more-than-MHV amplitudes in QCD

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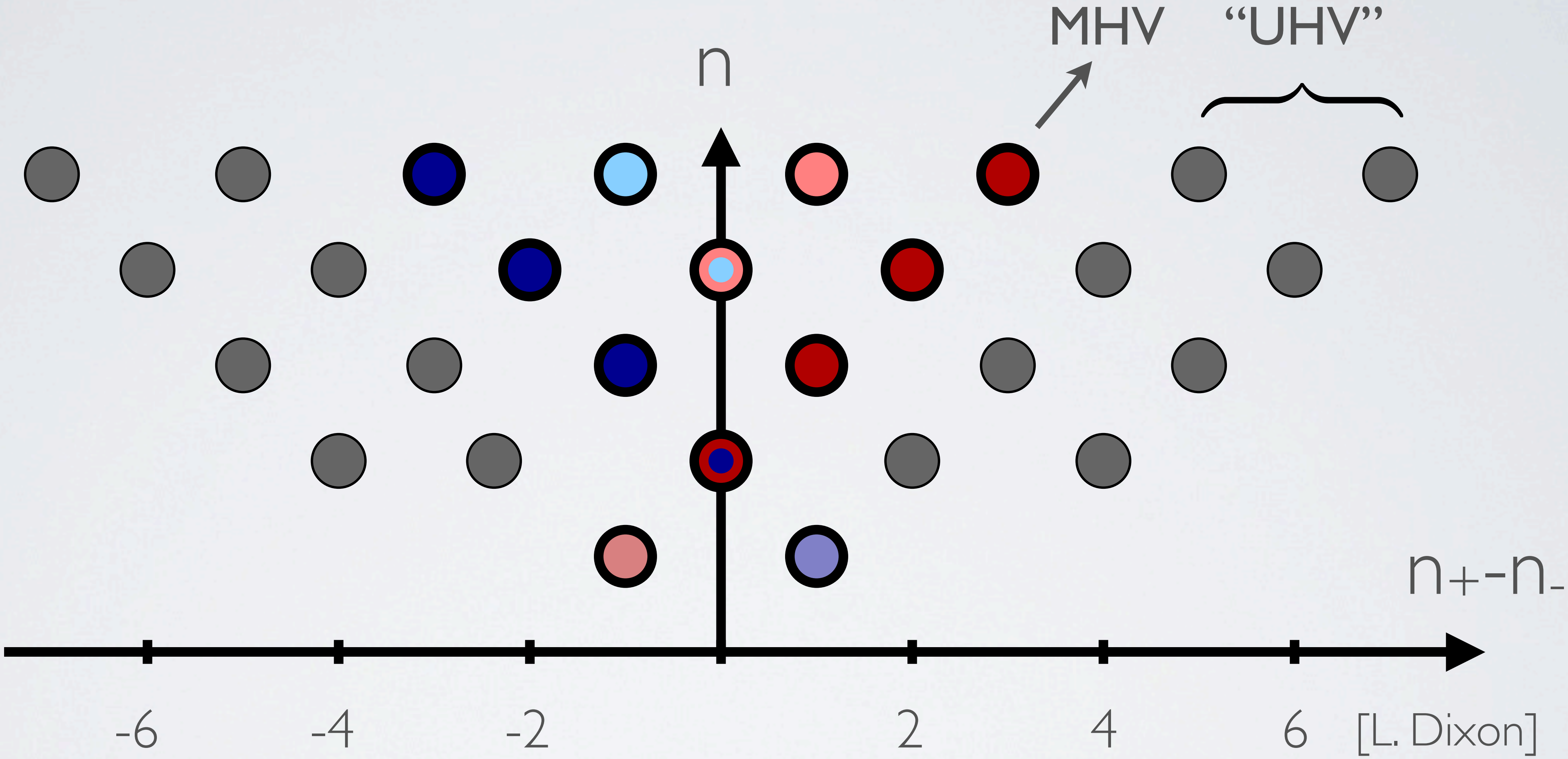
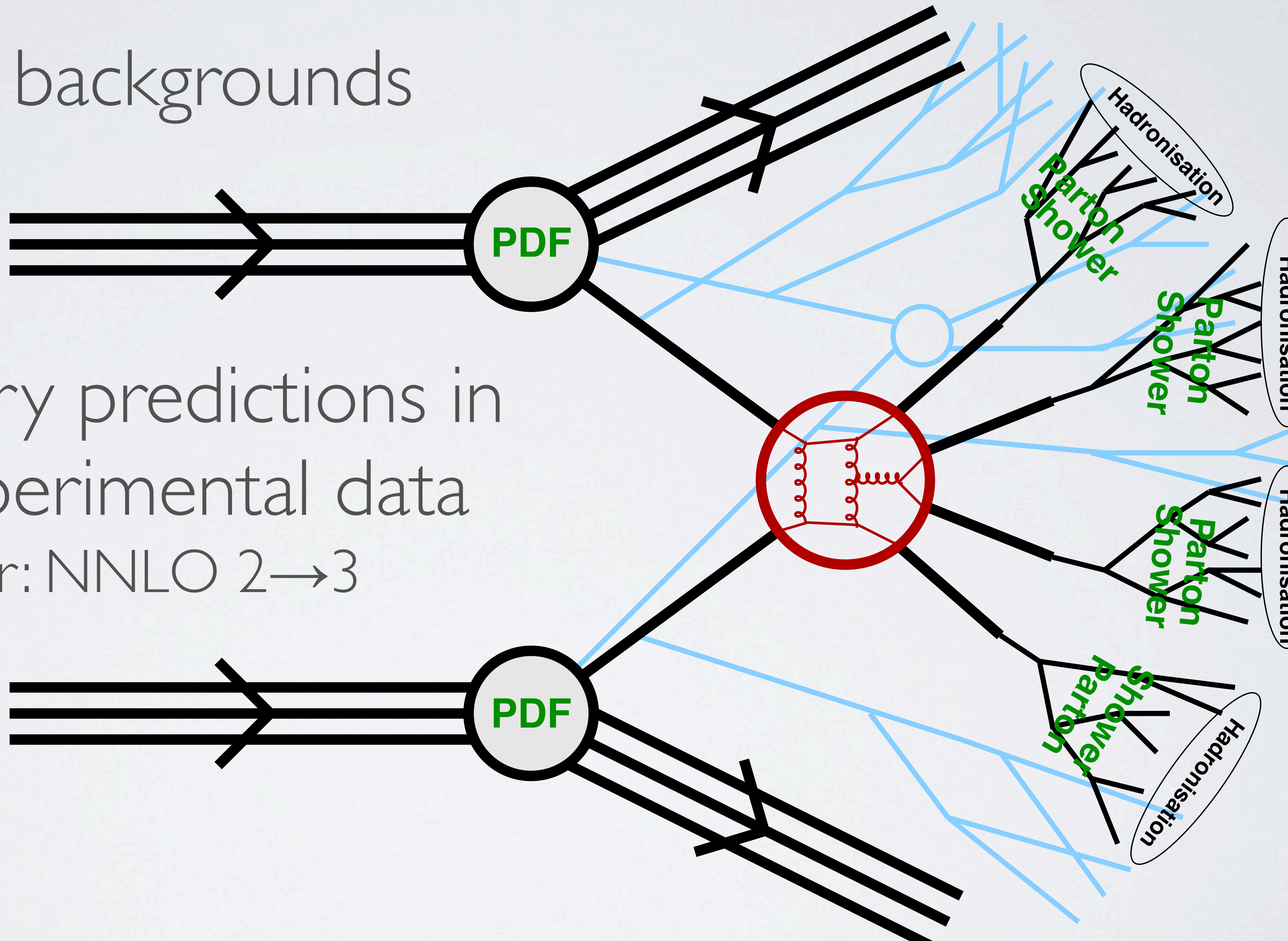
helicity amplitudes



vanish at tree-level and to
all loops in super-symmetric theories

Modelling hadron collisions

New physics backgrounds

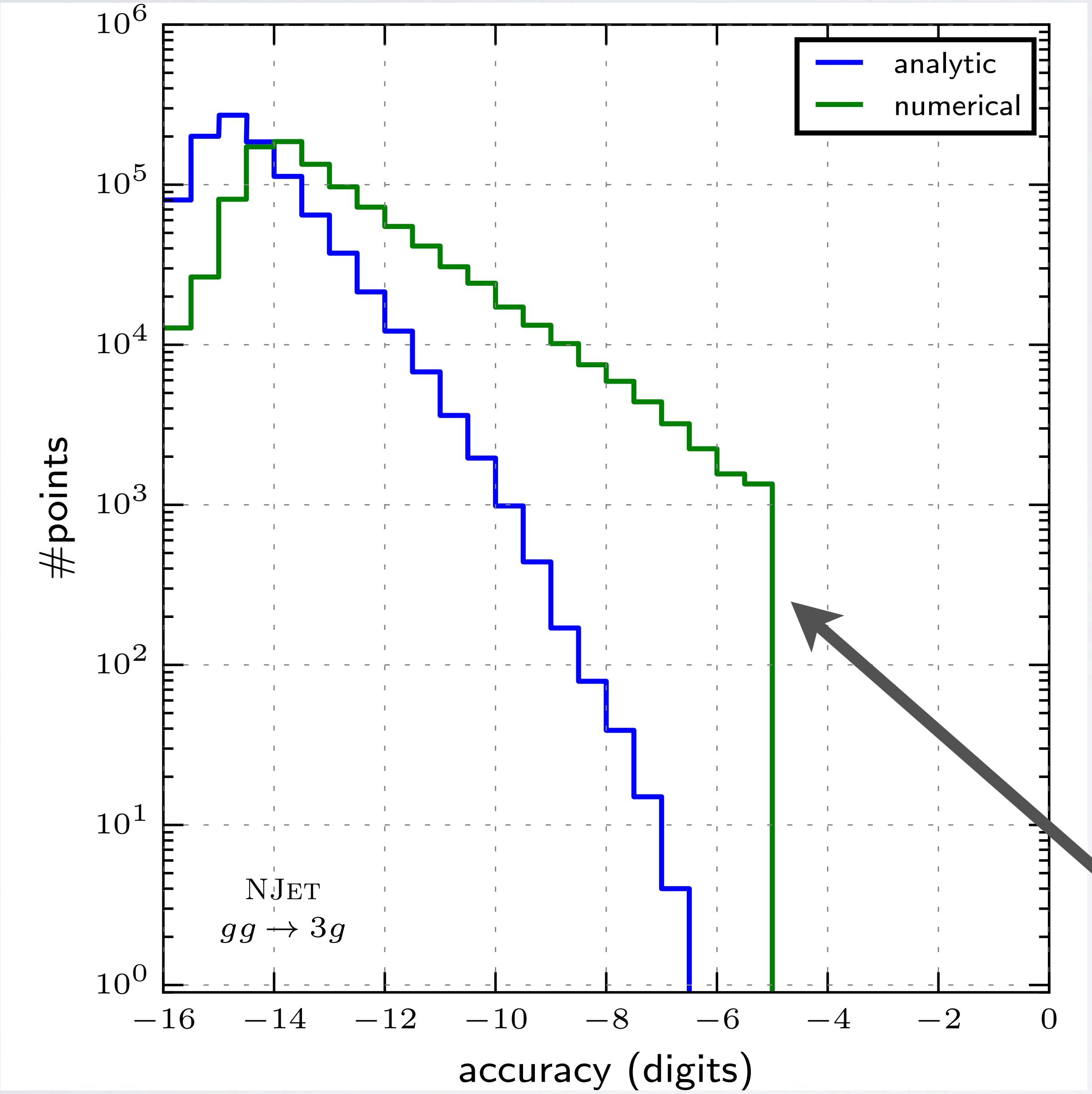


Keeping theory predictions in line with experimental data

next frontier: NNLO 2→3

Determination of SM parameters

spurious singularities



applications of loop amplitudes in NNLO computations more intensive

numerical D-dimensional generalised unitarity

vs

analytic computation with finite integrals basis (e.g. Bern, Dixon, Kosower [hep-ph/9302280])

removing spurious poles also simplifies coefficients ⇒ faster

(~100x in this case)

need to switch to quadruple precision evaluation

one-loop finite amplitudes in QCD

all multiplicity expressions!

- Off-shell currents for one-loop amplitudes to $\mathcal{O}(\epsilon^0)$

[Mahlon (1993)]

- all-plus ↔ $\mathcal{N} = 4$ D-dimensional unitarity cuts

[Bern, Dixon, Dunbar, Kosower (1996)]

- BCFW recursion for $\mathcal{O}(\epsilon^0)$

[Bern, Dixon, Kosower (2005)]

- CSW(MHV) rules for D-dimensional integrands

[Elvang, Freedman, Kiermaier (2012)]

outline

- d-dimensional generalised unitarity

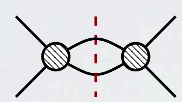
    $\Rightarrow$ multi-loops from trees

- two-loop all-plus amplitudes
- local integrands in d-dimensional amplitudes

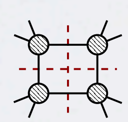
automated one-loop amplitudes

solving on-shell conditions requires **complex** momenta
⇒ factorise residues into **tree amplitudes**

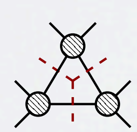
Unitarity: double cuts
[BDDK '94]
[triple cuts BDK '97]



Generalized unitarity:
quadruple cuts [BCF '04]

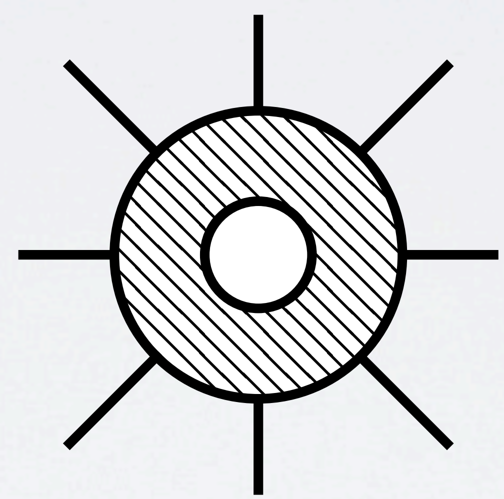


triple cuts [e.g. Forde '07]



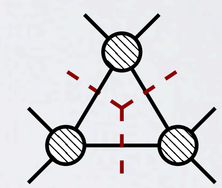
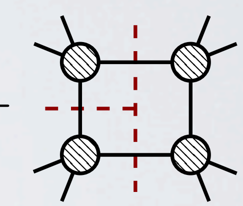
$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

find complex contour to isolate integral coefficient



multi-scale kinematic algebra performed **numerically**

Integrand reduction [OPP '05]

$\Delta_3 =$  $-$ 

D-dim. generalized unitarity [GKM '08]

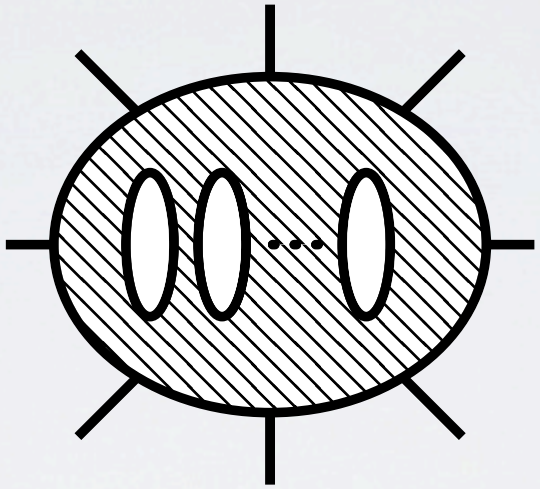
$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

explicitly remove poles

multi-loop amplitudes from trees

Maximal unitarity

[Kosower, Larsen, Johansson, Caron-Huot, Zhang, Søgaard]



Integrand reduction via polynomial division

[Mastrolia, Ossola, SB, Frellesvig, Zhang, Mirabella, Peraro, Malamos, Kleiss, Papadopolous, Verheyen, Feng, Huang]

e.g. IBPs

$$A = \sum_i (\text{rational})_i (\text{integral})_i \longleftarrow A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

IBPs must be free of doubled propagator MI

[Gluza, Kosower, Kajda 1009.0472] [Schabinger 1111.4220] [Ita 1510.05626] [Larsen, Zhang 1511.01071]

a toolbox for multi-loop integrands

momentum twistors
[Hodges (2009)]

six-dimensional spinor-helicity

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

generalised unitarity cuts

integrand reduction

$$\mathcal{A} = \sum_i S_i \frac{C(\Delta_i)\Delta_i}{\prod D_\alpha}$$

colour/kinematics BCJ relations

multi-loop integrand reduction

[Mastrolia, Ossola 1107.6041] [SB, Frellesvig, Zhang 1202.2019]

[Zhang 1205.5707] [Mastrolia, Mirabella, Ossola, Peraro 1205.7087]

ISP monomials

$$\Delta_T(\{k\}) = \sum_{\{\alpha\}} c_{T;\alpha_1\ldots\alpha_n} (k_1 \cdot p_5)^{\alpha_1} (k_2 \cdot p_2)^{\alpha_2} \ldots (k_1 \cdot \omega_2)^{\alpha_m} \ldots \mu_{12}^{\alpha_n}$$

rational coefficients

spurious directions

extra-dimensional ISPs

$$\mu_{ij} = -k_i^{-2\epsilon} \cdot k_j^{-2\epsilon}$$

integrand reduction

$$\Delta_{c;T}\Big|_{\mathrm{cut}} = \prod_i A_i^{(0)} - \sum_{T'} \frac{\Delta_{c;T'}}{\prod_{l\in T'/T} D_l}\Bigg|_{\mathrm{cut}}$$

on-shell the numerators can be written as products of tree-level amplitudes

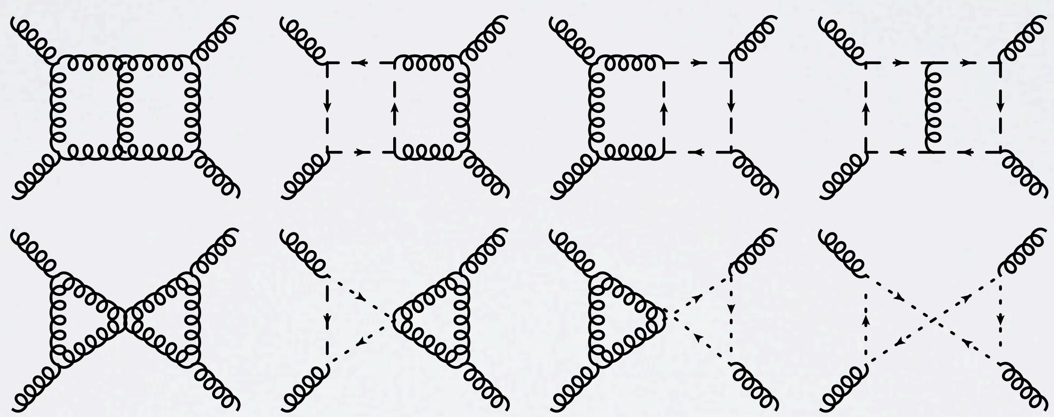
fix basis of **monomials** in **irreducible scalar products** via **polynomial division** (Gröbner basis)

integrand parameterisations not unique - freedom in the choices of ISP monomials

six-dimensional trees

six dimensional spinor helicity [Cheung, O'Connell 0902.0981]
(one-loop applications [Bern at al. 1010.0494] [Davies 1108.0398])

dimensional reduction to $4-2\epsilon$ using additional scalar amplitudes



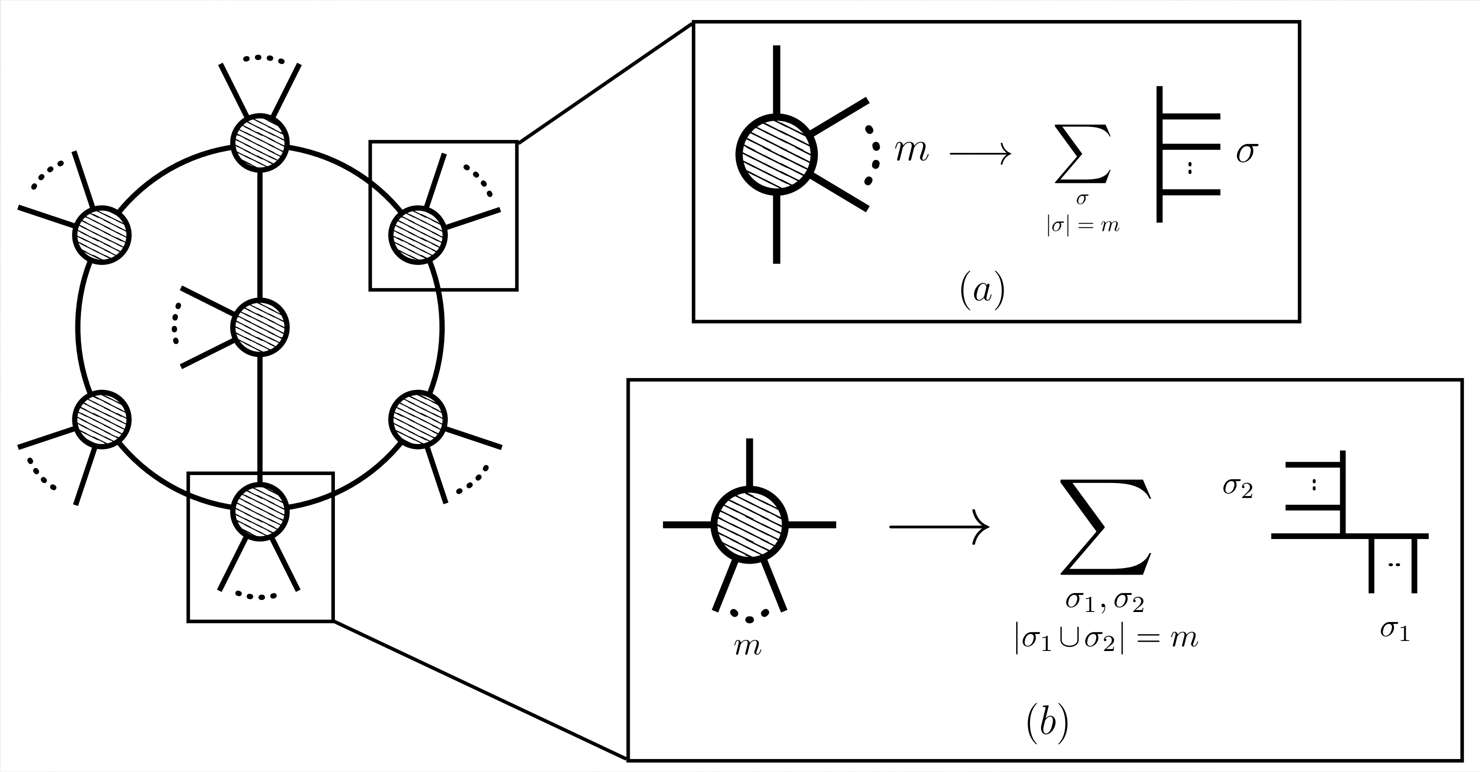
6D is a convenient way to manage all helicities simultaneously

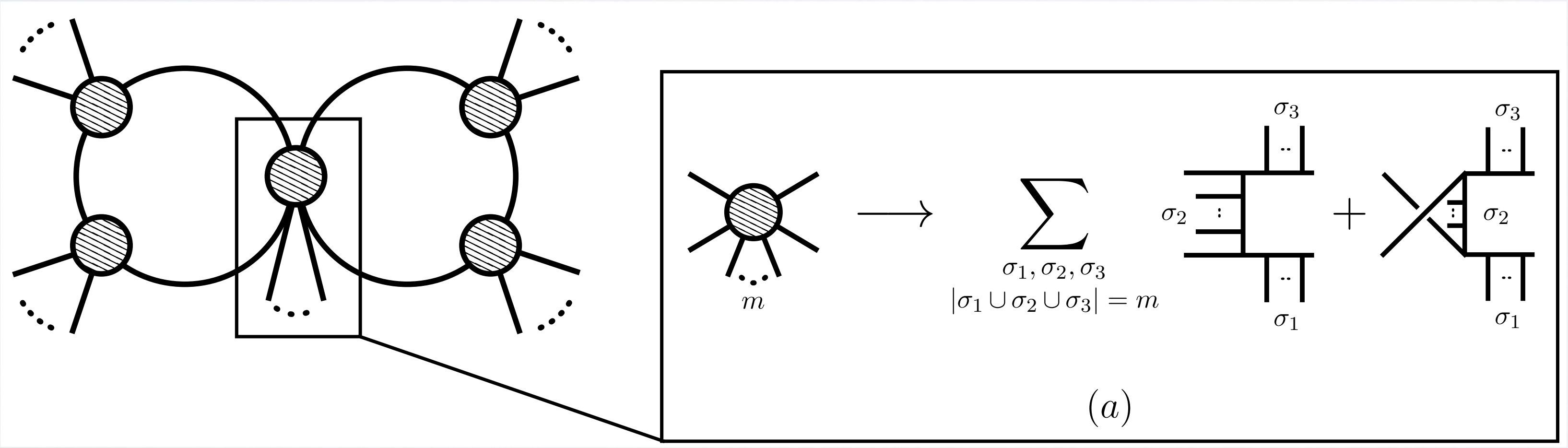
$$A(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{d\dot{d}}, 4_{d\dot{d}}) = \frac{i}{st}\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}]$$

General colour decompositions

[Dixon, Del Duca, Maltoni (1999)]

Inserting the DDM decomposition into colour dressed cuts leads to a compact loop decomposition





general tree-level DDM colour bases including fermions [Johansson, Ochirov arXiv:1507.00332]

non-planar from planar

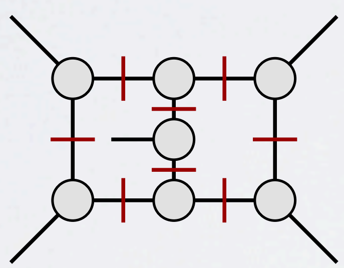
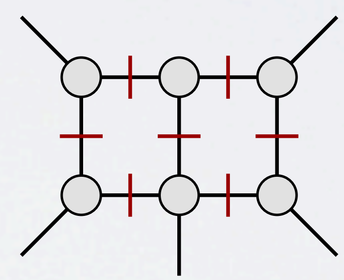
e.g. [Bern, Carrasco, Dixon, Johansson, Roiban 1201.5366]

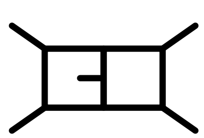
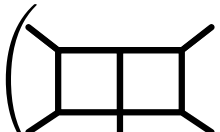
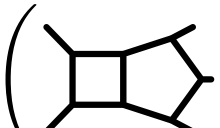
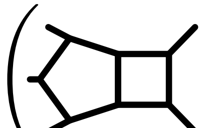
$$A_4(1,2,3,4) = \frac{s_{13}}{s_{12}} A_4(1,3,2,4)$$

see talks by Carrasco and Johansson

factorization

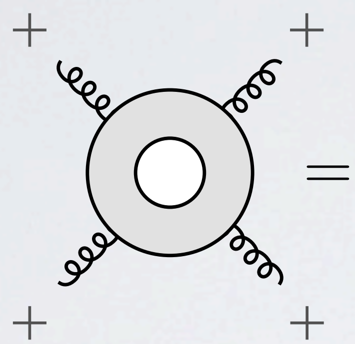
$$\Rightarrow \quad A_3(1,2,-P_{12})A_3(P_{12},3,4) = \operatorname*{Res}_{s_{12}=0}\left(A_4(1,2,3,4)\right) = s_{13}A_4(1,3,2,4)\Big|_{s_{12}=0}$$

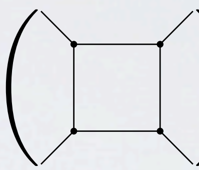
$\Rightarrow$  $= (k_1 - P_{123})^2$  $\Big|_{(k_1+k_2+p_3)^2}$

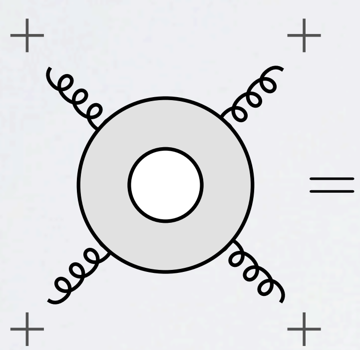
$\Rightarrow$ $\Delta\Big($  $\Big)\Big|_{\text{cut}} = \Big((k_1 - P_{123})^2 \Delta\Big($  $\Big) + \Delta\Big($  $\Big) - \Delta\Big($  $\Big)\Big)\Big|_{\text{cut}}$

applications:
all-plus amplitudes in QCD

one-loop 4pt all-plus



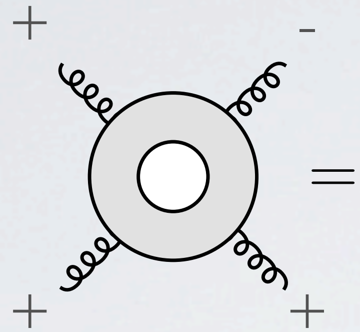
$= \frac{D_s - 2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\{-st\} \cdot \Big\{ I \Big($  $\Big) [\mu_{11}^2] \Big\}$

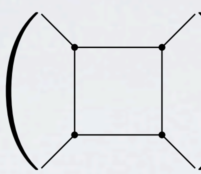
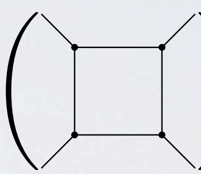


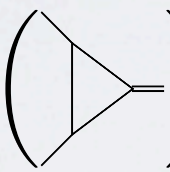
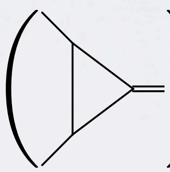
$$= \frac{D_s - 2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{-st}{6} + \mathcal{O}(\epsilon)$$

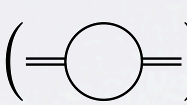
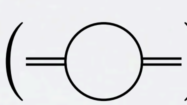
[Bern, Kosower (1991)]

one-loop 4pt single minus



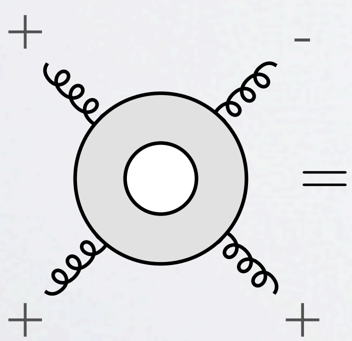
$= \vec{c} \cdot \Big\{ I\Big($  $\Big)[\mu_{11}^2],\ I\Big($  $\Big)[\mu_{11}],$

$I\Big($  $\Big)[\mu_{11}],\ I\Big($  $\Big)[\mu_{11}],$

$I\big($  $\big)[\mu_{11}],\ I\big($  $\big)[\mu_{11}] \Big\}$

$$\vec{c} = \frac{(D_s-2)[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]} \left\{ \frac{st}{u}, \frac{s^2t^2}{u^2}, \frac{t^2(s-u)}{u^2}, \frac{s^2(t-u)}{u^2}, -\frac{t(t-u)}{tu}, -\frac{s(s-u)}{su} \right\}$$

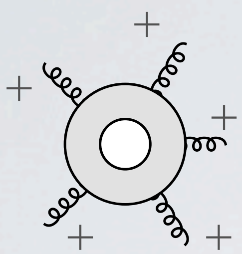
[Bern, Morgan (1995)]

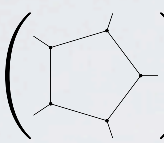


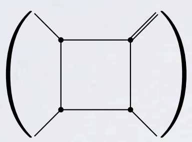
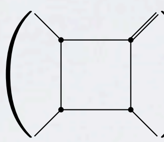
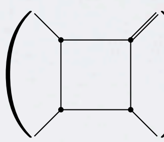
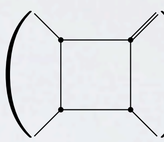
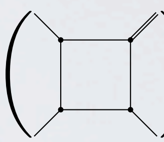
$$= \frac{(D_s-2)[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]} \frac{u}{6} + \mathcal{O}(\epsilon)$$

[Bern, Kosower (1991)]

one-loop 5pt all-plus



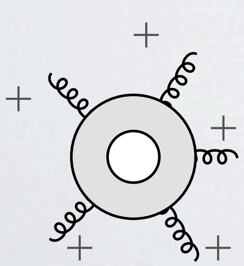
$= \vec{c} \cdot \Big\{ I\Big($  $\Big)[\mu_{11}^2],$

$I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2]\Big\}$

$$\vec{c} = \frac{(D_s - 2)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5(1234)}\{-s_{12}s_{23}s_{34}s_{45}s_{51},$$

$$s_{12}s_{23}\mathrm{tr}_-(1345), s_{23}s_{34}\mathrm{tr}_-(2451), s_{34}s_{45}\mathrm{tr}_-(3512), s_{45}s_{51}\mathrm{tr}_-(4123), s_{51}s_{12}\mathrm{tr}_-(5234)\}$$

[Bern, Morgan (1995)]



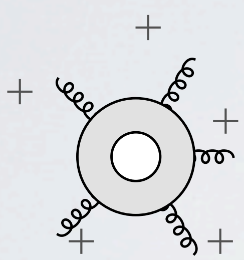
$$= \frac{(D_s - 2)}{6}\frac{s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} + \mathrm{tr}_5(1234)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \mathcal{O}(\epsilon)$$

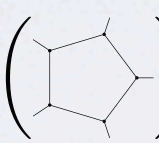
[Bern, Dixon, Kosower (1993)]

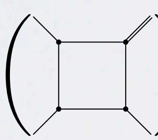
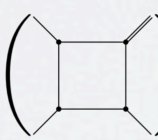
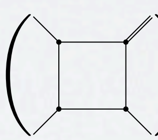
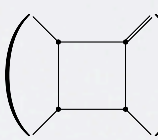
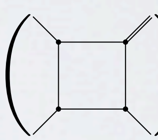
one-loop 5pt all-plus

$$\Delta^{(1)}(1^+,2^+,3^+,\ldots,n^+) = \frac{D_s-2}{\langle 12\rangle^4}\,\mu_{11}^2\,\Delta^{(1),[\mathcal{N}=4]}(1^-,2^-,3^+,\ldots,n^+)$$

[Bern, Dixon, Dunbar, Kosower (1996)]



$= \vec{c} \cdot \Big\{ I\Big($  $\Big)[\mu_{11}^3],$

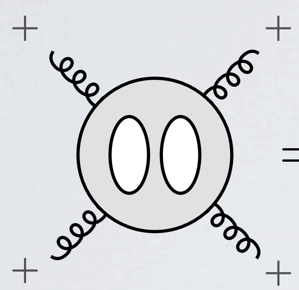
$I\Big($  $\Big)[\mu_{11}^2],\ I\Big($  $\Big)[\mu_{11}^2],\ I\Big($  $\Big)[\mu_{11}^2],\ I\Big($  $\Big)[\mu_{11}^2],\ I\Big($  $\Big)[\mu_{11}^2]\Big\}$

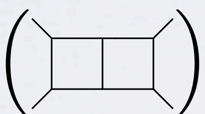
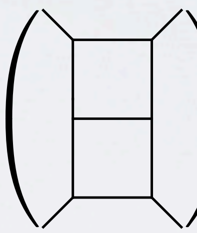
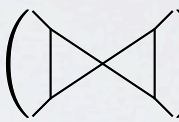
$$\vec{c} = \frac{(D_s-2)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\{2\mathrm{tr}_5(1234),\, s_{12}s_{23},\, s_{23}s_{34},\, s_{34}s_{45},\, s_{45}s_{51},\, s_{51}s_{12}\}$$

two-loop 4pt all-plus

(drop Parke-Taylor
pre-factors from now on....)

[Bern, Dixon, Kosower (2000)]



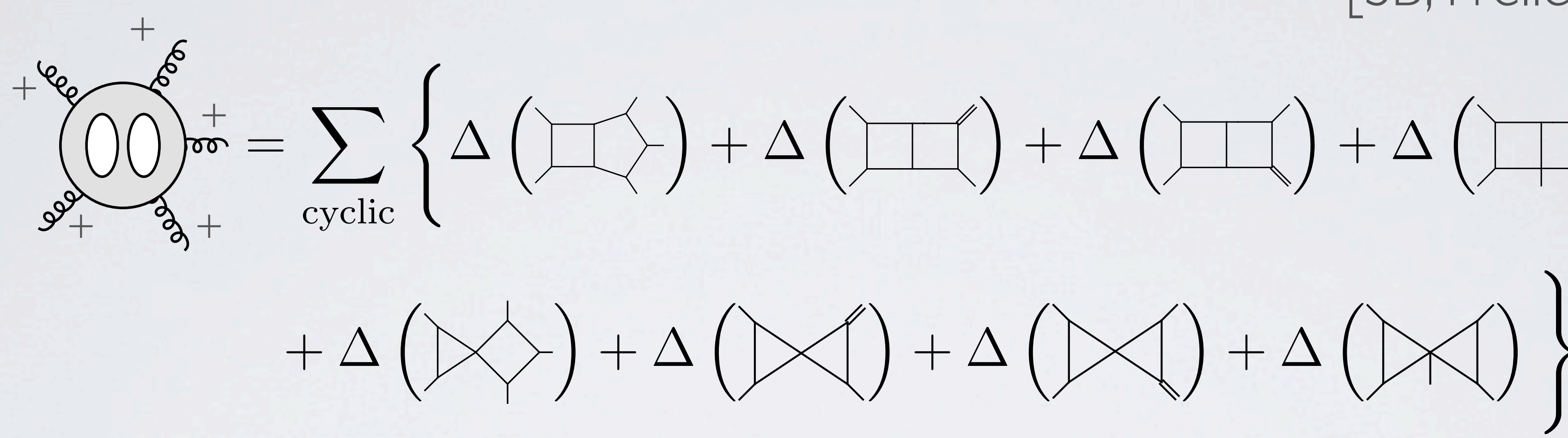
$= \{s^2 t, t^2 s, st\} \cdot \{I\Big($  $\Big)[F_1], I\Big($  $\Big)[F_1], I\Big($  $\Big)[F_2 + F_3 \frac{s+(l_1+l_2)^2}{s}]\}$

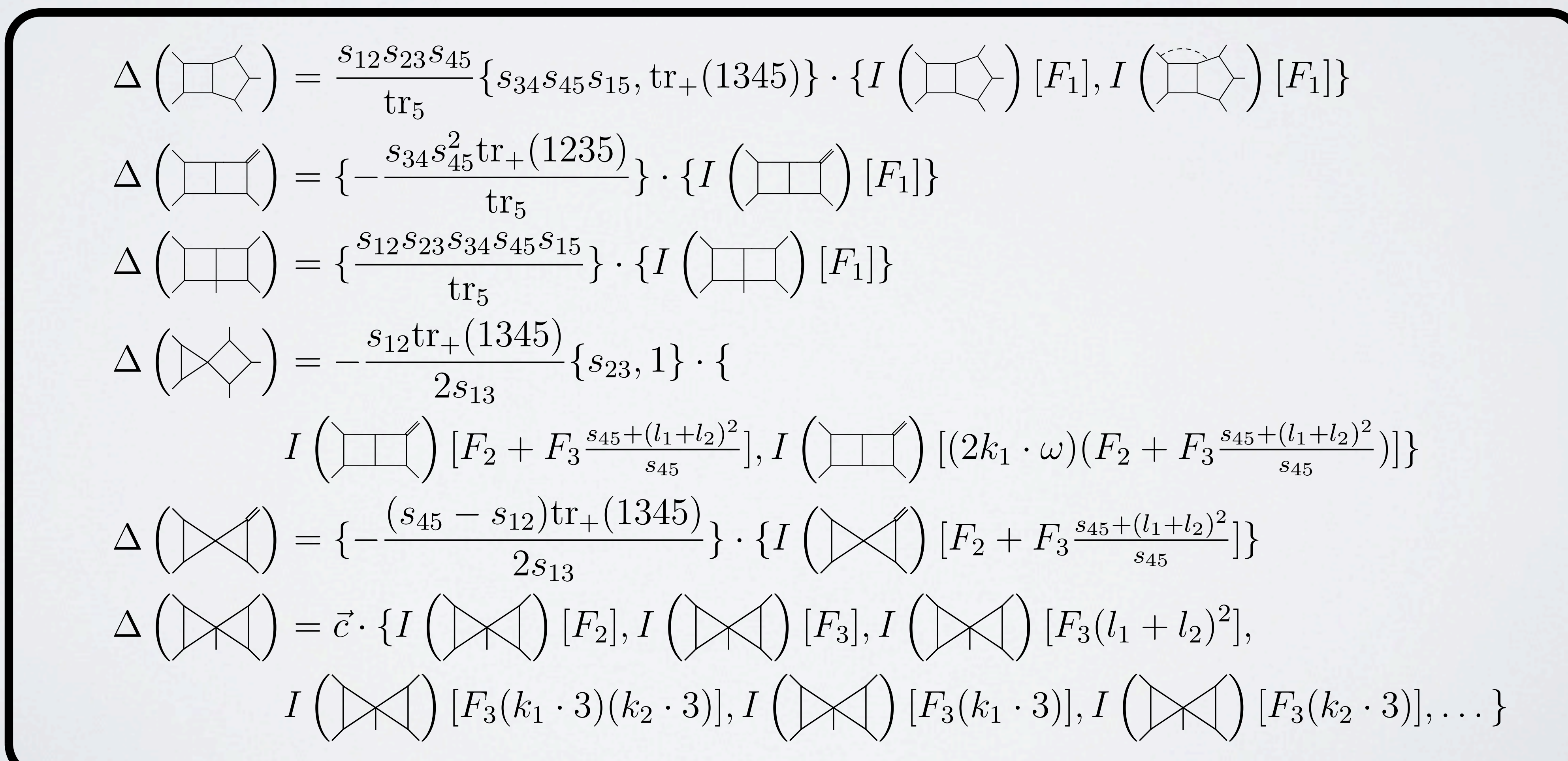
dimension shifting numerators

$$F_1 = (D_s - 2)\big(\mu_{11}\mu_{22} + (\mu_{11} + \mu_{22})^2 + 2(\mu_{11} + \mu_{22})\mu_{12}\big) + 16(\mu_{12}^2 - \mu_{11}\mu_{22}),$$
$$F_2 = 4(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12},$$
$$F_3 = (D_s - 2)^2 \mu_{11}\mu_{22}.$$

two-loop 5pt all-plus

[SB, Frellesvig, Zhang (2013)]





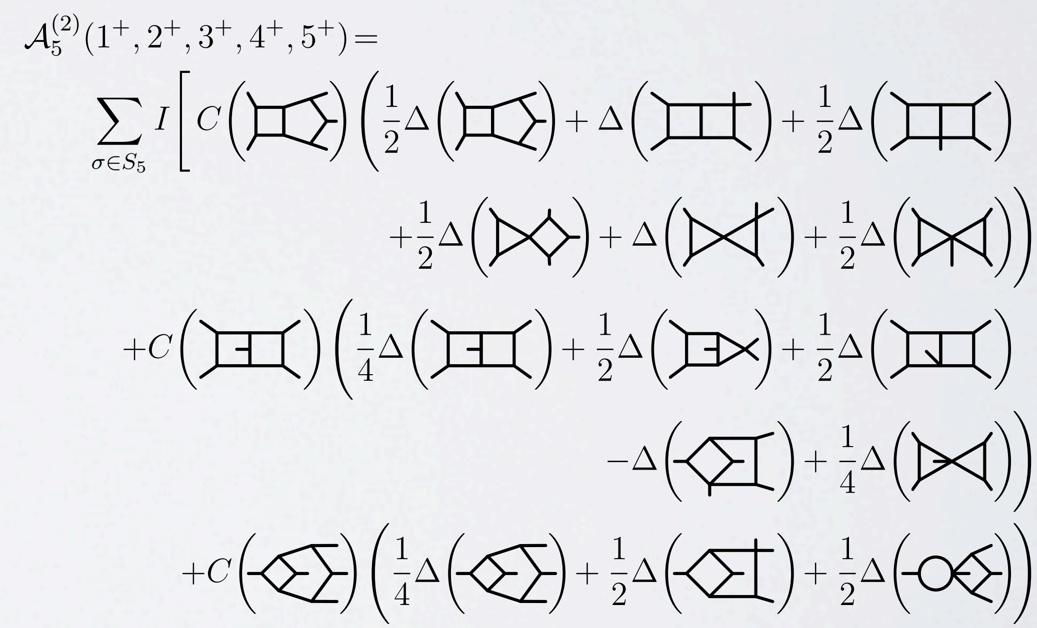
two-loop 5pt all-plus

$$\boxed{\Delta^{(2)}(1^+,2^+,3^+,\ldots,n^+) = \frac{D_s-2}{\langle 12\rangle^4}\,F_1\,\Delta^{(2),[\mathcal{N}=4]}(1^-,2^-,3^+,\ldots,n^+) + (\text{1-loop})^2}$$

all genuine two-loop topologies related to $\mathcal{N}$=4 MHV

non-planar cuts via BCJ
[SB, Mogull, Ochirov, O'Connell (2015)]

complete BCJ numerator representation
[Mogull, O'Connell (2015)]



two-loop 5pt all-plus amplitude

[Gehrmann, Henn, Lo Presti 1511.05409]

planar master integrals using canonical differential equation approach

$$A_5^{(2)} = A_5^{(1)} \left[-\sum_{i=1}^{5} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-v_i} \right)^{\epsilon} \right] + R\, F_5^{(2)} + \mathcal{O}(\epsilon)$$

$$F_5^{(2)} = \frac{5\pi^2}{12} F_5^{(1)} + \sum_{i=0}^{4} \sigma^i \left\{ \frac{v_5 \mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]}{(v_2+v_3-v_5)} I_{23,5} + \frac{1}{6} \frac{\mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\}. \quad (8)$$

simple function of Li_2

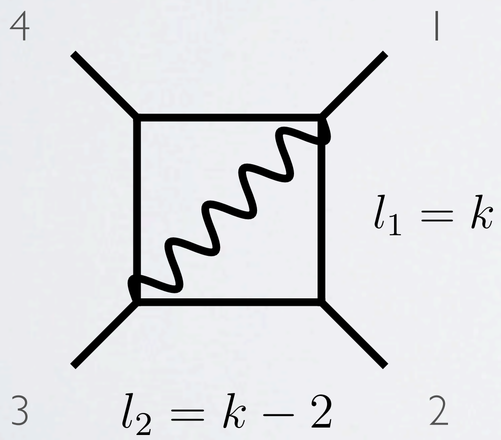
local integrands

[Arkani-Hamed, Bourjailly, Cachazo, Trnka 1012.6032]

[Arkani-Hamed, Bourjailly, Cachazo, Caron-Huot, Trnka 1008.2958]

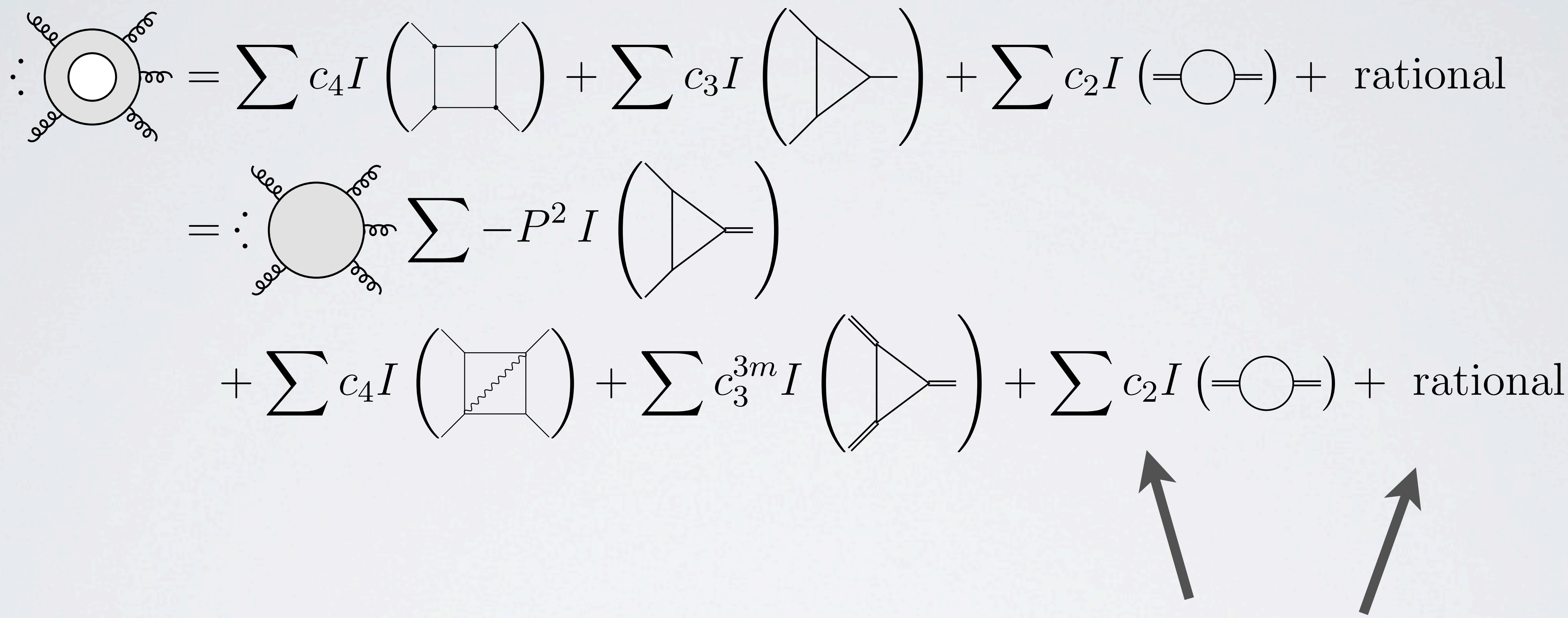
manage infra-red divergences at the integrand level

simple integrals with unit leading singularities



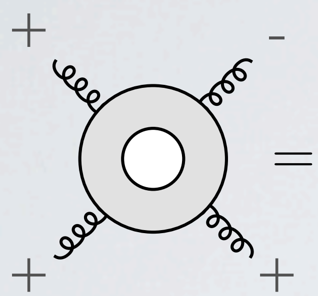
$$= I_4^{4-2\epsilon}\left[\mathrm{tr}\left(1 l_1 l_2 3\right)\right] = I_4^{6-2\epsilon}[1]$$

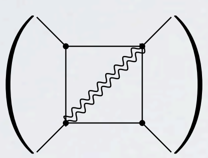
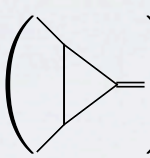
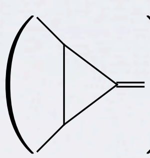
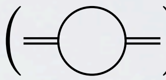
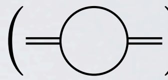
one-loop integrand bases



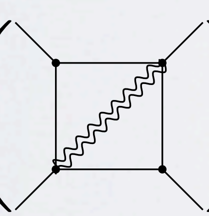
still complicated functions at high multiplicity

one-loop 4pt single minus

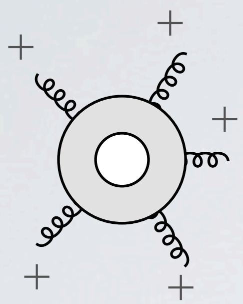


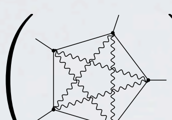
$= \vec{c} \cdot \Big\{ I\Big($  $\Big)[\mu_{11}], I\Big($  $\Big)[\mu_{11}], I\Big($  $\Big)[\mu_{11}], I\big($  $\big)[\mu_{11}], I\big($  $\big)[\mu_{11}] \Big\}$

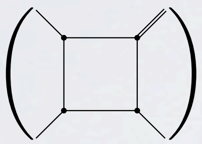
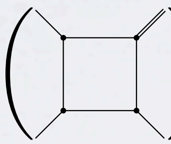
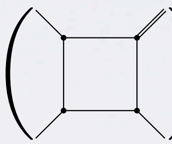
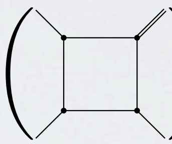
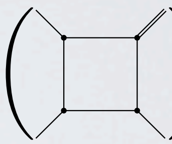
$$\vec{c} = \frac{(D_s-2)[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]} \left\{ \frac{1}{u^2}, s, t, \frac{2t}{s}, \frac{2s}{t} \right\}$$

$I\Big($  $\Big)[\mu_{11}] = \mu_{11}\mathrm{tr}_+(1 l_1 l_2 3)^2 - \text{bubbles}$

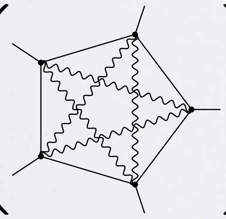
one-loop 5pt all-plus



$= \vec{c} \cdot \Big\{ I\Big($  $\Big)[1],$

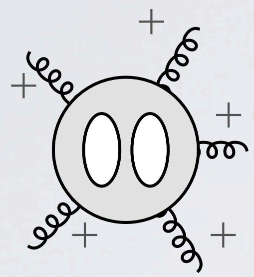
$I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], \Big\}$

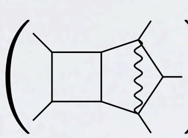
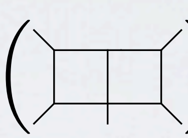
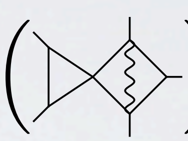
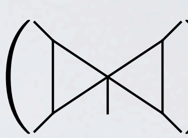
$$\vec{c} = \frac{(D_s-2)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\{\frac{\langle 13\rangle\langle 24\rangle\langle 35\rangle\langle 41\rangle\langle 52\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}, s_{12}s_{23}, s_{23}s_{34}, s_{34}s_{45}, s_{45}s_{51}, s_{51}s_{12}\}$$

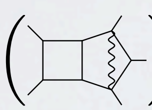
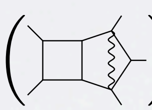
$I\Big($  $\Big)[1] = \prod_{i=1}^{5} \mathrm{tr}_+(k l_k l_{k+1}(k+2)) - \text{bubbles}$

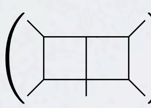
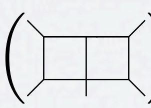
two-loop 5pt all-plus

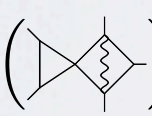
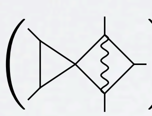
[SB, Mogull (in progress)]

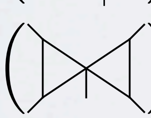


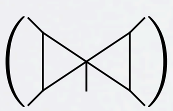
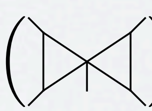
$= \sum_{\text{cyclic}} \Delta\Big($  $\Big) + \Delta\Big($  $\Big) + \Delta\Big($  $\Big) + \Delta\Big($  $\Big)$

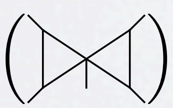
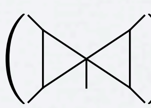
$\Delta\Big($  $\Big) = \{s_{45}\} \cdot \{I\Big($  $\Big)[F_1]\}$

$\Delta\Big($  $\Big) = \{s_{12}s_{45}s_{15}\} \cdot \{I\Big($  $\Big)[F_1]\}$

$\Delta\Big($  $\Big) = \{1\} \cdot \{I\Big($  $\Big)[F_2 + F_3\frac{s_{45}+(l_1+l_2)^2}{s_{45}}]\}$

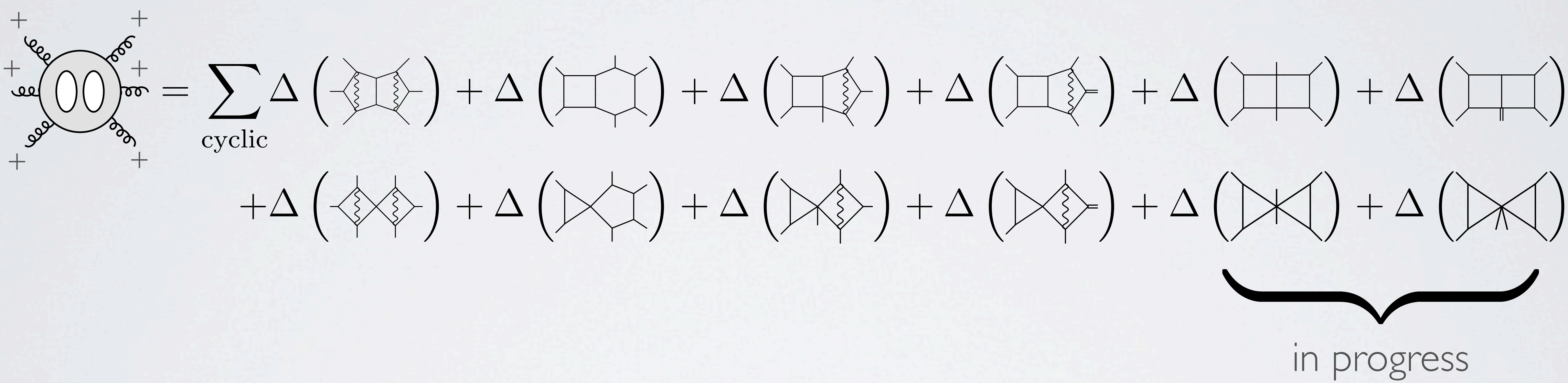
$\Delta\Big($  $\Big) = \{\mathrm{tr}_+(1245), s_{15}, -\mathrm{tr}_+(1345), -\mathrm{tr}_+(1235)\} \cdot \{$

$\Big($  $\Big)[F_2 + F_3\frac{4(l_1\cdot 3)(l_1\cdot 3)+s_{12}s_{45}+(s_{12}+s_{45})(l_1+l_2)^2}{s_{12}s_{45}}]$, $\Big($  $\Big)[F_3(l_1+l_2)^2]$,

$\Big($  $\Big)[F_3 2(l_2\cdot 3)(s_{12}+\frac{\mathrm{tr}_+(1l_2l_33)}{s_{13}})]$, $\Big($  $\Big)[F_3 2(l_1\cdot 3)(s_{12}+\frac{\mathrm{tr}_+(5l_5l_43)}{s_{53}})]\}$

towards two-loop 6pt all-plus

[SB, Mogull (in progress)]



follows the expected $\mathcal{N}$=4 x F_1 structure

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich (2008)]

[Arkani-Hamed, Bourjailly, Cachazo, Caron-Huot, Trnka (2010)]

outlook

- Automated loop amplitude methods
  - purely algebraic algorithms with rational momenta (momentum twistors)
  - automated D-dimensional cuts with 6D spinor-helicity
- First complete five-point amplitude in Yang-Mills
- Aim: apply to high multiplicity amplitudes at one and two loops
  - compact analytic expressions using local integrands: all multiplicity all-plus
  - general helicity configurations for $2 \to 3$: LHC physics at NNLO

Backup

momentum twistors

[Hodges (2009)]

an all multiplicity parameterisation (not unique)

$$Z_{iA} = \left(\frac{\Sigma_i}{s_{12}},\, 1-\delta_{1i},\, \frac{\langle 123i\rangle\langle 34\rangle[23]}{\langle 1234\rangle\langle 1i\rangle[12]},\, \frac{-\langle 13\rangle\langle 124i\rangle + \langle 14\rangle\langle 123i\rangle}{\langle 1234\rangle\langle 1i\rangle} \right)$$

$$\Sigma_i = \begin{cases} s_{12} & i=1 \\ \frac{\langle 13\rangle\langle 2i\rangle}{\langle 23\rangle\langle 1i\rangle} & i \neq 1 \end{cases}$$

In[21]:=

```
NN = 6;
GetMTrepALT[NN];
mtZ // MatrixForm // SymbolForm
```

Out[23]//DisplayForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\sigma_4}{s_{12}} & \frac{\sigma_5}{s_{12}} & \frac{\sigma_6}{s_{12}} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_{1234} & x_{1235} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_{1235}-x_{1245}}{x_{1234}} & \frac{1-x_{1246}}{x_{1234}} \end{pmatrix}$$

build spinors products etc. from rational phase-space points

momentum twistors

example: BCFW using rational kinematics

In[143]:=

```
NN = 7;
GetMTrepALT[NN];
mtZ // MatrixForm // SymbolForm
mtZ = mtZ /. lpS[1,2]→1; mtW = mtW /. lpS[1,2]→1;
{p1,p2,p3,p4,p5,p6,p7} = Map[ToSPN4[MT[#]]&,Range@NN];
Map[Simplify,BCFW4[AMP[{p1,1,gluon},{p2,1,gluon},{p3,1,gluon},{p4,2,gluon},{p5,2,gluon},{p6,2,gluon},{p7,2,gluon}],1,2]] // SymbolForm
```

Out[145]//DisplayForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\sigma_4}{s_{12}} & \frac{\sigma_5}{s_{12}} & \frac{\sigma_6}{s_{12}} & \frac{\sigma_7}{s_{12}} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_{1234} & x_{1235} & x_{1236} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_{1235}-x_{1245}}{x_{1234}} & \frac{x_{1236}-x_{1246}}{x_{1234}} & \frac{1-x_{1247}}{x_{1234}} \end{pmatrix}$$

Out[148]//DisplayForm=

$$-\frac{i\,(x_{1234}-x_{1235}+x_{1245})^3}{(-1+\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)(\sigma_6-\sigma_7)\,x_{1234}\,x_{1235}\,x_{1245}\,(\sigma_5 x_{1234}-\sigma_4 x_{1235}+x_{1245})} - \frac{i\,(x_{1234}(x_{1235}-x_{1236})-x_{1236}x_{1245}+x_{1235}x_{1246})^3}{(-1+\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)(\sigma_6-\sigma_7)\,x_{1234}\,x_{1235}\,x_{1236}\,(-x_{1236}x_{1245}+x_{1235}x_{1246})\,(-x_{1236}(\sigma_5 x_{1234}+x_{1245})+x_{1235}(\sigma_6 x_{1234}+x_{1246}))} + \frac{i\,(x_{1234}(-1+x_{1236})-x_{1246}+x_{1236}x_{1247})^3}{(-1+\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)(\sigma_6-\sigma_7)\,x_{1234}\,x_{1236}\,(-x_{1246}+x_{1236}x_{1247})\,(\sigma_6 x_{1234}+x_{1246}-x_{1236}(\sigma_7 x_{1234}+x_{1247}))}$$

[c.f. bcfw mathematica package Bourjaily (2010)]