



# Twistors and Amplitudes

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## A confluence of ideas in fundamental physics

The Standard Model is based on *massless fields*. Massless SU(3) gluons are essential to predictions and observations in high energy physics.

The geometry of General Relativity and cosmology has also developed in the direction of seeing *light-cone structure* as fundamental.

For massless fields, the dependence on *gauge choice* in Feynman summation is a major problem. There must be better ways of understanding the complete gauge-invariant amplitudes.

Massless fields, null momenta and null geodesics can be described by new and *better coordinates*.

Also, spin-1/2 and complex structure is fundamental to quantum theory.

Everyone here will know about 2-component spinors and the helicity representation.

## Zero-mass fields and conformal symmetry

You may not know... that this spinor description was used for amplitudes in the early 1970s. Partly to simplify amplitudes, but also because the fundamental spinor description of fields arises in General Relativity (reformulation due to Penrose and others, 1960s).

The zero-mass fields for spin 0, 1/2, 1, 2...

$$\square\phi = 0$$

$$\nabla^{AA'}\phi_{A'} = 0$$

$$F_{ab} = \epsilon_{A'B'}\phi_{AB} + \epsilon_{AB}\phi_{A'B'}, \quad \nabla^{AA'}\phi_{A'B'} = 0$$

$$\nabla^{AA'}\psi_{A'B'C'D'} = 0$$

These fields also satisfy a hidden symmetry: conformal symmetry.

They are invariant under Poincaré transformations (rotations, boosts, translations)

but under *inversion* as well

$$x^a \rightarrow \frac{x^a - x^b t_b t^a}{x^2}, \quad t^2 = +1$$

The structure is very similar to that of Möbius transformations.

Möbius transformations, on the compactified Riemann sphere, preserve circles.

The conformal group of transformation preserves light cones, on the compactified Minkowski space, with its extra *light cone at infinity*.

## Twistor geometry

Twistor space (Penrose, 1968...) naturally embodies all these ideas.  
The 15-parameter conformal symmetry group is fundamental.  
Parity symmetry is much less important (we are forced into a chiral choice).

Twistor space is a  $C^4$  such that the 2-dimensional-subspaces of the  $C^4$  correspond to points of (complexified, compactified) Minkowski space-time. The connecting formula is:

$$Z^\alpha = (Z^0, Z^1, Z^2, Z^3) \leftrightarrow (ix^{AA'} \pi_{A'}, \pi_{A'})$$

Given a twistor  $Z$ , the point  $x$  is not uniquely defined. It can lie anywhere on a 2-complex-dimensional plane.

If  $x$  is *real*, then there is a entire *real null line* in this plane.

A twistor is a certain kind of complexification of a light ray.

Conformal transformations on real Minkowski space correspond to  $SU(2,2)$  acting on twistor space; the *complexified* conformal transformations are just  $SL(4, \mathbb{C})$ : extremely simple.

The representation of the Poincare transformations is easy; less obvious is the inversion which corresponds to the linear map  $(z^0, z^1, z^2, z^3) \rightarrow (z^2, z^3, z^1, z^0)$ .

*Complex conjugate* structure depends acutely on the Minkowski metric (as opposed to Euclidean or split-signature). The complex conjugate of a twistor is a *dual* twistor. This induces an *indefinite norm* on twistor space. The twistors which are null, as measured by this norm, are those which contain real  $x$ .

## Points in space-time and lines in projective twistor space

*Incidence relations* between twistor space and Minkowski space are fundamental.

The 2-subspaces of twistor space are *lines* in projective twistor space  $CP^3$ , and their incidence properties are easily visualised — essentially the same as for lines in Euclidean 3-space.

In general, two such lines are skew, but if they meet, the corresponding points of CM are null-separated.

One special line in projective twistor space corresponds to the point at infinity in compactified Minkowski space. Points in CM are finite or non-finite points according to whether the corresponding lines meet this special line  $I$  in PT.

But the definition of the *metric* on CM requires a *non-projective* definition of the 2-plane I in non-projective twistor space, i.e. the skew two-twistor I has a definite scale.

Explicitly: distance is given by

$$(x - y)^2 = -2 \frac{\epsilon_{\lambda\mu\nu\sigma} P^\lambda Q^\mu R^\nu S^\sigma}{(I_{\lambda\mu} P^\lambda Q^\mu)(I_{\nu\sigma} R^\nu S^\sigma)}$$

Reinforced by the Penrose *non-linear graviton* (1976), non-perturbative twistor geometry which depends on *non-projective* I structure to define the metric.

Classical twistor theory programme (RP, 1970s): fundamental physical ideas should be re-expressed in terms of twistor space and new concepts should emerge.

Perhaps, if points are no longer primary, the UV divergence question might be resolved within twistor geometry; perhaps QFT in curved space would make new sense...

A program with similarity with string theory, *but* with a commitment to 4 dimensions! There is a new dimension: twistor scale. This has NOT been used significantly in recent developments.

## Twistor representation of zero-mass fields

Penrose correspondence: z.r.m. fields correspond to homogeneous twistor functions, more precisely, first-cohomology elements. This is consistent with quantization scheme in which classical  $[p, x]$  commutation relations go over to  $[Z, d/dZ]$  and helicity to homogeneity.

Thus: Penrose correspondence for spin 1:

$$\psi_{A'B'} \leftrightarrow f_{-4}(Z^\alpha) \leftrightarrow \tilde{f}_0(W_\alpha)$$

Helicity representation of fields is *essential* for the twistor-space representation.

Very important in motivation: positive/negative frequency fields are given by cohomology on the two halves of twistor space ( $|Z| > 0$ ,  $|Z| < 0$ ) without reference to momenta or Fourier analysis.

This purely twistor-geometric definition continues to an inner product formula for gauge fields — manifestly finite and gauge-independent:

$$\langle \phi | \psi \rangle = \oint g_0(Z^\alpha) f_{-4}(Z^\alpha) DZ$$

Eigenstates of momentum do not fit into this scheme. They are not finite-normed. From the point of view of twistor geometry, they have essential singularities at I.

So hard to make contact with standard scattering theory.

This is just one aspect of a general problem: how to combine the elegant expression of conformally invariant structures with the many aspects of conformal symmetry *breaking* in physical reality.

## Twistor-geometric scattering amplitudes

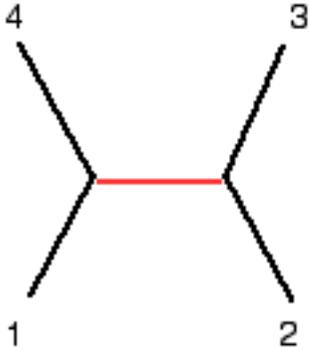
Idea: the inner product is zeroth-order scattering.

Can we get actual scattering amplitudes in a similar form, as functionals of the external twistor-represented fields, thus manifestly conformally invariant and manifestly finite?

Penrose's 1972 results for massless QED scattering amplitudes began this programme.

# Möller scattering ( $ee \rightarrow ee$ ) and crossing-related processes

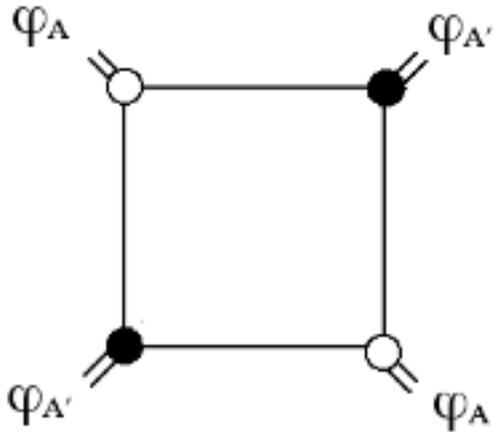
Feynman diagram



Corresponding Helicity Amplitude

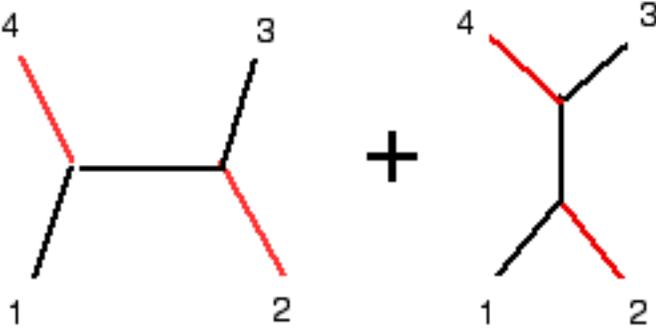
$$\frac{\langle 13 \rangle [24] \delta^4(\sum p_i)}{p_1 \cdot p_4} = \frac{\langle 13 \rangle^2 \delta^4(\sum p_i)}{\langle 14 \rangle \langle 23 \rangle}$$

Twistor diagram is of form



# Compton scattering ( $e\gamma \rightarrow e\gamma$ and crossing-related processes)

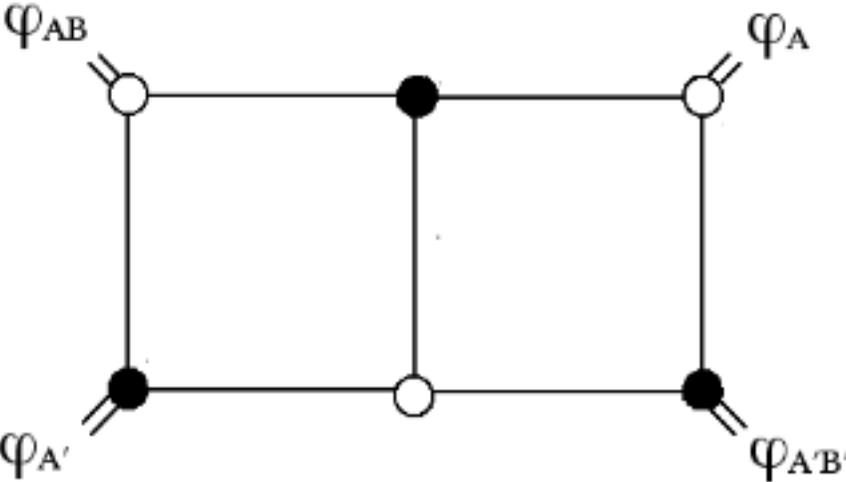
Feynman diagram:



Amplitude:

$$\frac{\langle 12 \rangle^2 \delta^4(\sum p_i)}{\langle 14 \rangle \langle 34 \rangle}$$

Penrose's original twistor diagram was of form:



The *twistor diagrams* here are *many-dimensional contour integrals* defined by:

white vertices, integration  $DW$  over dual twistor  $W$

black vertices, integration  $DZ$  over twistor  $Z$

edges represent boundary in  $W.Z=0$ , or poles  $(W.Z)^{-1}$ ,  $(W.Z)^{-2}$ ,  $(W.Z)^{-3}$  ...  
(details don't matter; in the supersymmetric version just *pure boundaries*).

Momenta never appear in the formalism.

The Compton scattering twistor diagram is *gauge-invariant*, and it corresponds to the *sum* of Feynman diagrams (each of them, by itself, gauge-dependent).

Everything in these diagrams is 'on-shell', i.e. associated with homogeneous zero-mass free field equations. The 'virtual' fields have disappeared.

## Development of twistor diagrams until 2000

The goal of the twistor diagram formalism was to express scattering amplitudes as manifestly finite, manifestly gauge-invariant, functionals of the positive/negative frequency external fields.

It needs detailed theory of many-dimensional compact contour integration of holomorphic functions.

But the amplitudes are *not* well-defined functionals, and the desired contours don't exist!

This is because of the *infra-red* or forward scattering problem.

On top of this, far too many cases of different channels, no simple basis of states.

## A missed opportunity

1990s: Massless fields more fundamental than ever

Pure massless SU(3) gauge interaction in field theory, not anticipated in 1970!

Twistor diagrams for color-stripped amplitudes in pure gauge theory were recognised as the fundamental case, and similarities with string pictures noted.

Confluence with the idea of on-shell methods in field theory could have been made earlier.

But this was missed:

Ignorance of the Parke-Taylor formula and Nair's work!

## Witten twistor-string model 2003

This made the connection twistor geometry, string theory and advanced field theory.

Revealed the vital importance of the *Parke-Taylor amplitude formula*.  
that everyone here knows: the colour-stripped amplitude for MHV is

$$A_{n;0}^{\text{MHV}} = i \frac{\langle j k \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle},$$

All the interactions earlier studied as twistor diagrams can be regarded as special cases.

Its simplicity of form depends crucially on the decomposition of null momenta into spinors.  
Dominating role of the homogeneities (+2 or -2) all entirely consistent with twistor representation.

## Confluence achieved at last

Developments after 2004 went in a new direction.

Twistor diagrams were used to resolve the *combinatorial complexity* of many-particle gauge-invariant expressions, as well as making explicit their *conformal invariance*.

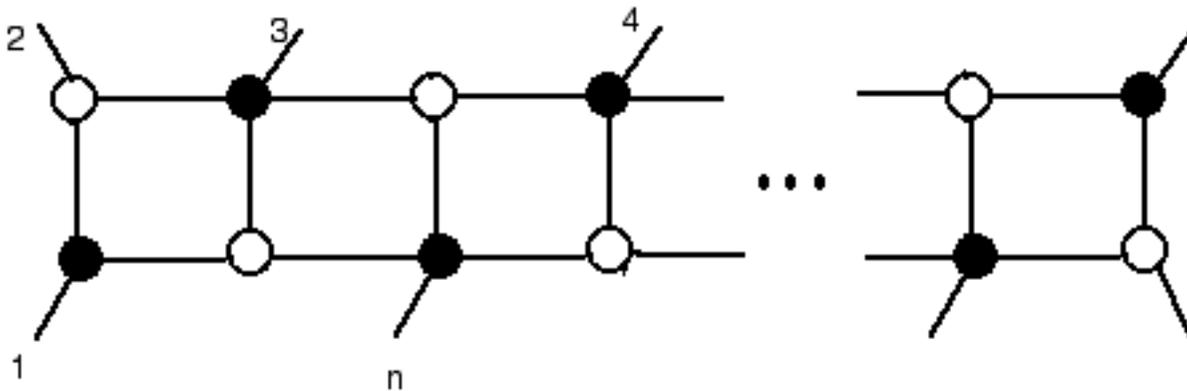
This can be seen as a vast extension of the gauge-invariance of the original Compton scattering diagram, while setting aside the infra-red problem and more general problems of definition.

## Translating tree amplitudes into twistor diagrams (1)

Set aside concerns about the infra-red problem. Regard the twistor diagrams as encoding amplitudes as momentum-space expressions without worrying about their meaning for finite-normed external wave functions (this approach also goes back to RP 1971).

Consider the part of the MHV amplitude defined by the ordering  $(123\dots n)$ .

Then the Parke-Taylor formula corresponds to a twistor diagram like:



and this fact absorbs all previous results (Möller, Compton, pure gauge....)

Twistor diagram representation implies *manifest conformal invariance*.

## BCFW recursion, 2004-5

A simple recursive formula for general tree amplitudes going beyond MHV, due to Britto-Cachazo-Feng-Witten.

Proof by partial fractions in momentum space, using the knowledge that the amplitude can only have simple poles. It depends crucially on spinor decomposition of null momenta in the *complex*, because it needs formal 3-field amplitudes to be non-zero.

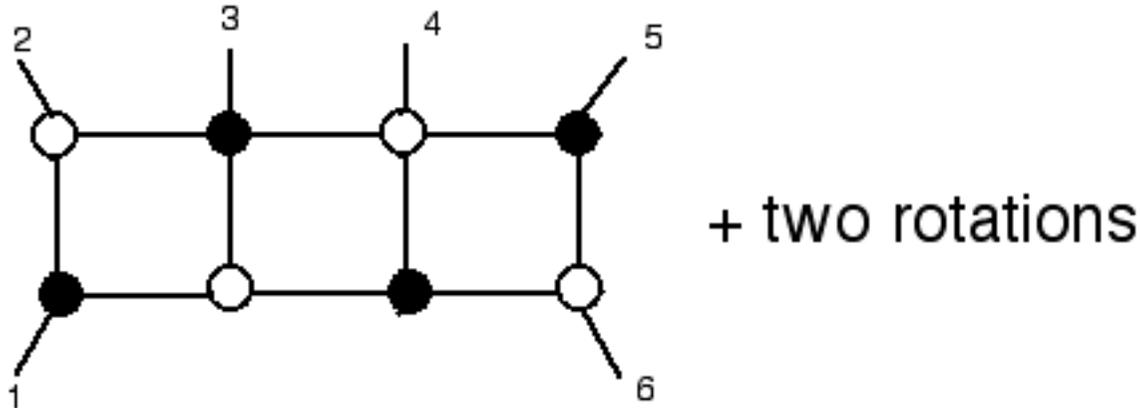
It appears as a summation over sub-amplitudes, including such 3-amplitudes.

## Translating tree amplitudes into twistor diagrams (2)

The BCFW formula becomes a *purely graph-theoretic rule* for joining twistor diagrams.

This takes us beyond MHV to all tree amplitudes (and by extension, to loops).

Simplest application:  $A(- - - + + +)$ : obtain sum of three twistor diagrams:



This gave the first *new* formulas for amplitudes based on twistor methods, as well as making the conformal invariance manifest.

## The twistor Grassmannian

Nima Arkani-Hamed, Freddy Cachazo and collaborators were already asking how BCFW recursion embodied (super-)conformal symmetry. These 2005-6 diagrams suggested the answer.

But they left the mapping from diagrams to rational functions of momentum spinors as an *ad hoc* procedure.

N A-H et al transformed this into a systematic formalism *for on-shell diagrams* based on a *Grassmannian* structure. Twistor integration problems largely sidestepped by working in real twistor space, (i.e.  $++--$  signature in space-time).

Enormous extension to leading singularities and then integrands of loop integrals, all well related to Witten string-motivated structures.

But this still left amplitudes as (non-unique) sums over diagrams...

## The problem of spurious poles

Spurious poles (i.e. removable singularities) are introduced by the partial-fractions method on which BCFW recursion is based. The twistor diagram representation retained these spurious singularities, which vanish on the summation of (in general, many) twistor diagrams.

For six fields, in the case  $A(1^- 2^+ 3^- 4^+ 5^- 6^+)$  we have two three-term expressions:

$$(1 + g^2 + g^4) \frac{[13]^4 \langle 46 \rangle^4}{[12][23] \langle 45 \rangle \langle 56 \rangle [1|2 + 3|4] [3|4 + 5|6] S_{123}} \delta\left(\sum_{i=1}^6 p_i\right)$$

$$(1 + g^2 + g^4) \frac{[3|2 + 4|6]^4}{\langle 56 \rangle \langle 61 \rangle [23][34] [2|3 + 4|5] [4|5 + 6|1] S_{234}} \delta\left(\sum_{i=1}^6 p_i\right)$$

A different six-term expression was known earlier (Berends and Giele, 1988).

## Momentum-twistor space

Not easy to show the equivalence of these expressions *on the subspace defined by the delta-function*.

The key to simplifying it is *dual conformal symmetry*

By choosing twistor coordinates *for the region space*, the dual conformal symmetry of the amplitude expression is made manifest.

This is a non-linear transformation of the momenta, in such a way that the condition expressed by the delta-function is absorbed into simple linear algebra.

An NMHV amplitude can then be written as the 4-volume of a *polytope* in this momentum-twistor space — augmented to a  $CP^4$  by a fifth dimension which encodes the possible helicities of the fields using supersymmetry.

For  $n$  particles, the polytope is (the dual of) the cyclic polytope  $P_n = \sum (*, i, i+1, j, j+1)$ .

This has vertices only of form  $(i, i+1, j, j+1)$ , and these correspond precisely to the physical singularities.

The amplitude is simply a 4-volume:

$$M_n^{\text{NMHV}} = 4! \int d^4\phi \int_{\tilde{P}_n} \frac{D^4\mathcal{W}}{(\mathcal{Z}_0 \cdot \mathcal{W})^5}.$$

For the case  $n=6$  the polytope can be represented as the sum of three 4-simplices:

$$P_6 = (12345) + (13456) + (12356)$$

and the spurious poles are artifacts of this choice of 'triangulation'.

By triangulating the polytope into *six* pieces, without spurious vertices, we obtain the older 1988 expression.

## The Positive Grassmannian

The polytopes for  $A(- - - + + + \dots +)$  have a special *positivity property*, analogous to convex polygons in real plane.

This requires using 'real' twistor spaces, so not clearly related to Minkowski space. Positivity allows the *ordering* of points on an  $RP^1$  to extend to higher dimensions: The *Positive Grassmannian*.

All this implicitly has (pseudo-)momenta for defining in/out states and crossing properties.

Remarks:

The most computational useful form of twistor coordinates, has been in a hybrid setting where momentum-state basis is fundamental.

These radical developments seem to lose the connection with Minkowski space and the geometrical features of positive and negative frequency which RP saw as fundamental.

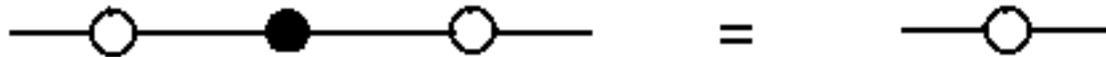
But...

## Twistor diagrams are reincarnated

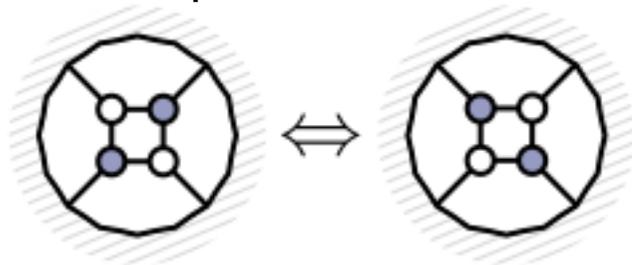
... as representations of cells of the positive Grassmannian.

They can also be defined as abstract graphs:

Define equivalence classes of such graphs by



and the 'square move' relation:



This structure explains the ambiguity of representation noted by Penrose (1972) for Compton scattering. Nima Arkani-Hamed et al (2012) give a bijection between [classes of] *planar* graphs and the *permutations*.

## Not a complete theory

Can we get away from the momentum-states and make the theory totally twistor-geometric?

Can the graphs still be interpretable as *twistor integrals* as in the original Penrose programme, for genuine Minkowski-space scattering of finite-normed fields?

(Note: The helicity *deformation* ideas of Studacher et al., relate to ideas from that original investigation. Study of many-dimensional contour integration still relevant.)

But L-loop amplitudes are given as *integrand*s leaving  $4L$  dimensions to be integrated out. Extension to *non-maximal supersymmetry* is needed. And a treatment of *mass*.

This picture also depends on separating the total amplitude into a sum over different colour-orderings. We must express (1) Non-planar amplitudes and graphs (2) relations between different orderings and (3) gravity.

## Some recent twistor-related developments

Determinant formula for MHV gravity tree amplitudes, developed rapidly by Freddy Cachazo, David Skinner et al to all gravitational tree amplitudes.

The strong feature of twistor-geometric  $\mathbb{I}$  twistor as a numerator factor.  
This may be a non-simple  $\mathbb{I}$  carrying a cosmological constant.

Determinantal formulas greatly extended: the Cachazo-He-Yuan scattering equations of wide application, and improved picture of gravity as square of gauge theory (KLT relations).

Lionel Mason et al: ambitwistor strings, going back to null geodesic as primary object, relating directly to CHY scattering equations.

## Back to the Future

Most radical: Penrose's 2015 ideas for 'Palatial Twistors', suggests non-projective I appears in the fundamental geometry as the commutator of Z's in a non-commutative geometry.

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = \epsilon I^{\alpha\beta}, \quad \bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = \bar{\epsilon} I_{\alpha\beta},$$

Idea: a *non-projective* I is an essential ingredient, and should be embodied in fundamental geometry.