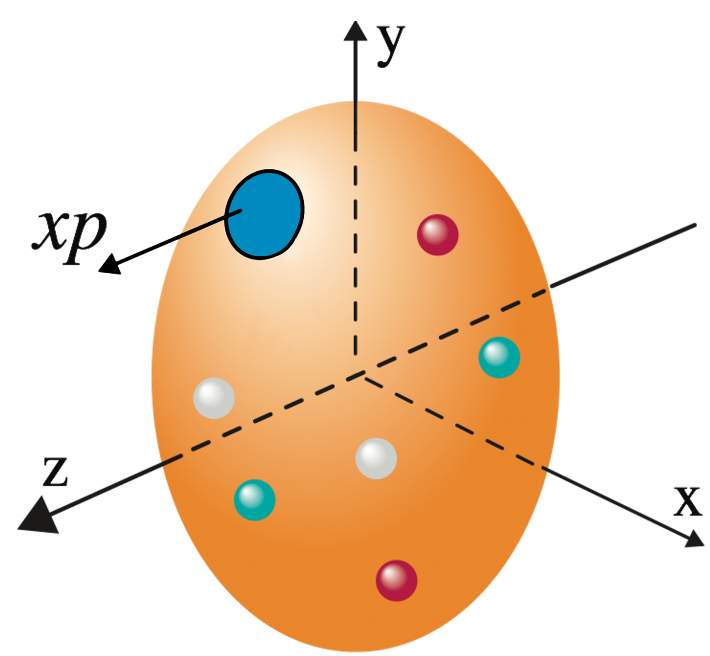
Factorization

Confined system of strongly interacting quarks and gluons



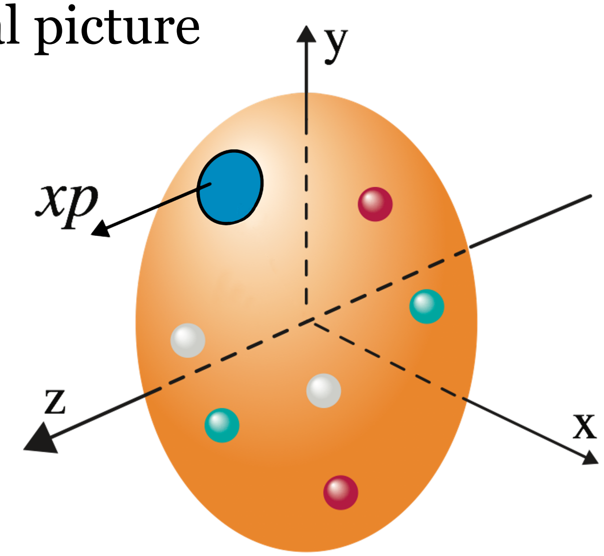


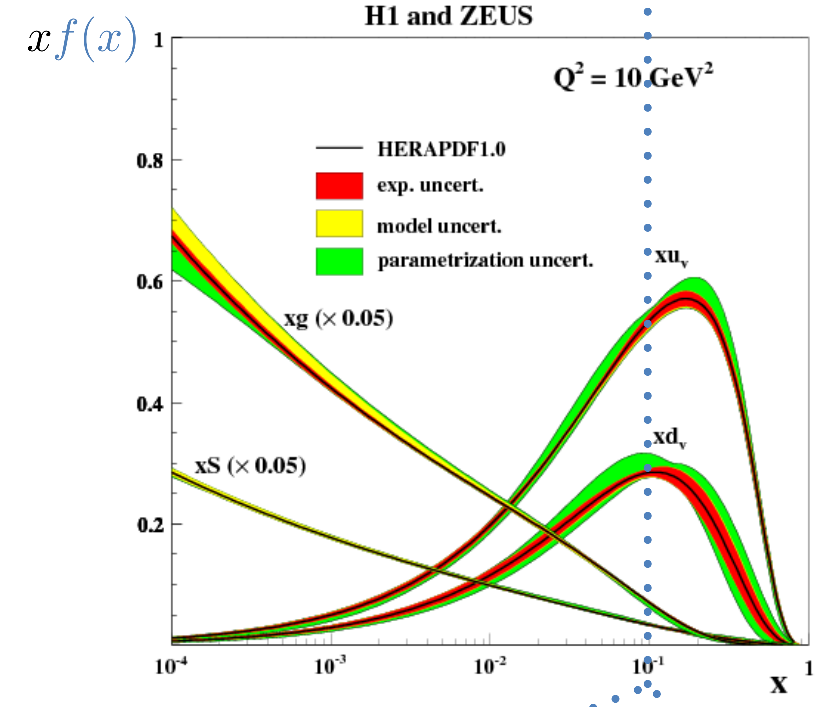




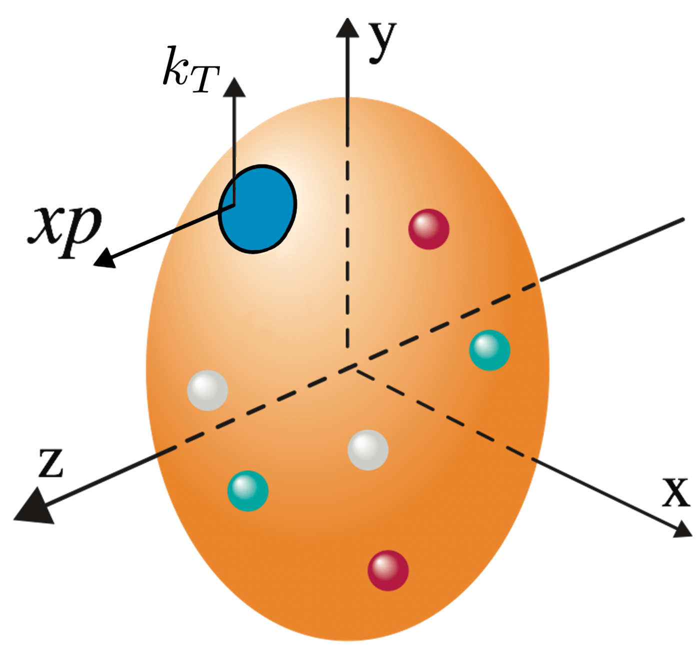
TMD DISTRIBUTION

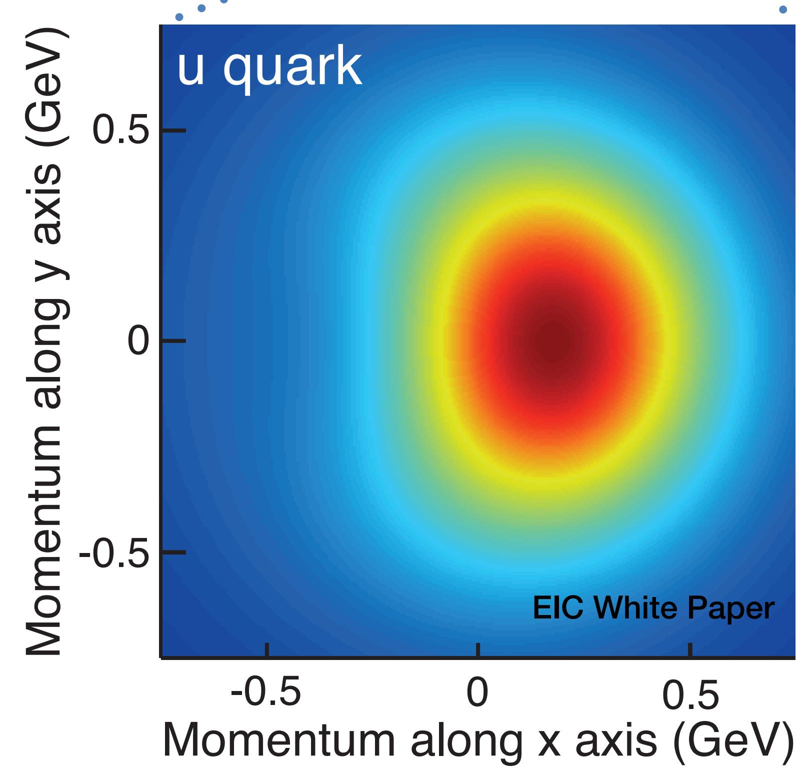
One-dimensional picture



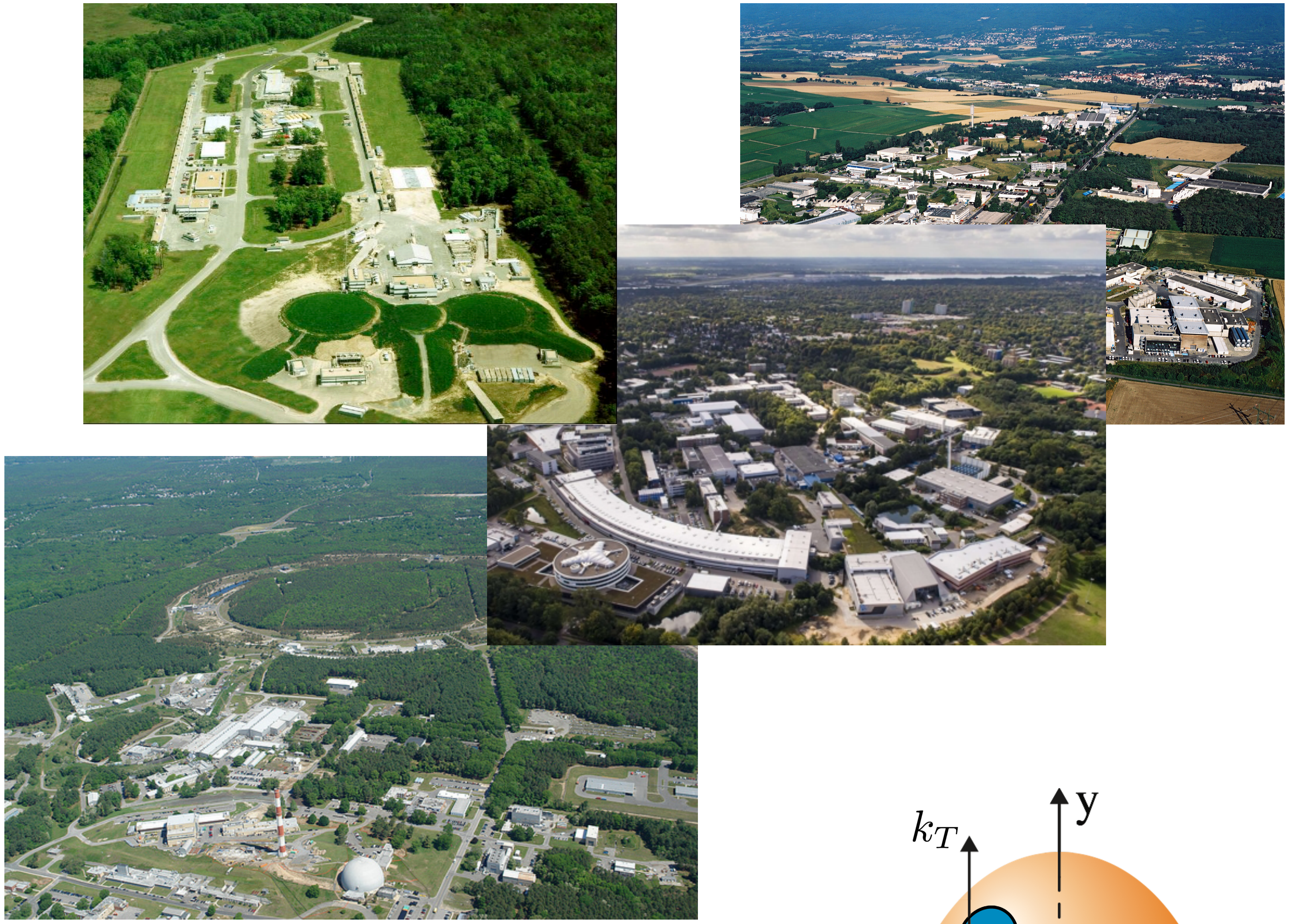


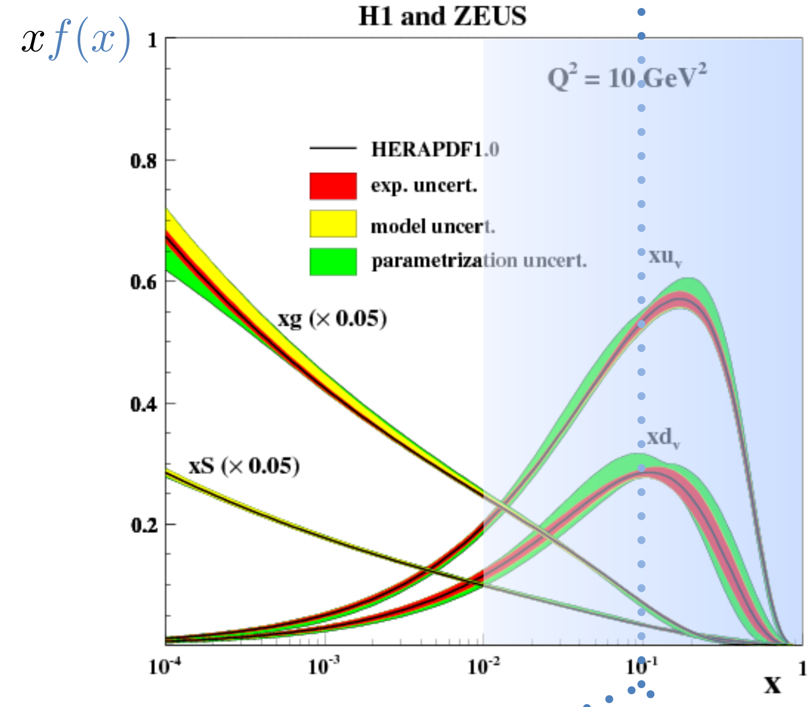
Three-dimensional picture
(Transverse momentum dependent distribution functions)

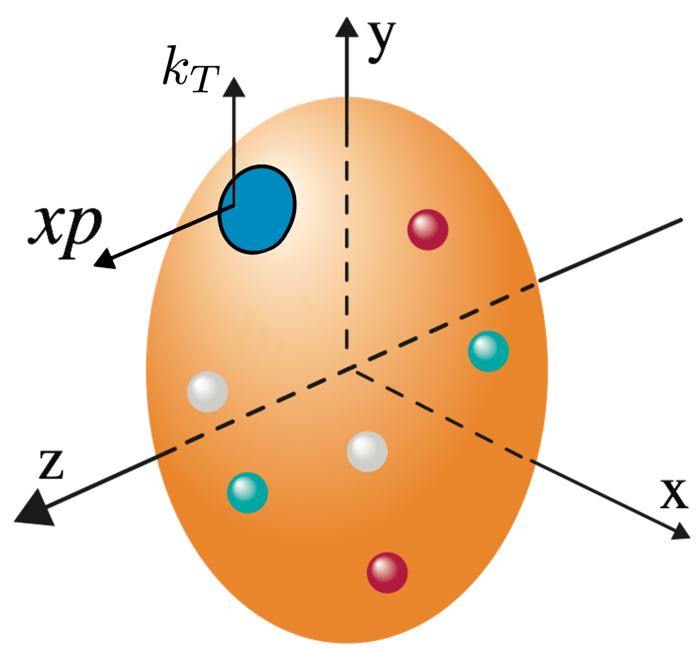


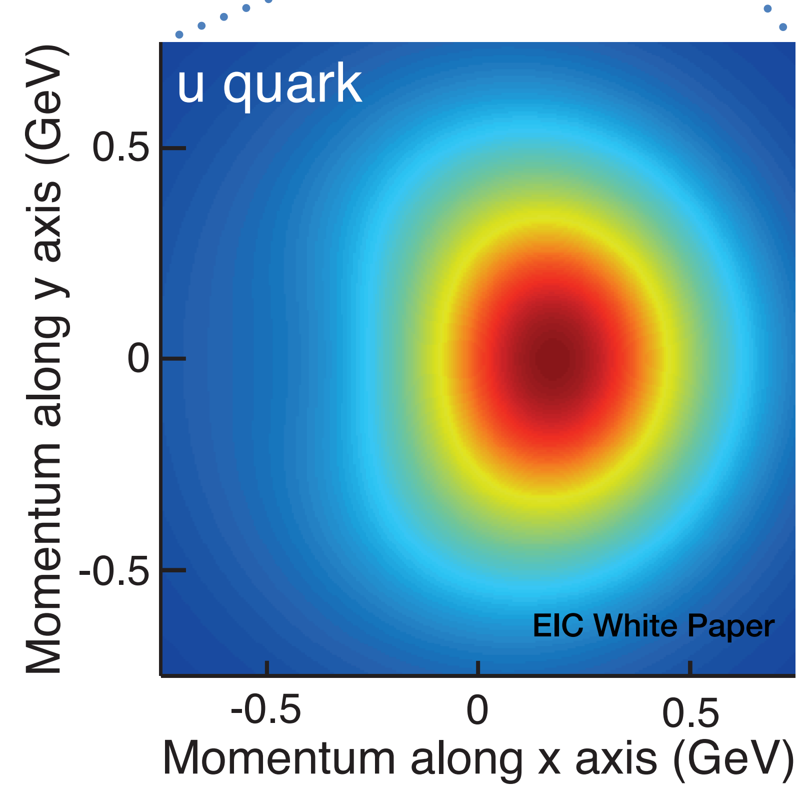


QUARK TMD DISTRIBUTIONS

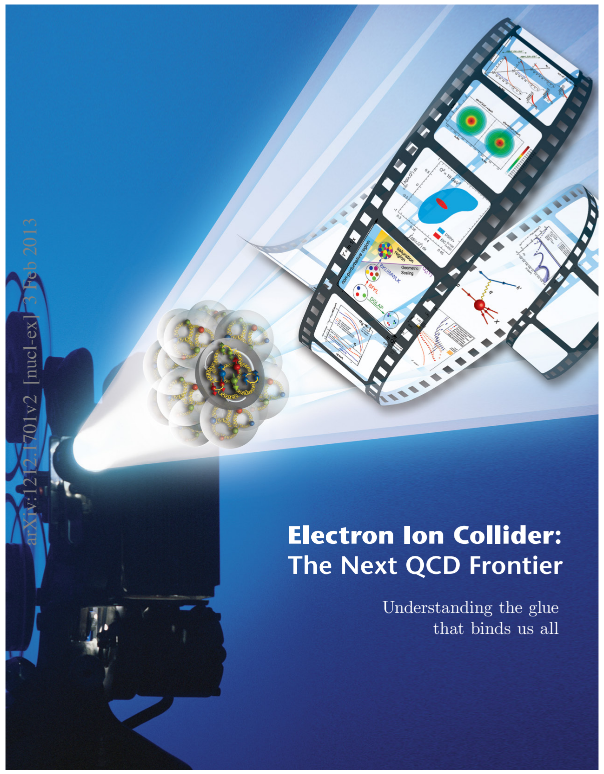


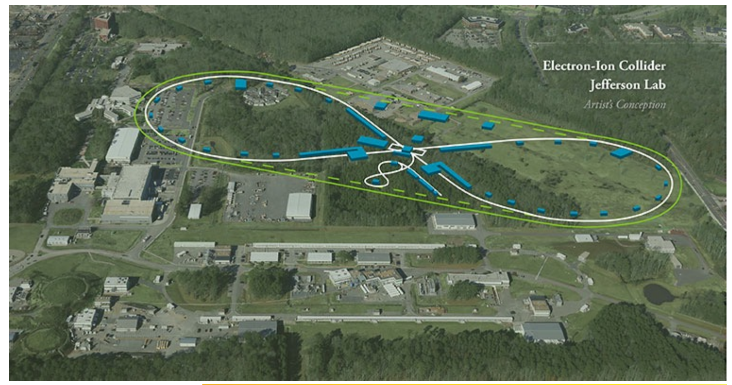


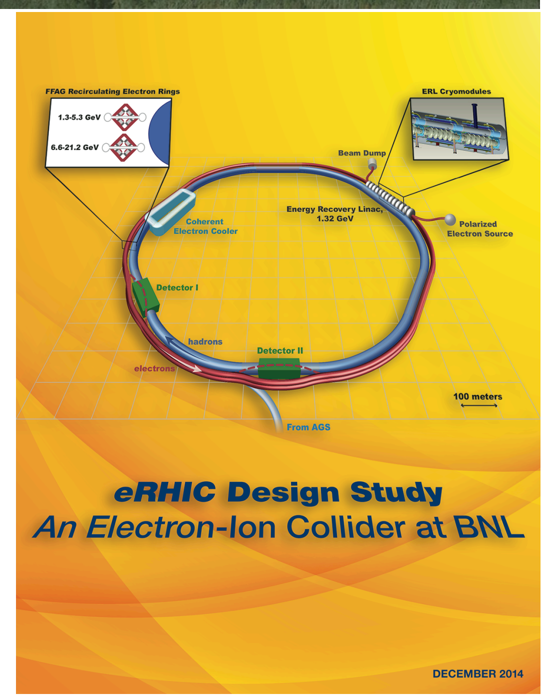


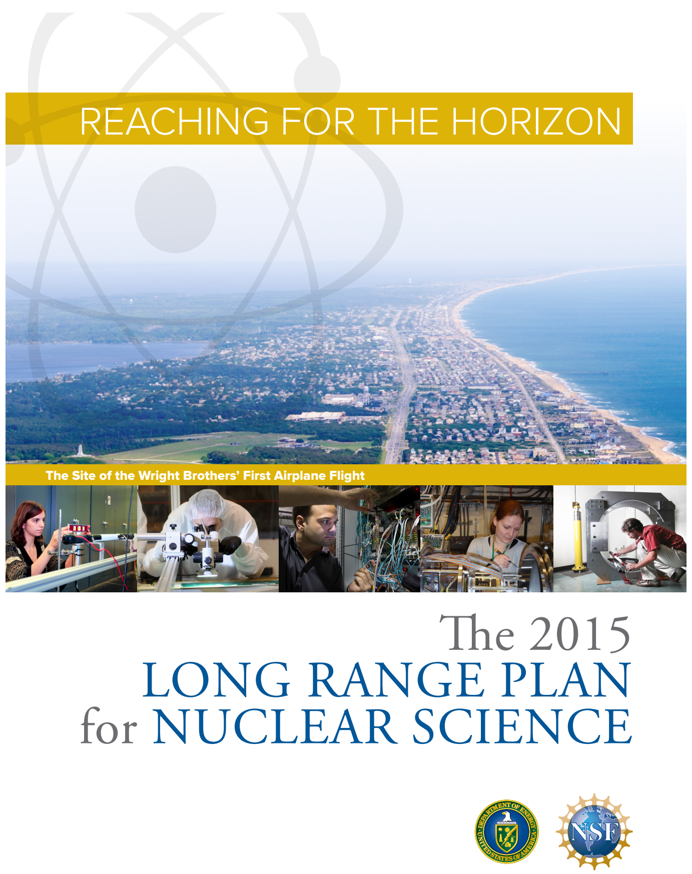


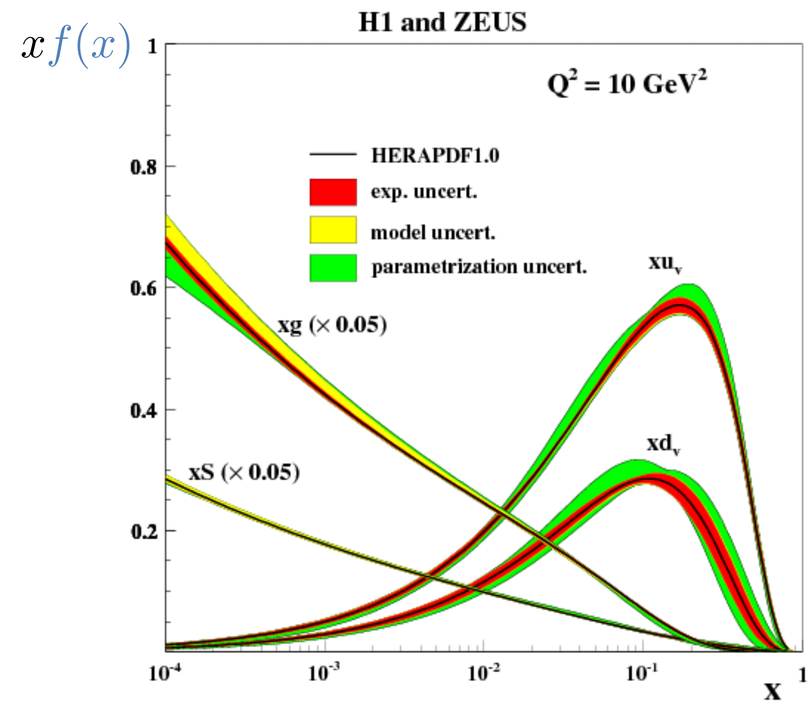
TMD DISTRIBUTIONS AT EIC



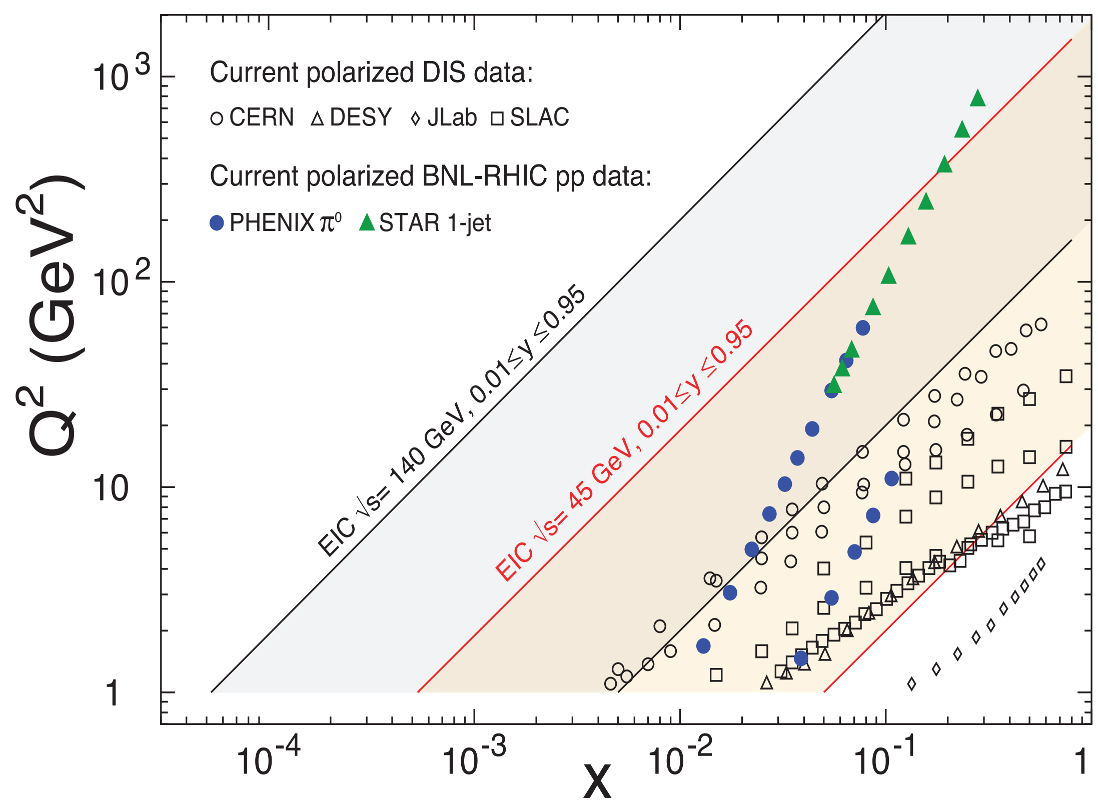




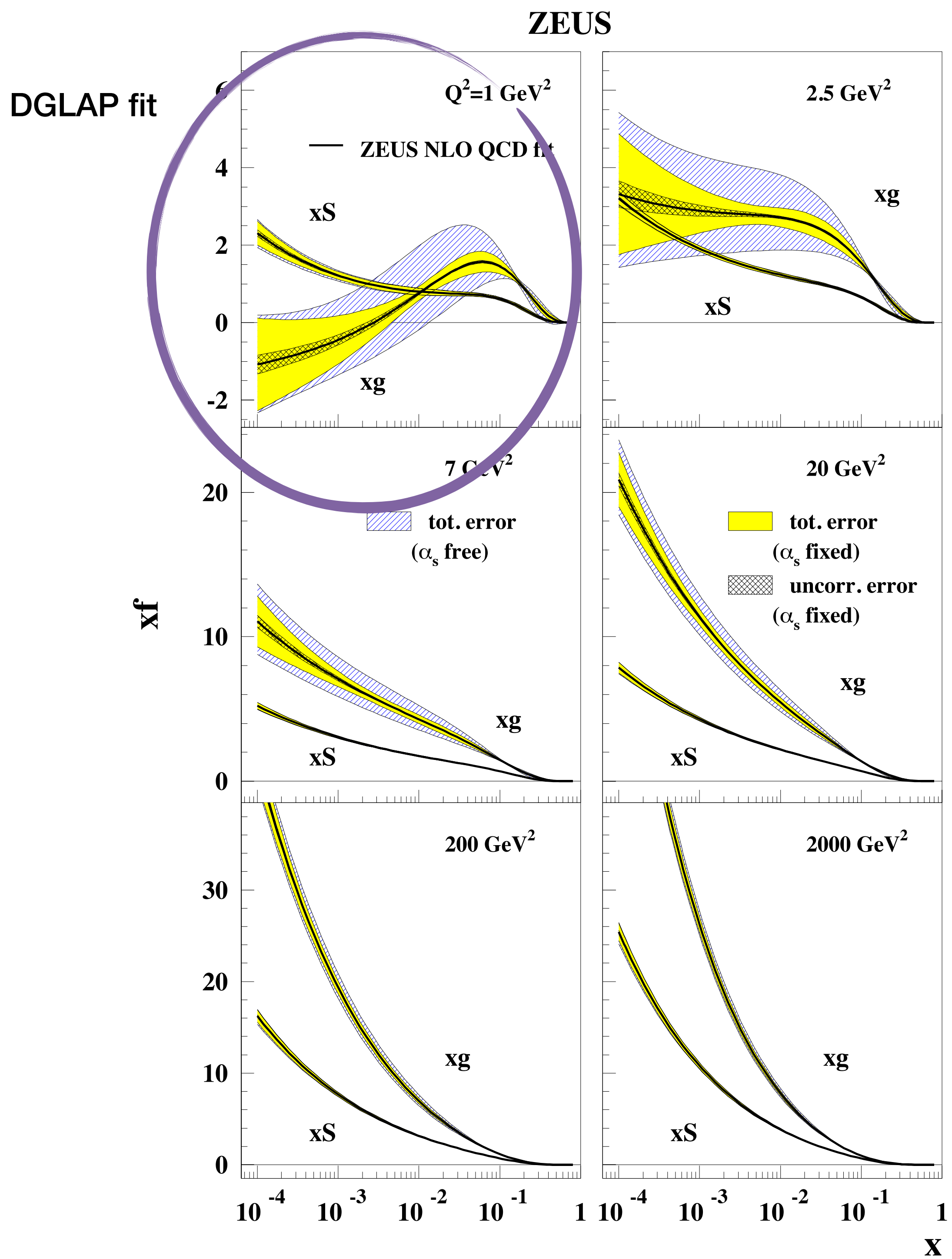


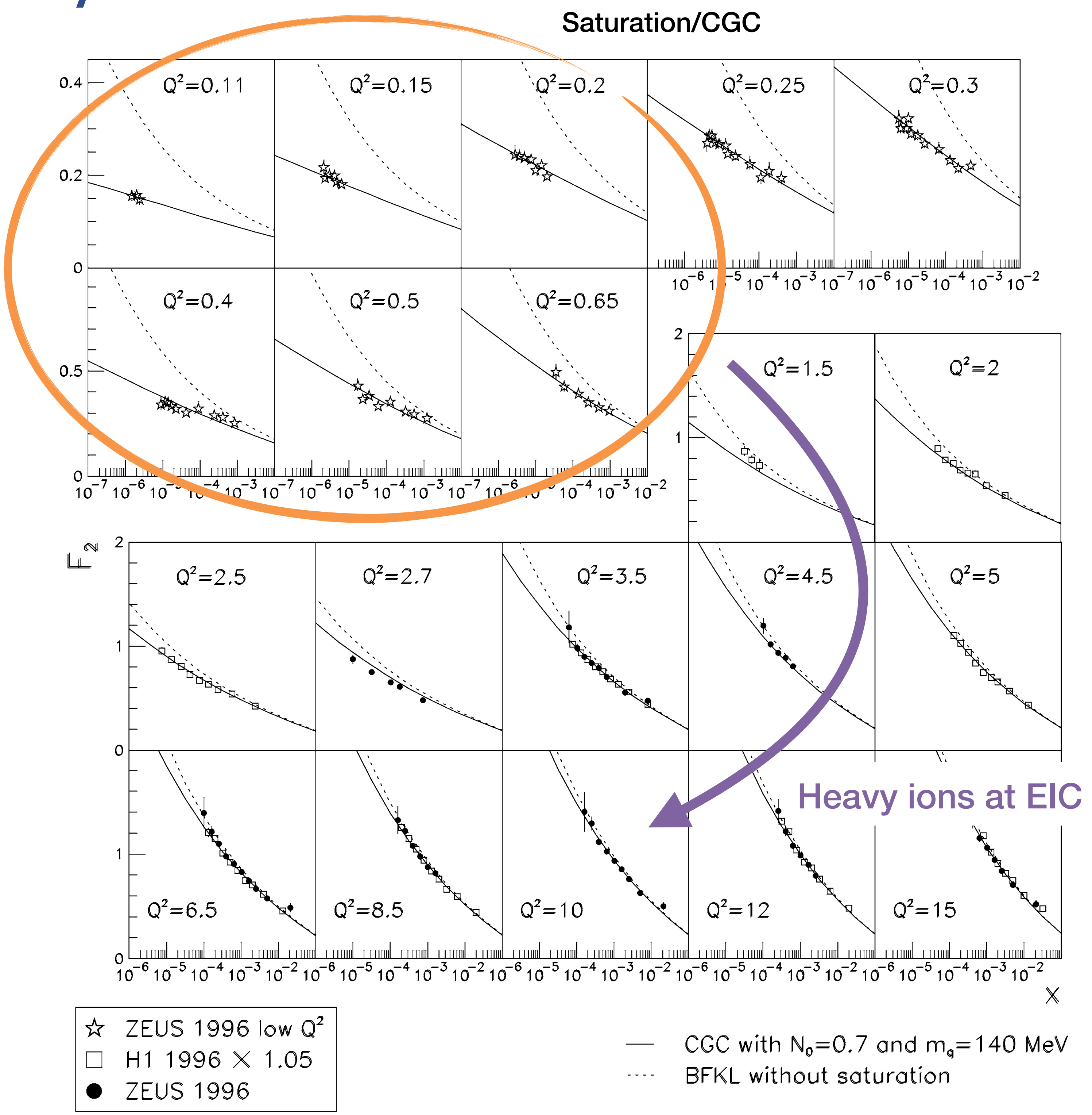


- Region of much smaller x
- We will be able to study gluon-matter distributions



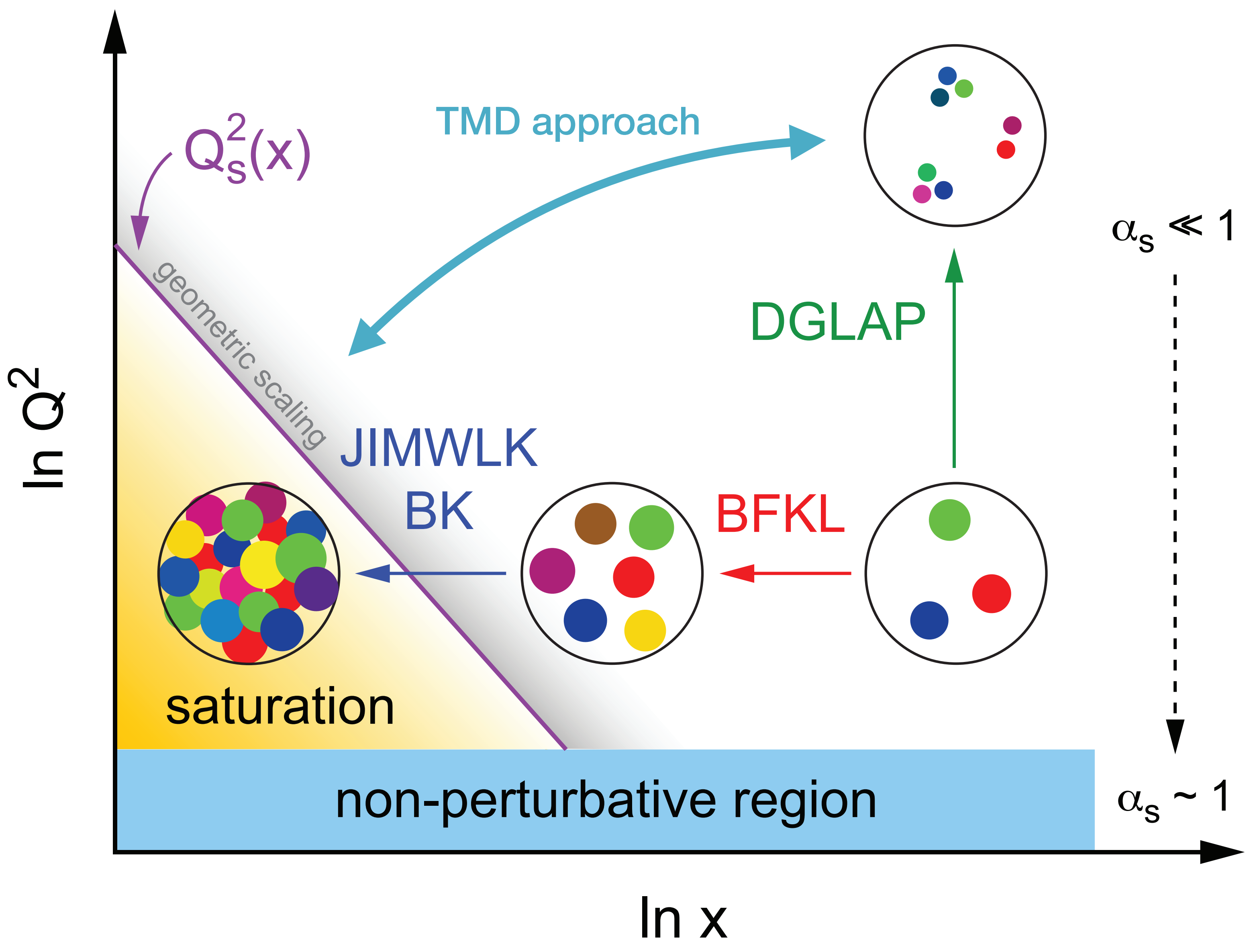
DGLAP vs. BFKL/BK

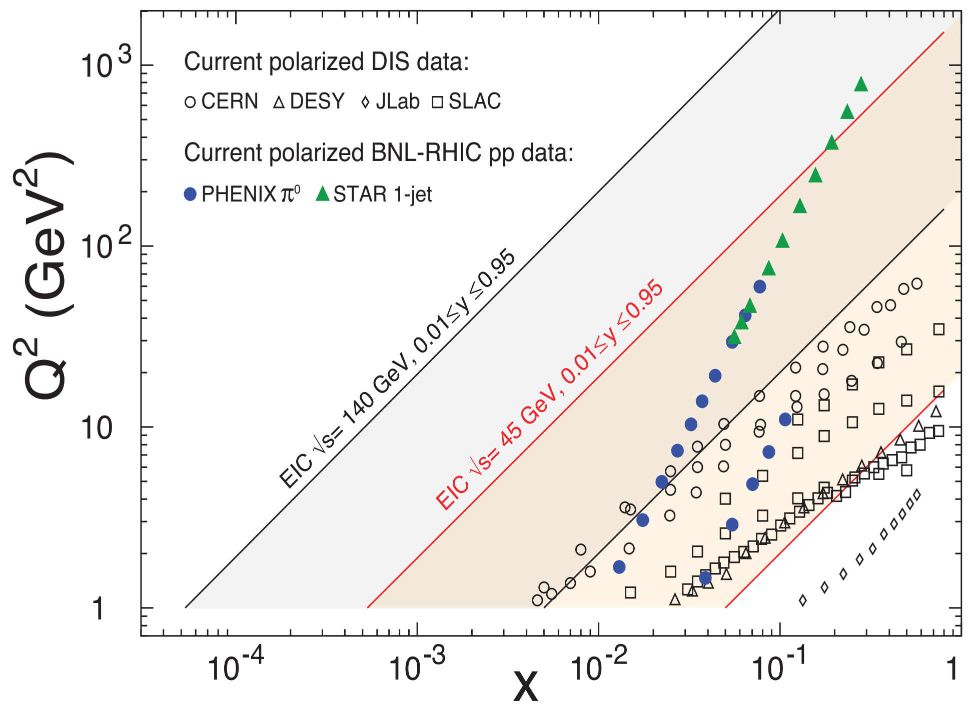




E. Iancu, K. Itakura, S. Munier
Phys. Lett. B 590 (2004) 199

DGLAP AND BFKL/BK

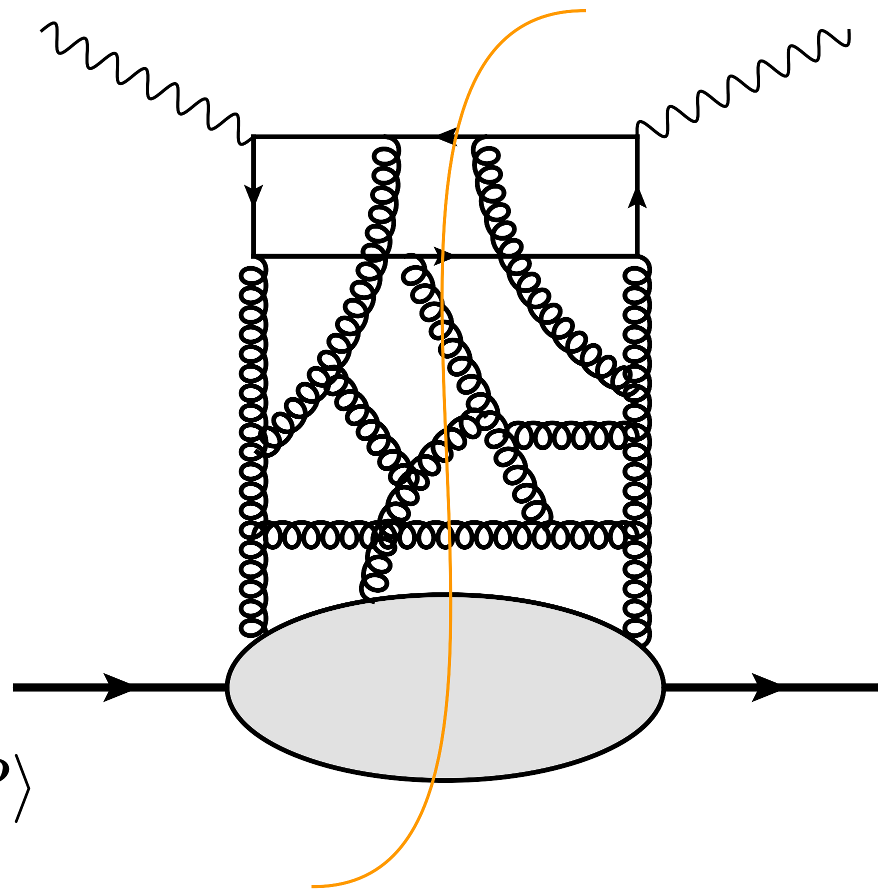




Particle production

Total cross section (collinear distributions):

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X \int d^4z e^{iqz}\langle P|j^\mu(z)|X\rangle\langle X|j^\nu(0)|P\rangle$$

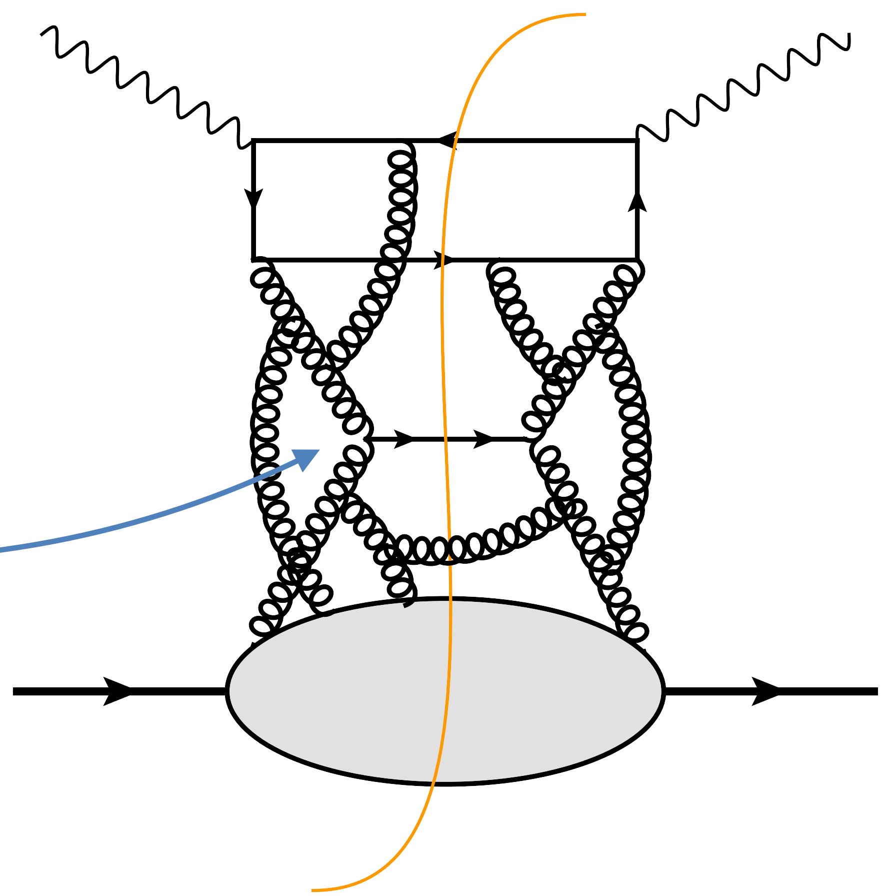


Particle production (TMD distributions):

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X \int d^4z e^{iqz}\langle P|j^\mu(z)|X+p\rangle\langle X+p|j^\nu(0)|P\rangle$$

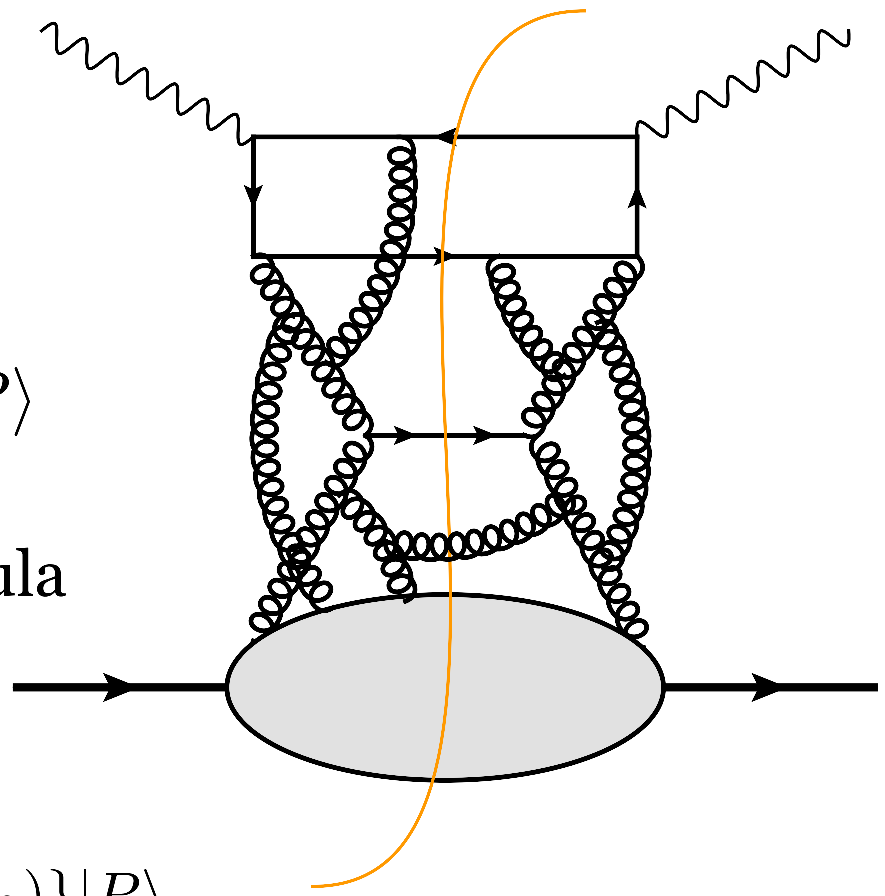
Scalar particle production:

$$\mathcal{L}_h = -\frac{1}{4}g_h F^{a\mu\nu}(z)F^a_{\mu\nu}(z)\phi(z)$$



Particle production

Particle production (TMD distributions):



$$W^{\mu\nu} = \frac{1}{4\pi} \sum_X \int d^4 z e^{iqz} \langle P|j^\mu(z)|X+p\rangle\langle X+p|j^\nu(0)|P\rangle$$

reduction formula

$$W^{\mu\nu} = \frac{g_h^2}{64\pi} \int d^4 z \int d^4 z_1 \int d^4 z_2 e^{iqz - ipz_1 + ipz_2}$$
$$\times \sum_X \langle P|\tilde{T}\{j^\mu(z) F^{a\rho\sigma}(z_1) F^a_{\rho\sigma}(z_1)\}|X\rangle\langle X|T\{j^\nu(0) F^{b\lambda\tau}(z_2) F^b_{\lambda\tau}(z_2)\}|P\rangle$$

different time-ordering

$$\sum_X |X\rangle\langle X|$$

$$W^{\mu\nu} = \frac{g_h^2}{64\pi} \int d^4 z \int d^4 z_1 \int d^4 z_2 e^{iqz - ipz_1 + ipz_2}$$
$$\times \int D\tilde{A} D\tilde{\psi}\ \Psi_p^*(\tilde{A},\tilde{\psi}) e^{-iS_{QCD}(\tilde{A},\tilde{\psi})} \tilde{j}^\mu(z) \tilde{F}^{a\rho\sigma}(z_1) \tilde{F}^a_{\rho\sigma}(z_1)$$
$$\times \int DAD\psi \Psi_p(A,\psi) e^{iS_{QCD}(A,\psi)} j^\nu(0) F^{b\lambda\tau}(z_2) F^b_{\lambda\tau}(z_2) \times \delta(\tilde{A}|_{t=+\infty} - A|_{t=+\infty}) \delta(\tilde{\psi}|_{t=+\infty} - \psi|_{t=+\infty})$$

KINEMATIC VARIABLES

Light-like vectors:

$$p_1^\mu \simeq q^\mu + \frac{Q^2}{s} P^\mu \qquad p_2^\mu \simeq P^\mu - \frac{M^2}{s} q^\mu$$

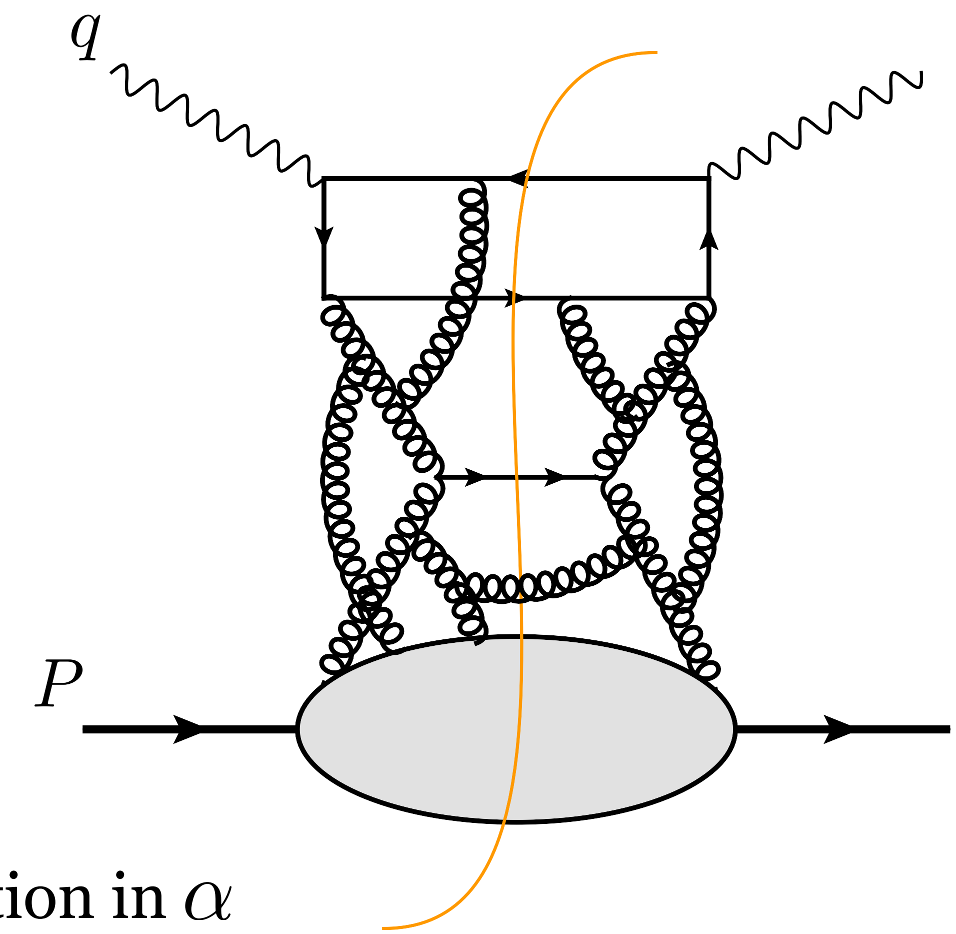
Sudakov momentum decomposition:

$$p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

Coordinate vector decomposition:

$$z^\mu = \frac{2}{s} z_* p_1^\mu + \frac{2}{s} z_\bullet p_2^\mu + z_\perp^\mu$$

$$z_* = \sqrt{\tfrac{s}{2}} z_+ \qquad z_\bullet = \sqrt{\tfrac{s}{2}} z_-$$



separation in α

rapidity: $y = \log \alpha$

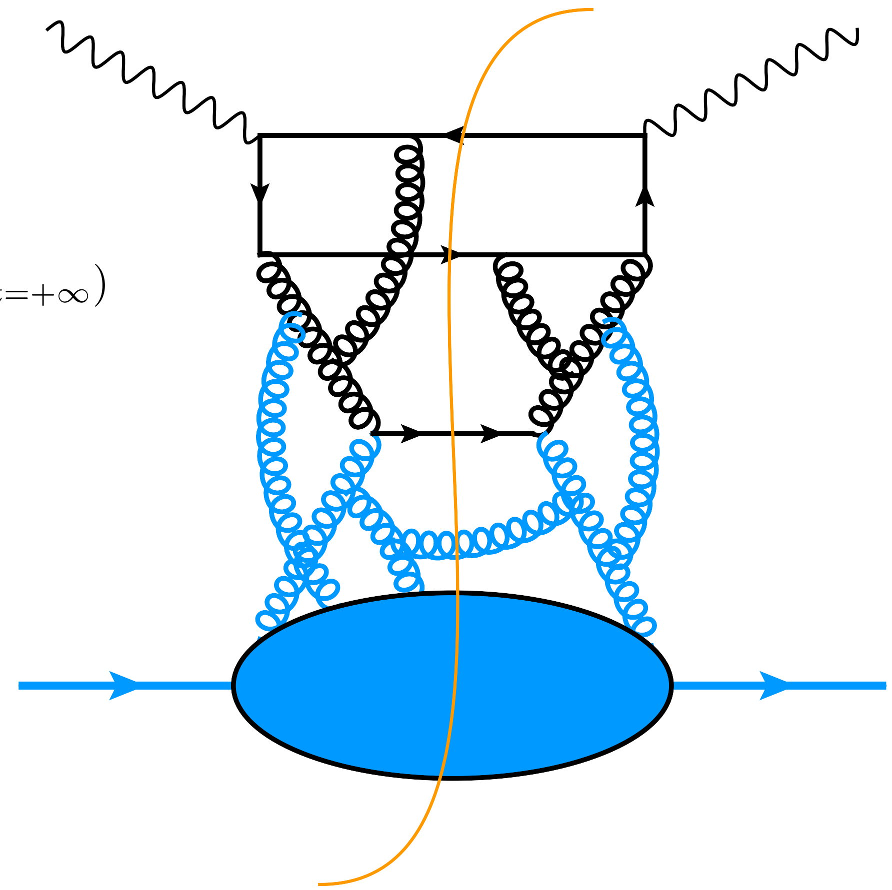
$$W^{\mu\nu} = \frac{g_h^2}{64\pi} \int d^4 z \int d^4 z_1 \int d^4 z_2 e^{iqz - ipz_1 + ipz_2}$$

$$\times \int D\tilde{A} D\tilde{\psi}\ \Psi_p^*(\tilde{A}, \tilde{\psi}) e^{-iS_{QCD}(\tilde{A}, \tilde{\psi})} \tilde{j}^\mu(z) \tilde{F}^{a\rho\sigma}(z_1) \tilde{F}^a_{\rho\sigma}(z_1)$$

$$\times \int DAD\psi \Psi_p(A, \psi) e^{iS_{QCD(A,\psi)}} j^\nu(0) F^{b\lambda\tau}(z_2) F^b_{\lambda\tau}(z_2) \times \delta(\tilde{A}|_{t=+\infty} - A|_{t=+\infty}) \delta(\tilde{\psi}|_{t=+\infty} - \psi|_{t=+\infty})$$

RAPIDITY FACTORIZATION

$$W^{\mu\nu} = \frac{g_h^2}{64\pi}\int d^4z \int d^4z_1 \int d^4z_2 e^{iqz-ipz_1+ipz_2}$$
$$\times \int D\tilde{A}D\tilde{\psi}\ \Psi_p^*(\tilde{A},\tilde{\psi})e^{-iS_{QCD}(\tilde{A},\tilde{\psi})}\tilde{j}^\mu(z)\tilde{F}^{a\rho\sigma}(z_1)\tilde{F}^a_{\rho\sigma}(z_1)$$
$$\times \int DAD\psi\Psi_p(A,\psi)e^{iS_{QCD}(A,\psi)}j^\nu(0)F^{b\lambda\tau}(z_2)F^b_{\lambda\tau}(z_2) \times \delta(\tilde{A}|_{t=+\infty} - A|_{t=+\infty})\delta(\tilde{\psi}|_{t=+\infty} - \psi|_{t=+\infty})$$



$$A \to A|_{\alpha>\sigma} + A|_{\alpha<\sigma}$$

$$W^{\mu\nu} = \frac{g_h^2}{64\pi}\int d^4z \int d^4z_1 \int d^4z_2 e^{iqz-ipz_1+ipz_2}$$
$$\times \int D\tilde{A}D\tilde{\psi}\int DAD\psi e^{-iS_{QCD}(\tilde{A},\tilde{\psi})}e^{iS_{QCD}(\tilde{A},\tilde{\psi})}\Psi_p^*(\tilde{A},\tilde{\psi})\Psi_p(A,\psi)\delta(\tilde{A}|_{t=+\infty} - A|_{t=+\infty})\delta(\tilde{\psi}|_{t=+\infty} - A|_{t=+\infty})$$
$$\times \sum_X \langle 0|\tilde{T}\{j^\mu(z)F^{a\rho\sigma}(z_1)\tilde{F}^a_{\rho\sigma}(z_1)\}|X\rangle_A\langle X|T\{j^\nu(0)F^{b\lambda\tau}(z_2)F^b_{\lambda\tau}(z_2)\}|0\rangle_A$$

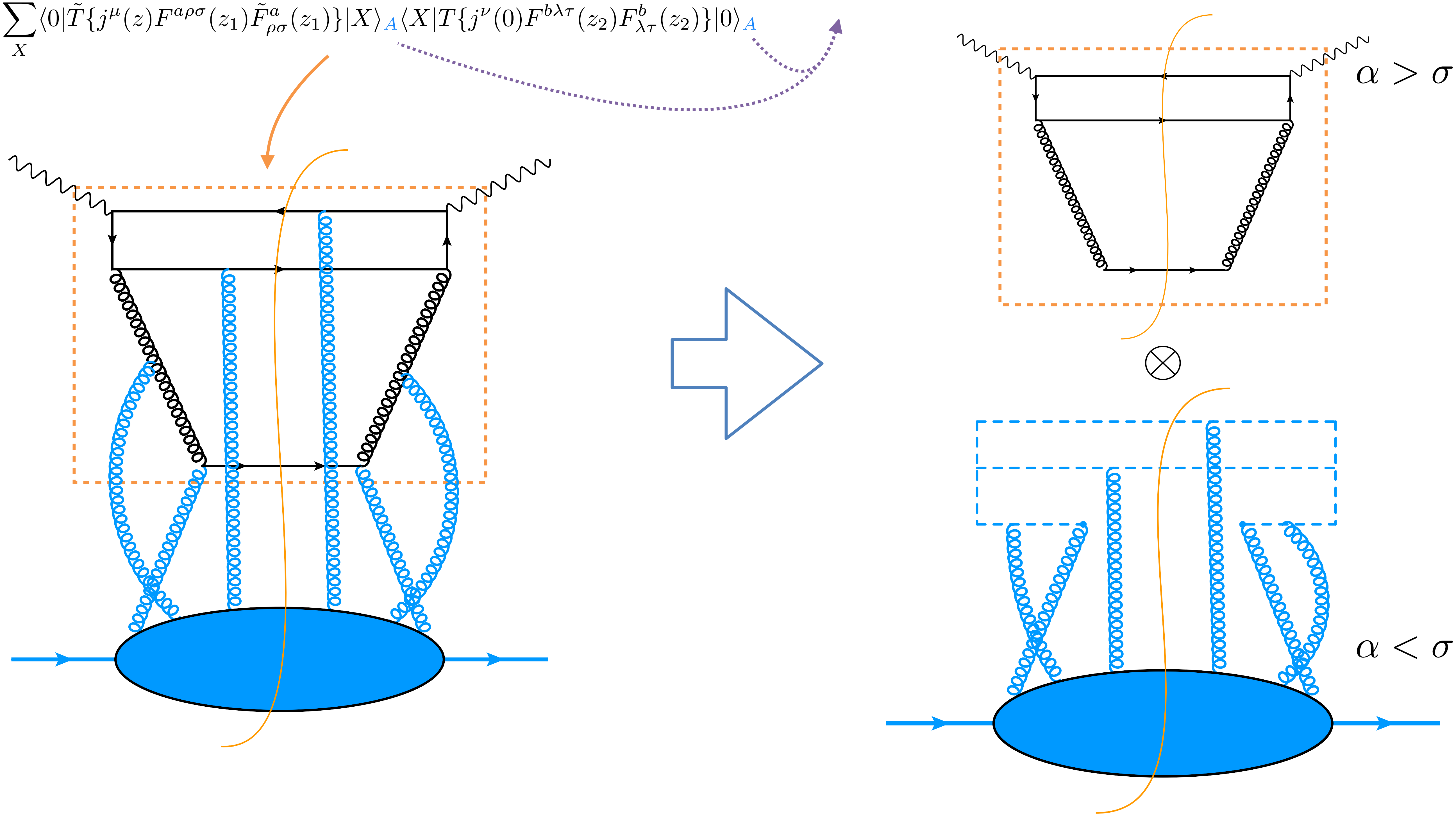
separate dependence on A fields

Rapidity factorization

$$W^{\mu\nu} = \frac{g_h^2}{64\pi}\int d^4z\int d^4z_1\int d^4z_2 e^{iqz-ipz_1+ipz_2}$$

$$\times\int D\tilde{A}D\tilde{\psi}\int DAD\psi e^{-iS_{QCD}(\tilde{A},\tilde{\psi})}e^{iS_{QCD}(\tilde{A},\tilde{\psi})}\Psi_p^*(\tilde{A},\tilde{\psi})\Psi_p(A,\psi)\delta(\tilde{A}|_{t=+\infty}-A|_{t=+\infty})\delta(\tilde{\psi}|_{t=+\infty}-A|_{t=+\infty})$$

$$\times\sum_X\langle 0|\tilde{T}\{j^\mu(z)F^{a\rho\sigma}(z_1)\tilde{F}^a_{\rho\sigma}(z_1)\}|X\rangle_A\langle X|T\{j^\nu(0)F^{b\lambda\tau}(z_2)F^b_{\lambda\tau}(z_2)\}|0\rangle_A$$

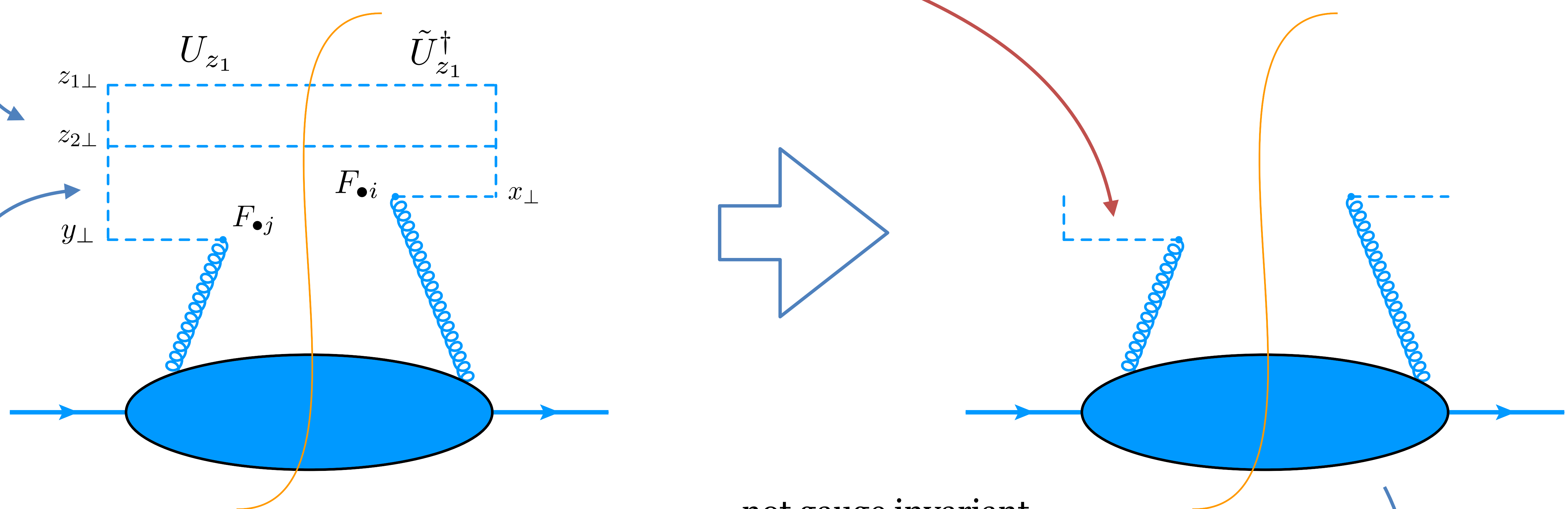


WILSON LINES

$$tr\langle P|\tilde{T}\{U_{z_2}[z_{2\perp},x_\perp]_{-\infty}[-\infty,x_*]_x F_{\bullet i}(x_*,x_\perp)[x_*,-\infty]_x[x_\perp,z_{1\perp}]_{-\infty}U^\dagger_{z_1}\}$$
$$\times T\{U_{z_1}[z_{1\perp},y_\perp]_{-\infty}\underline{[-\infty,y_*]_y F_{\bullet j}(y_*,y_\perp)[y_*,-\infty]_y}[y_\perp,z_{2\perp}]_{-\infty}U^\dagger_{z_2}\}|P\rangle$$

two different T-products

gauge invariant



not gauge invariant

space(light)-like distance

$$tr\langle P|\tilde{T}\{[-\infty,x_*]_x F_{\bullet i}(x_*,x_\perp)[x_*,-\infty]_x\}T\{[-\infty,y_*]_y F_{\bullet j}(y_*,y_\perp)[y_*,-\infty]_y[x_\perp,y_\perp]_{-\infty}\}|P\rangle$$

TMD DISTRIBUTION

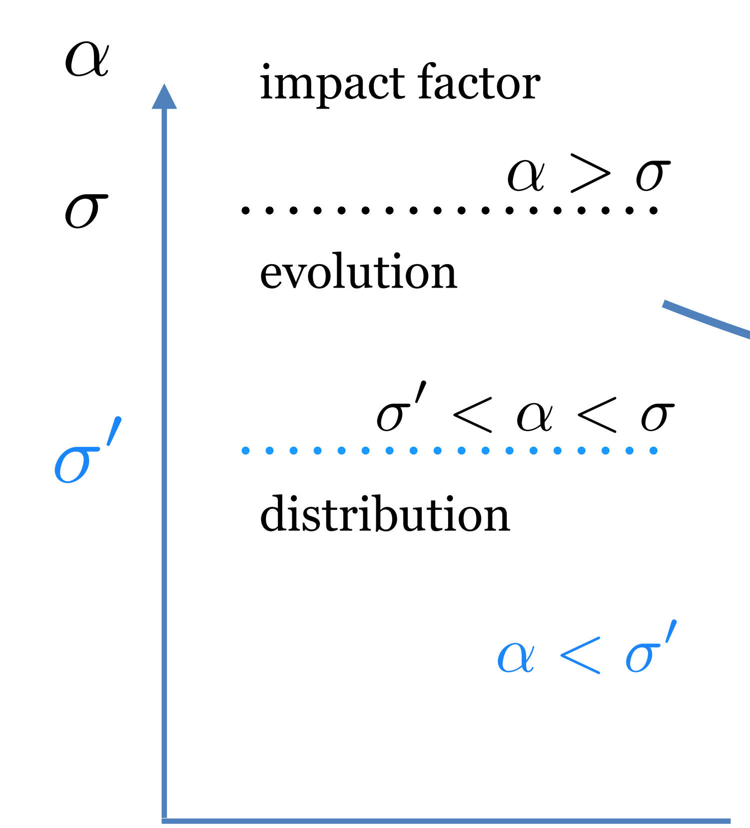
$$\langle P|\mathcal{G}_i^a(x_*,x_\perp)[x_\perp,y_\perp]^{ab}_{-\infty}\mathcal{G}_j^b(y_*,y_\perp)|P\rangle^\sigma$$

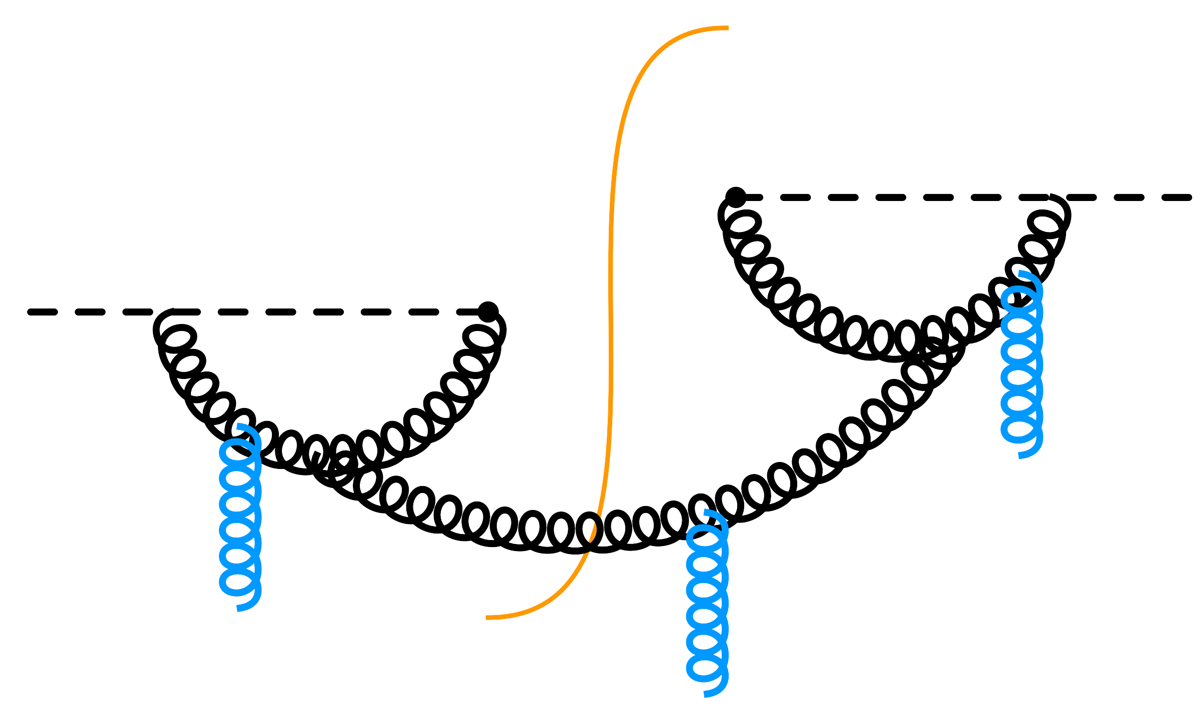
regulator

$$\mathcal{G}_i^a(x_B,x_\perp) \equiv \frac{2}{s}\int dx_*\; e^{ix_Bx_*}[-\infty,x_*]_x^{am}\, gF^m_{\bullet i}(x_*,x_\perp)$$

x_B

$$\tilde{\mathcal{G}}_j^a(x_B,y_\perp) \equiv \frac{2}{s}\int dy_*\; e^{-ix_By_*}\, g\tilde{F}^m_{\bullet j}(y_*,y_\perp)[y_*,-\infty]_y^{ma}$$

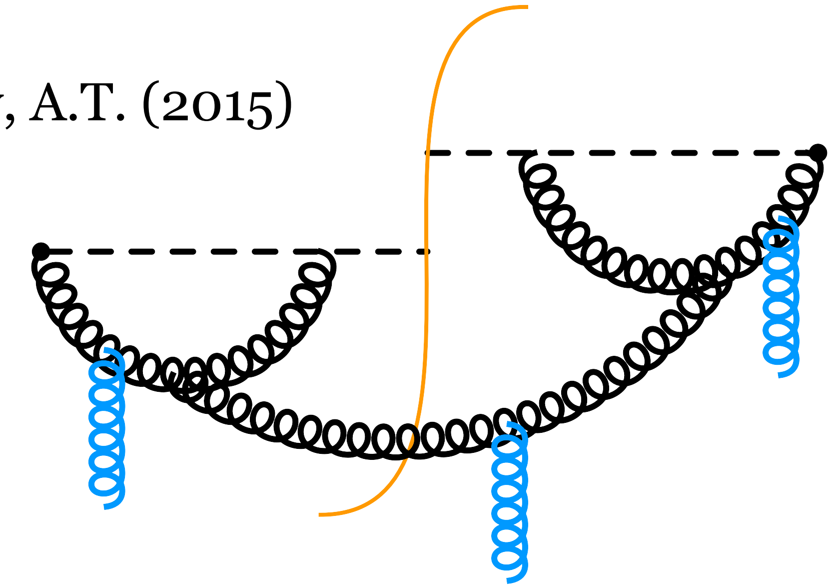




Evolution for future-point Wilson lines

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\mathcal{F}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle \\
&= -\alpha_s \mathrm{Tr}\Big\{\langle p|\int d^2k_\perp\, L_i{}^\mu(k,x_\perp,x_B)^{\mathrm{light-like}}\theta\Big(1-x_B-\frac{k_\perp^2}{\sigma s}\Big)L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}} \\
&+ 2\mathcal{F}_i(x_B,x_\perp)(y_\perp| -\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l - g_{jm}g^{kl})U\frac{1}{\sigma x_B s+p_\perp^2}U^\dagger \\
&+ \mathcal{F}_j(x_B)\frac{\sigma x_B s}{p_\perp^2(\sigma x_B s+p_\perp^2)}|y_\perp) \\
&+ 2(x_\perp|U\frac{1}{\sigma x_B s+p_\perp^2}U^\dagger(2\delta_i^k\delta_m^l - g_{im}g^{kl})(i\partial_k - U_k)\mathcal{F}_l(x_B)\frac{p^m}{p_\perp^2} \\
&+ \mathcal{F}_i(x_B)\frac{\sigma x_B s}{p_\perp^2(\sigma x_B s+p_\perp^2)}|x_\perp)\mathcal{F}_j(x_B,y_\perp)|p\rangle\Big\} + O(\alpha_s^2)
\end{aligned}
$$

Ian Balitsky, A.T. (2015)



Moderate-x (DGLAP). Linear evolution

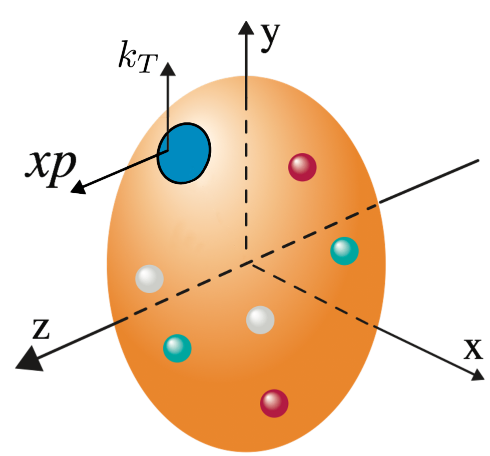
$x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim 1$

Sudakov evolution

$x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \ll s$

Small-x. Non-linear evolution

$x_B \ll 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \ll s$



The equation describes the rapidity evolution of gluon TMD operator for any x_B and transverse momenta

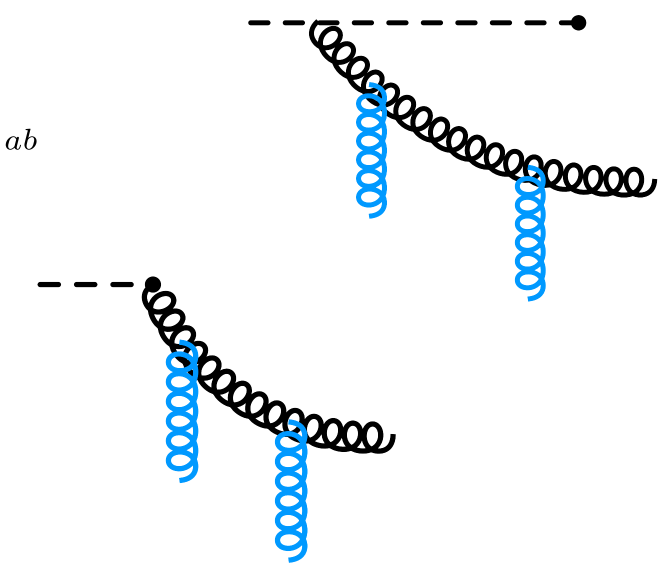
This expression is UV and IR convergent

LIPATOV'S VERTEX

$$\langle \tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp)\rangle^{\ln\sigma} = -\int_{\sigma'}^{\sigma}\frac{\bar{d}\alpha}{2\alpha}\,\bar{d}^2k_\perp\,\tilde{L}^{am;\rho}_i(k,x_\perp,\beta_B)L^{ma}_{\rho j}(k,y_\perp,\beta_B)^{\ln\sigma'}$$

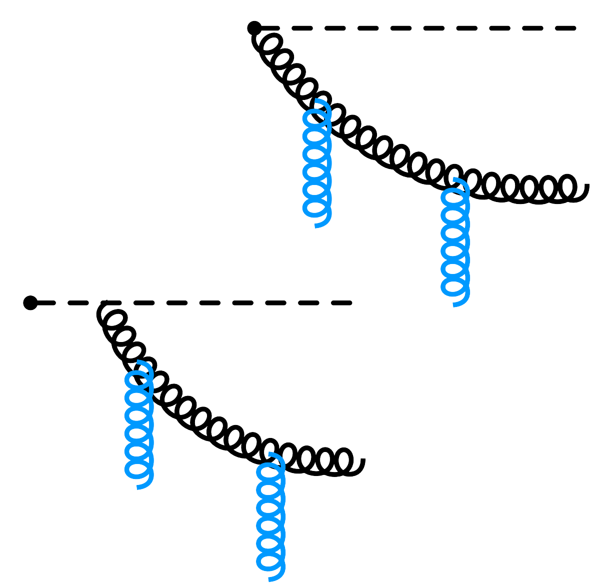
Past-point Wilson lines:

$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{light-like} = i\lim_{k^2\to 0}k^2\langle A^a_\mu(k)\mathcal{G}^b_i(\beta_B,y_\perp)\rangle \overset{light-like}{=} (k_\perp|2\Big\{-\frac{k^\perp_\mu}{k^2_\perp}U + U\frac{p^\perp_\mu}{p^2_\perp}\Big\}\mathcal{G}_i(\beta_B,y_\perp)|y_\perp)^{ab}$$
$$+g(k_\perp|U\frac{p^2_\perp g_{\mu i}+2p^\perp_\mu p_i}{\alpha\beta_B s+p^2_\perp} - \frac{k^2_\perp g_{\mu i}+2k^\perp_\mu k_i}{\alpha\beta_B s+k^2_\perp}U|y_\perp)^{ab} + \frac{2gk^\perp_\mu}{k^2_\perp}e^{-i(k,y)_\perp}U^{ae}_{y_\perp}\mathcal{G}^{eb}_i(\beta_B+\frac{k^2_\perp}{\alpha s},y_\perp)$$
$$-2ge^{-i(k,y)_\perp}\Big\{\frac{\delta^j_\mu k_i+\delta^j_i k^\perp_\mu - g_{\mu i}k^j}{\alpha\beta_B s+k^2_\perp} + \frac{g_{\mu i}k^2_\perp k^j + 2k^\perp_\mu k_i k^j}{(\alpha\beta_B s+k^2_\perp)^2}\Big\}\Big(U^{ae}_{y_\perp}\mathcal{G}^{en}_j(\beta_B+\frac{k^2_\perp}{\alpha s},y_\perp) - i\partial_j U^{an}\Big) + O(p_{2\mu})$$



Future-point Wilson lines:

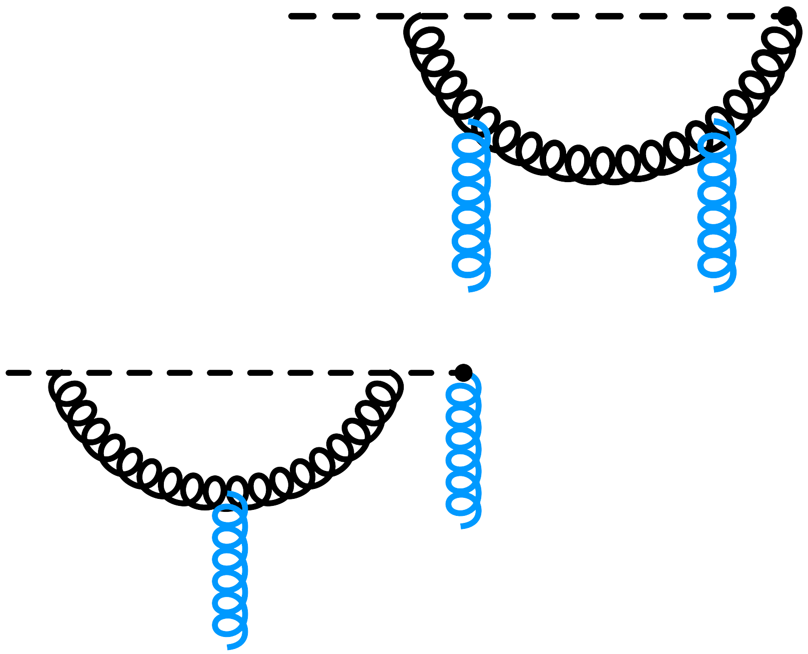
$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{light-like} = i\lim_{k^2\to 0}k^2\langle A^a_\mu(k)\mathcal{F}^b_i(\beta_B,y_\perp)\rangle^{light-like}$$
$$= g(k_\perp|U\frac{p^2_\perp g_{\mu i}+2p^\perp_\mu p_i}{\alpha\beta_B s+p^2_\perp}U^\dagger - \frac{k^2_\perp g_{\mu i}+2k^\perp_\mu k_i}{\alpha\beta_B s+k^2_\perp}|y_\perp)^{ab} + \frac{2gk^\perp_\mu}{k^2_\perp}e^{-i(k,y)_\perp}\mathcal{F}^{ab}_i(\beta_B+\frac{k^2_\perp}{\alpha s},y_\perp)$$
$$-2ge^{-i(k,y)_\perp}\Big[\frac{\delta^j_\mu k_i+\delta^j_i k^\perp_\mu - g_{\mu i}k^j}{\alpha\beta_B s+k^2_\perp} + \frac{g_{\mu i}k^2_\perp k^j + 2k^\perp_\mu k_i k^j}{(\alpha\beta_B s+k^2_\perp)^2}\Big]\Big(\mathcal{F}_j(\beta_B+\frac{k^2_\perp}{\alpha s},y_\perp) - i\partial_j U_y U^\dagger_y\Big) + O(p_{2\mu})$$



VIRTUAL CORRECTION

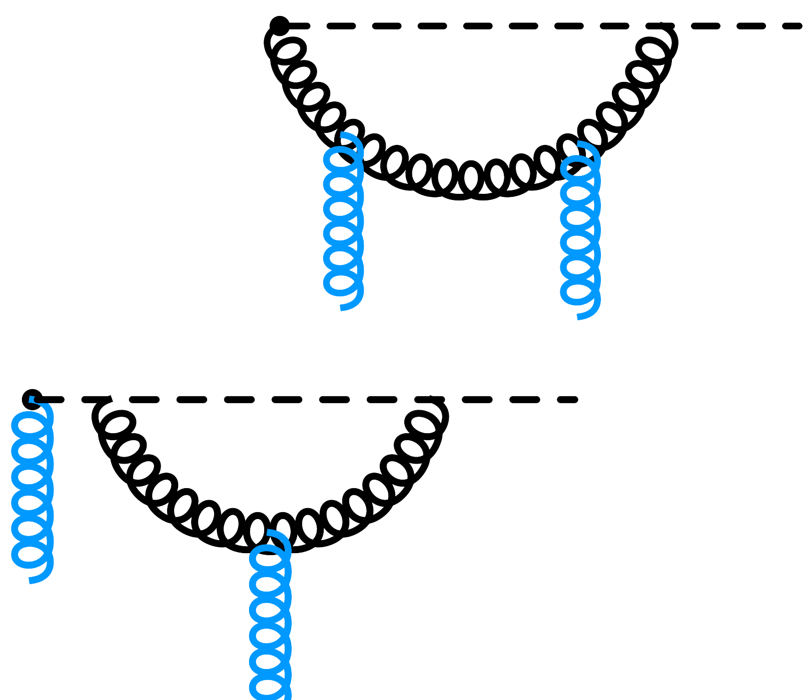
Past-point Wilson lines:

$$\langle \mathcal{G}_i^n(\beta_B, y_\perp)\rangle = -ig^2 f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!^{-}\alpha}{\alpha}(y_\perp| - U^\dagger \frac{1}{\alpha\beta_B s - p_\perp^2}U(2\delta_i^k\delta_j^l - g_{ij}g^{kl})$$
$$\times(i\overrightarrow{\partial}_k + iU^\dagger\partial_k U)\mathcal{G}_l(\beta_B)\frac{p^j}{p_\perp^2} + \mathcal{G}_i^{kl}(\beta_B)(y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s - p_\perp^2 + i\epsilon)}|y_\perp)^{kl}$$



Future-point Wilson lines:

$$\langle \mathcal{F}_i^n(\beta_B, y_\perp)\rangle \;=\; -ig^2 f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!^{-}\alpha}{\alpha}(y_\perp| - \frac{p^j}{p_\perp^2}\mathcal{F}_l(\beta_B)(i\overleftarrow{\partial}_k + i\partial_k U U^\dagger)$$
$$\times\;(2\delta_i^k\delta_j^l - g_{ij}g^{kl})U\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger + \mathcal{F}_i(\beta_B)\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s + p_\perp^2)}|y_\perp)^{kl}$$

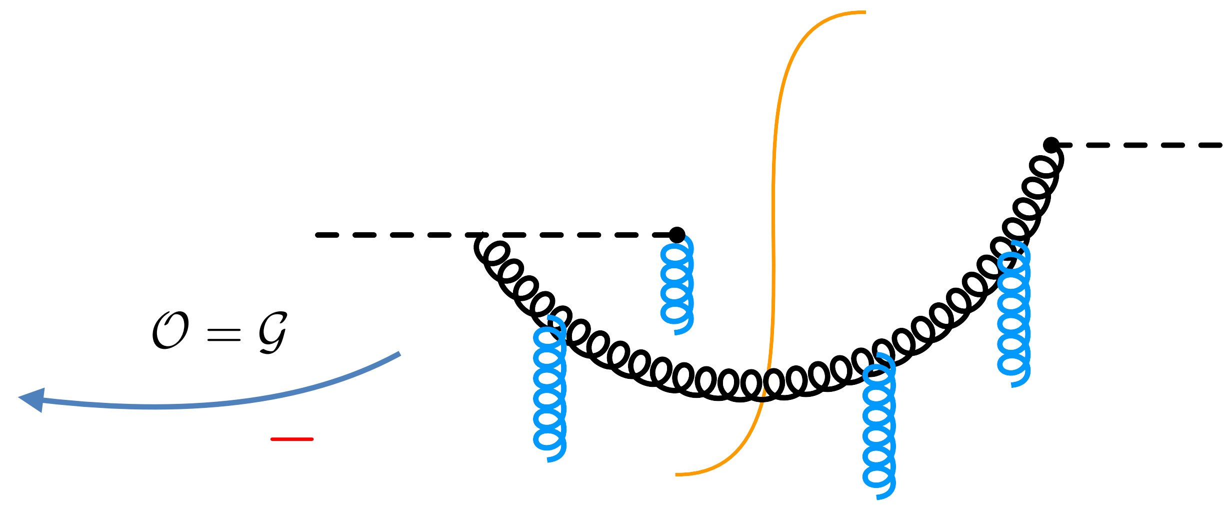


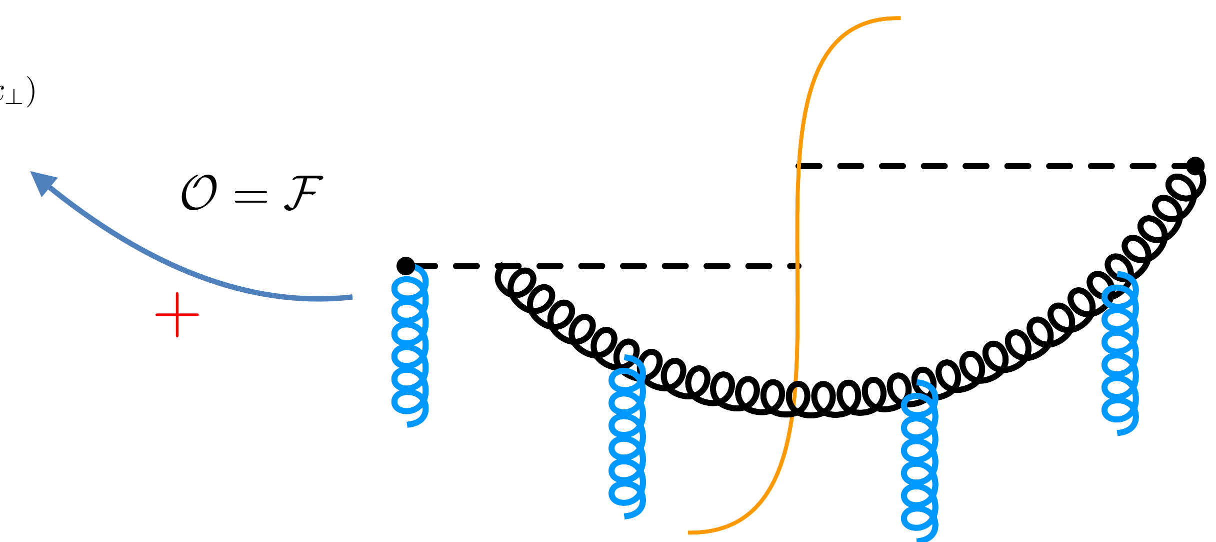
Non-linear part

This result is valid for all β_B and $k_\perp$

General evolution equation

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\mathcal{O}_i^a(\beta_B,x_\perp)\mathcal{O}_j^a(\beta_B,y_\perp)|p\rangle \\
&= -\alpha_s\langle p|Tr\Big\{\int \bar{d}^2k_\perp\theta(1-\beta_B-\frac{k_\perp^2}{\sigma s})\Big[(x_\perp|\Big([\pm\infty,-\infty]\frac{1}{\sigma\beta_B s+p_\perp^2}([-\infty,\pm\infty]k_k \\
&+p_k[-\infty,\pm\infty])\frac{\sigma\beta_B s\delta_i^\mu-2k_\perp^\mu k_i}{\sigma\beta_B s+k_\perp^2}-2k_\perp^\mu g_{ik}[\pm\infty,-\infty]\frac{1}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty] \\
&-2\delta_k^\mu[\pm\infty,-\infty]\frac{p_i}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]\Big)\mathcal{O}^k(\beta_B+\frac{k_\perp^2}{\sigma s})[\pm\infty,\infty]|k_\perp) \\
&\times(k_\perp|[\infty,\pm\infty]\mathcal{O}^l(\beta_B+\frac{k_\perp^2}{\sigma s})\Big(\frac{\sigma\beta_B s g_{\mu j}-2k_\mu^\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_l[\pm\infty,-\infty]+[\pm\infty,-\infty]p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty] \\
&-2k_\mu^\perp g_{jl}[\pm\infty,-\infty]\frac{1}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]-2g_{\mu l}[\pm\infty,-\infty]\frac{p_j}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]\Big)|y_\perp) \\
&+2(x_\perp|\mathcal{O}_i(\beta_B+\frac{p_\perp^2}{\sigma s})\frac{p_\perp^\mu}{p_\perp^2}[\pm\infty,\infty]|k_\perp) \\
&\times(k_\perp|[\infty,\pm\infty]\mathcal{O}^l(\beta_B+\frac{k_\perp^2}{\sigma s})\Big(\frac{\sigma\beta_B s g_{\mu j}-2k_\mu^\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_l[\pm\infty,-\infty]+[\pm\infty,-\infty]p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty] \\
&-2k_\mu^\perp[\pm\infty,-\infty]\frac{g_{jl}}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]-2g_{\mu l}[\pm\infty,-\infty]\frac{p_j}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]\Big)|y_\perp) \\
&+2(x_\perp|\Big([\pm\infty,-\infty]\frac{1}{\sigma\beta_B s+p_\perp^2}([-\infty,\pm\infty]k_k+p_k[-\infty,\pm\infty])\frac{\sigma\beta_B s\delta_i^\mu-2k_\perp^\mu k_i}{\sigma\beta_B s+k_\perp^2} \\
&-2k_\perp^\mu[\pm\infty,-\infty]\frac{g_{ik}}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]-2\delta_k^\mu[\pm\infty,-\infty]\frac{p_i}{\sigma\beta_B s+p_\perp^2}[-\infty,\pm\infty]\Big)\mathcal{O}^k(\beta_B+\frac{k_\perp^2}{\sigma s})[\pm\infty,\infty]|k_\perp) \\
&\times(k_\perp|[\infty,\pm\infty]\frac{p_\mu^\perp}{p_\perp^2}\mathcal{O}_j(\beta_B+\frac{p_\perp^2}{\sigma s},y_\perp)|y_\perp)\Big] \\
&+2\mathcal{O}_i(\beta_B,x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{O}_k(\beta_B)(i\overleftarrow{\partial}_l+i\partial_l[\pm\infty,\mp\infty][\mp\infty,\pm\infty]) \\
&\times(2\delta_m^k\delta_j^l-g_{jm}g^{kl})[\pm\infty,\mp\infty]\frac{1}{\sigma\beta_B s\pm p_\perp^2}[\mp\infty,\pm\infty]|y_\perp) \\
&+2(x_\perp|[\pm\infty,\mp\infty]\frac{1}{\sigma\beta_B s\pm p_\perp^2}[\mp\infty,\pm\infty](2\delta_i^k\delta_m^l-g_{im}g^{kl}) \\
&\times(i\overrightarrow{\partial}_k-i\partial_k[\pm\infty,\mp\infty][\mp\infty,\pm\infty])\mathcal{O}_l(\beta_B)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{O}_j(y_\perp,\beta_B)\Big\} \\
&-4\int\frac{\bar{d}^2k_\perp}{k_\perp^2}\Big[\theta(1-\beta_B-\frac{k_\perp^2}{\sigma s})e^{ik_\perp(x-y)_\perp}\mathcal{O}_i(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp)\mathcal{O}_j(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp)-\frac{\sigma\beta_B s}{\sigma\beta_B s+k_\perp^2}\mathcal{O}_i(\beta_B,x_\perp)\mathcal{O}_j(\beta_B,y_\perp)
\end{aligned}
$$





Moderate-x (DGLAP). Linear evolution

$x_B\sim 1$ and $k_\perp^2\sim(x-y)_\perp^{-2}\sim 1$

Sudakov evolution

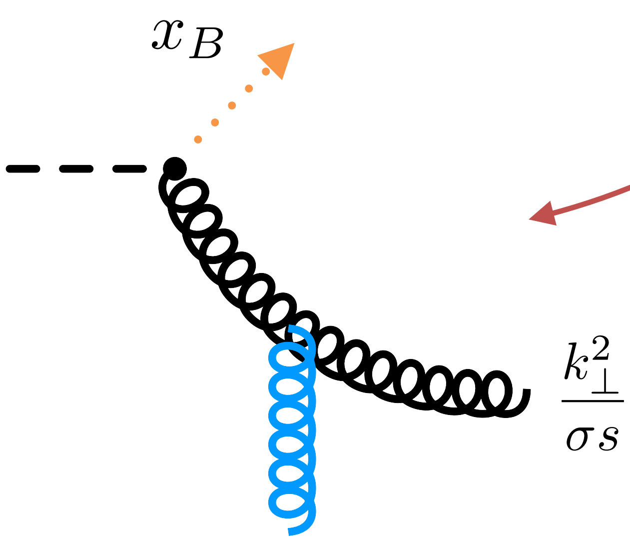
$x_B\sim 1$ and $k_\perp^2\sim(x-y)_\perp^{-2}\ll s$

Small-x. Non-linear evolution

$x_B\ll 1$ and $k_\perp^2\sim(x-y)_\perp^{-2}\ll s$

MODERATE-X LIMIT

$$\frac{d}{d\ln\sigma}\langle p|\mathcal{G}^a_i(x_B,x_\perp)\mathcal{G}^a_j(x_B,y_\perp)|p\rangle$$
$$= -\alpha_s\mathrm{Tr}\Big\{\langle p|\int {d\!\!\bar{}}^2k_\perp\, L_i^{\ \mu}(k,x_\perp,x_B)\theta\big(1-x_B-\frac{k_\perp^2}{\sigma s}\big)L_{\mu j}(k,y_\perp,x_B)|p\rangle\Big\} + \textit{virtual part}$$



$x_B \sim 1$

$\sigma' \ll \sigma$

$k'^2_\perp \ll k^2_\perp$

strict ordering

$$\langle p|\mathcal{F}^a_i(\beta_B,y_\perp)\mathcal{F}^a_j(\beta_B,0_\perp)|p+\xi p_2\rangle =$$
$$= 2\pi^2\delta(\xi)g^2\Big[-g_{ij}\mathcal{D}(\beta_B,y_\perp)-\frac{1}{m^2}(2\partial_i\partial_j+g_{ij}\partial_\perp^2)\mathcal{H}(\beta_B,y_\perp)\Big]$$

P. J. Mulders, J. Rodrigues (2001)

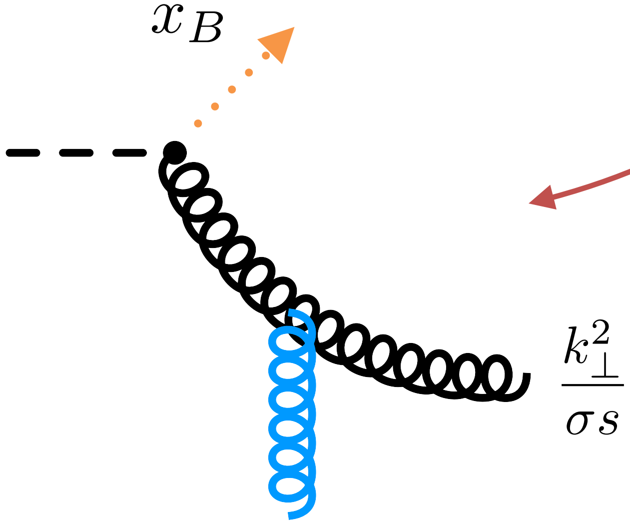
linearization of the evolution equation

$$\frac{d}{d\ln\sigma}\Big[g_{ij}\alpha_s\mathcal{D}(\beta_B,z_\perp)+\frac{4}{m^2}(2z_iz_j+g_{ij}z_\perp^2)\alpha_s\mathcal{H}''(\beta_B,z_\perp)\Big]$$
$$=\frac{\alpha_sN_c}{\pi}\int_{\beta_B}^1\frac{dz'}{z'}\Big\{g_{ij}J_0\Big(|z_\perp|\sqrt{\sigma\beta_Bs\frac{1-z'}{z'}}\Big)\alpha_s\mathcal{D}\big(\frac{\beta_B}{z'},z_\perp\big)\Big[\frac{1}{z'(1-z')}-2+z'-z'^2\Big]$$
$$+J_2\Big(|z_\perp|\sqrt{\sigma\beta_Bs\frac{1-z'}{z'}}\Big)\big(g_{ij}+\frac{2z_iz_j}{z^2}\big)\alpha_s\mathcal{D}\big(\frac{\beta_B}{z'},z_\perp\big)\Big[\frac{1}{z'}-1\Big]$$
$$+\frac{4}{m^2}J_0\Big(|z_\perp|\sqrt{\sigma\beta_Bs\frac{1-z'}{z'}}\Big)(2z_iz_j+g_{ij}z_\perp^2)\alpha_s\mathcal{H}''\big(\frac{\beta_B}{z'},z_\perp\big)\Big[\frac{1}{z'(1-z')}-\frac{1}{z'}-1\Big]$$
$$-\frac{4g_{ij}}{m^2}J_2\Big(|z_\perp|\sqrt{\sigma\beta_Bs\frac{1-z'}{z'}}\Big)z_\perp^2\alpha_s\mathcal{H}''\big(\frac{\beta_B}{z'},z_\perp\big)\Big[-z'+z'^2\Big]\Big\}$$
$$-\frac{\alpha_sN_c}{\pi}\int_0^1\frac{dz'}{(1-z')}\Big[g_{ij}\alpha_s\mathcal{D}(\beta_B,z_\perp)+\frac{4}{m^2}(2z_iz_j+g_{ij}z_\perp^2)\alpha_s\mathcal{H}''(\beta_B,z_\perp)\Big]$$

Coincides with the equation for future-point Wilson lines

Moderate-x limit

$$\frac{d}{d\ln\sigma}\langle p|\mathcal{G}_i^a(x_B,x_\perp)\mathcal{G}_j^a(x_B,y_\perp)|p\rangle$$
$$= -\alpha_s \mathrm{Tr}\Big\{\langle p|\int d\!\!\bar{}^{\,2}k_\perp\, L_i{}^{\mu}(k,x_\perp,x_B)\theta\Big(1-x_B-\frac{k_\perp^2}{\sigma s}\Big)L_{\mu j}(k,y_\perp,x_B)|p\rangle\Big\} + \textit{virtual part}$$



$x_B \sim 1$

$\sigma' \ll \sigma$

$k_\perp'^2 \ll k_\perp^2$

strict ordering

$$\langle p|\mathcal{F}_i^a(\beta_B,y_\perp)\mathcal{F}_j^a(\beta_B,0_\perp)|p+\xi p_2\rangle =$$
$$= 2\pi^2\delta(\xi)g^2\Big[-g_{ij}\mathcal{D}(\beta_B,y_\perp)-\frac{1}{m^2}(2\partial_i\partial_j+g_{ij}\partial_\perp^2)\mathcal{H}(\beta_B,y_\perp)\Big]$$

linearization of the evolution equation

$$\frac{d}{d\ln\sigma}\Big[\alpha_s\mathcal{D}(\beta_B,z_\perp)\Big] = \frac{\alpha_s N_c}{\pi}\int_{\beta_B}^1\frac{dz'}{z'}\Big\{J_0\Big(|z_\perp|\sqrt{\sigma\beta_B s\frac{1-z'}{z'}}\Big)\alpha_s\mathcal{D}\Big(\frac{\beta_B}{z'},z_\perp\Big)\Big[\frac{1}{z'(1-z')_+}-2+z'-z'^2\Big]$$
$$-\frac{4}{m^2}J_2\Big(|z_\perp|\sqrt{\sigma\beta_B s\frac{1-z'}{z'}}\Big)z_\perp^2\alpha_s\mathcal{H}''\Big(\frac{\beta_B}{z'},z_\perp\Big)\Big[-z'+z'^2\Big]\Big\}$$

$z_\perp = 0$

$$\frac{d}{d\ln\sigma}\alpha_s\mathcal{D}(\beta_B,0_\perp) = \frac{\alpha_s N_c}{\pi}\int_{\beta_B}^1\frac{dz'}{z'}\Big[\frac{1}{z'(1-z')_+}-2+z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{\beta_B}{z'},0_\perp\Big)$$

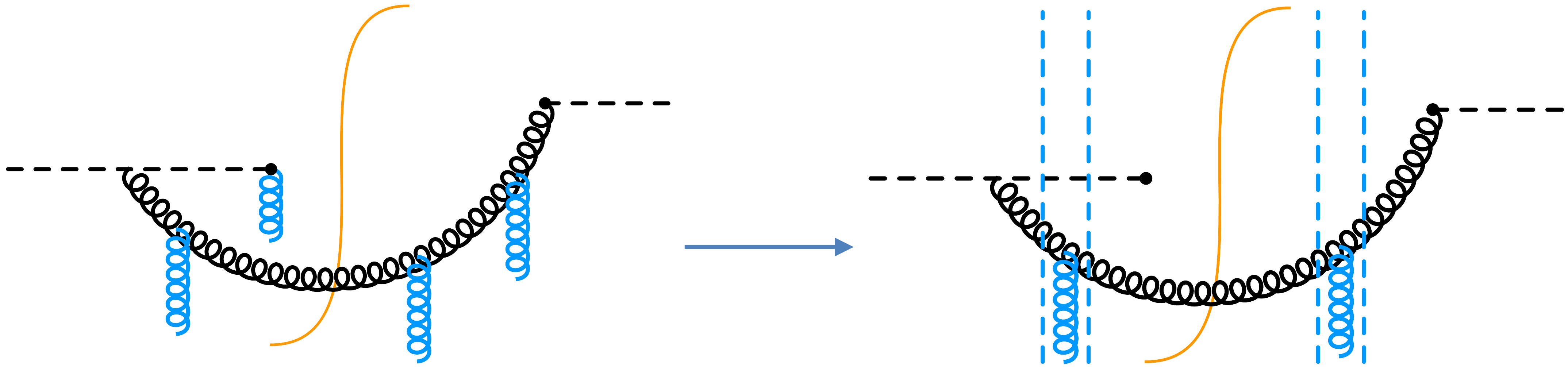
Coincides with the equation for future-point Wilson lines

DGLAP evolution kernel

SMALL-X LIMIT

$x_B \to 1$

$k_\perp^2 \ll s$



Past-point Wilson lines:

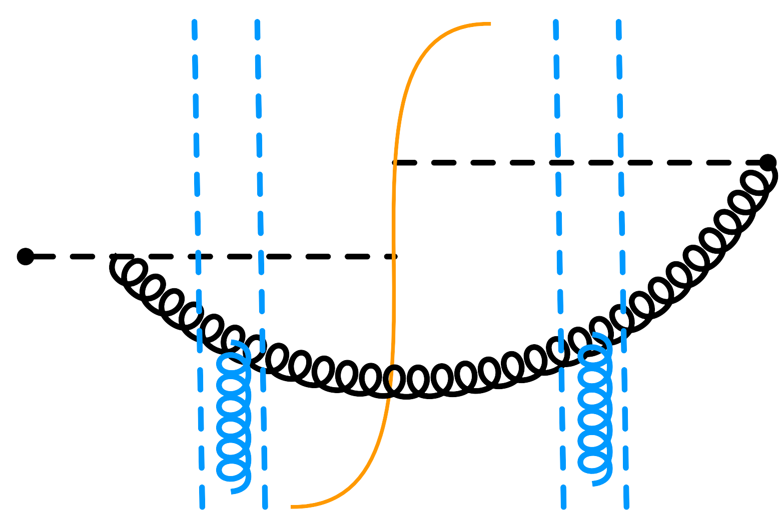
non-linear evolution equation

$$\frac{d}{d\ln\sigma}\langle \bar{U}_i^a(x_\perp)\bar{U}_j^a(y_\perp)\rangle = -\frac{g^2}{8\pi^3}\int d^2z_\perp Tr\Big\{(i\overrightarrow{\partial}_i^x + \bar{U}_i^x)\Big(U_x^\dagger U_z - 1\Big)\frac{(x-y)_\perp^2}{(x-z)_\perp^2(z-y)_\perp^2}\Big(U_z^\dagger U_y - 1\Big)(-i\overleftarrow{\partial}_j^y + \bar{U}_j^y)\Big\}$$

BFKL/BK kernel

Future-point Wilson lines:

$$\frac{d}{d\ln\sigma}\langle \tilde{U}_i^a(x_\perp)U_j^a(y_\perp)\rangle = -\frac{g^2}{8\pi^3}\int d^2z_\perp Tr\Big\{(-i\overrightarrow{\partial}_i^x + \tilde{U}_i^x)\Big(\tilde{U}_x\tilde{U}_z^\dagger - 1\Big)\frac{(x-y)_\perp^2}{(x-z)_\perp^2(z-y)_\perp^2}\Big(U_z U_y^\dagger - 1\Big)(i\overleftarrow{\partial}_j^y + U_j^y)\Big\}$$



Concussions. Rapidity factorization