Geometry of non-planar amplitudes

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Goal

Mathematical structures in planar N=4 SYM

Other theories
Goal

Mathematical structures in planar N=4 SYM

Non-planar N=4 SYM
Plan of the talk

- Geometric picture for integrand in planar N=4 SYM
- Singularity structure of non-planar amplitudes
- Towards supergravity amplitudes
Once upon a time there was a MHV amplitude….

\[ A = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \ldots \langle n1 \rangle} \]

First evidence for simplicity in scattering amplitudes

Amplitudes are more than sums of Feynman diagrams
Singularities of amplitudes

- Scattering amplitudes of massless particles in D=4
  \[ A = \sum_j \int dI_j = \int dI \]

- General idea: amplitudes are fixed from their singularities

- Locality: only \( \frac{1}{P^2} \) present

- Unitarity: factorization on poles

Integrand is an ideal object to construct/study
Unitarity methods

(Bern, Dixon, Kosower)

- Iterative use of the unitary cut

- Generate basis of integrals, fixing coefficients from cuts

- Tremendous success in calculations in 1990-today

Blackhat: QCD background
**BCFW recursion relations**

(Britto, Cachazo, Feng, Witten 2005)

- Large class of theories at tree-level
- Tree-level unitarity
- Shift momenta + Cauchy formula
  
  \[
p_1 \rightarrow p_1 + zq \\
p_2 \rightarrow p_2 - zq
\]
- Very efficient method:

<table>
<thead>
<tr>
<th></th>
<th>gg → 4g</th>
<th>gg → 5g</th>
<th>gg → 6g</th>
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<td>Feynman diagrams</td>
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<td>Recursion relations</td>
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</table>
Hydrogen atom of gauge theories

- N=4 Super Yang-Mills theory in the planar limit
- Great toy model for QCD
  - Tree-level amplitudes identical
  - Convergent perturbative series, no confinement
  - Hidden symmetries in the theory
- Past: new methods for amplitudes originated here
Planar $\mathcal{N}=4$ SYM theory

- Useful playground for many theoretical ideas

Diagram:

- Integrability
  - Yangian
- AdS/CFT
  - Strong coupling
- Wilson loops
  - OPE expansion
- Twistor strings
- Hexagon bootstrap
- BDS ansatz
Dual variables

- Generally, each diagram has its own variables
  - No global loop momenta
  - Each diagram: its own labels

- Planar limit: dual variables

  \[ p_i = x_{i+1} - x_i \]

  \[ k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3) \quad \text{etc} \]

  \[ \ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3) \quad \text{etc} \]

  Global variables
Dual conformal symmetry

Using these variables: define a single function

\[ \mathcal{M} = \int d^4y_1 \ldots d^4y_L \mathcal{I}(x_i, y_j) \]

Integrand

Tree-level amplitudes + integrand in planar N=4 SYM:

Dual conformal symmetry

Superconformal symmetry + Dual -> Yangian

(Drummond, Henn, Smirnov, Sokatchev 2007)

(Drummond, Henn, Korchemsky, Sokatchev 2008)

(Drummond, Henn, Plefka 2009)
Momentum twistors

(Hodges 2009)

New variables: points in $\mathbb{P}^3$

$$Z = \begin{pmatrix} \lambda_a \\ x_{a\dot{a}} \lambda_a \end{pmatrix}$$

Dual conformal symmetry acts as SL(4) on Z
Momentum twistors

- Dual conformal invariants: $\langle 1234 \rangle = \epsilon_{abcd} Z^a_1 Z^b_2 Z^c_3 Z^d_4$

- Functions of momenta only: projective

\[
\ell^2 = \frac{\langle AB41 \rangle}{\langle AB \rangle \langle 41 \rangle}
\]

- Infinity twistor $I^{ab}$ breaks dual conformal symmetry

\[
\langle 12 \rangle = \langle 12I \rangle = \epsilon_{abcd} Z^a_1 Z^b_2 I^{cd}
\]
Manifest Yangian symmetry

- Terms in BCFW recursion: products of on-shell amplitudes

\[ A_n = \sum_{L,R} \text{Tension between locality and symmetry} \]

- Each term separately Yangian invariant

- Iterate until all vertices are 3pt: on-shell diagrams
On-shell diagrams

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)
On-shell diagrams

- Draw arbitrary graph with three point vertices

Products of three point amplitudes

\[
\begin{align*}
P > 4L & : \text{Extra delta functions} \\
P = 4L & : \text{Function of external data only} \\
P < 4L & : \text{Unfixed parameters (forms)}
\end{align*}
\]
On-shell diagram expansion

- Example of 6pt amplitude

\[ \sum_{L,R} (P^2) \]

- Each diagram: on-shell, gauge invariant function
  Planar N=4: Yangian invariant \[ [12345] \]

- Same pictures: cuts of the loop amplitudes with \[ \delta(P^2) \]
Loop recursion relations

(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT 2010)

- Recursion relations for $\ell$-loop integrand

\[ A_n = A_{n+2} + \sum_{L,R} \]

- Example: 4pt 1-loop

5-loop on-shell diagram = 1-loop off-shell box
Permutations

• Represent graphically permutation

• Graph with only 3pt vertices
  
  - Turn right on blue
  - Turn left on white

Tree-level amplitudes: list of permutations
Positive Grassmannian

- Space of $n$ points in $k$-dim projective space with linear dependencies between consecutive points

\[
\begin{pmatrix}
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\end{pmatrix}
\]

- $(k \times n)$ real matrix with positive main $(k \times k)$ minors

- How to construct this matrix? Using the same diagrams
Amalgamation procedure

- Construct big positive matrix from small ones

\[
\begin{pmatrix}
* & * & *
\end{pmatrix}
\quad \begin{pmatrix}
* & * & *
\end{pmatrix}
\]

- Arbitrary graph: positive matrix

\[
\begin{pmatrix}
* & * & * & * & *
* & * & * & * & *
\end{pmatrix}
\quad \begin{pmatrix}
* & * & * & * & * & *
* & * & * & * & *
\end{pmatrix}
\]

Gluing preserves positivity of minors
Connection to amplitudes

- Building positive matrix: face or edge variables

- Same function as a product of 3pt amplitudes equal to

\[ \Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \delta(C \cdot Z) \]

Solves for \( \alpha_i \)
in terms of \( \lambda_i, \tilde{\lambda}_i \)
and gives \( \delta(P)\delta(Q) \)
Logarithmic singularities

- Amplitude = sum of on-shell diagrams
  \[ \Omega \sim \frac{d\alpha}{\alpha} \] near any pole \[ \alpha = F(\ell_j, p_i) \]

- More than single poles:
  \[ \frac{dx \, dy}{xy(x + y)} \xrightarrow{x=0} \frac{dy}{y^2} \]

- Logarithmic singularities specific for planar N=4 SYM
  Generic QFT: \[ \Omega = F(\alpha) \delta(C \cdot Z) \]

- Dlog form: close relation to maximal transcendentality
Geometric interpretation

- On-shell diagrams: regions (cells) in the Grassmannian
- Logarithmic form: “volume” of this region
- Amplitude: sum of on-shell diagrams
  Given by BCFW: unitarity
- Question: Is there a complete geometric definition?
Amplituhedron

(Arkani-Hamed, JT 2013)
Volume of polyhedron

(Hodges 2009)

- Tree-level process: $gg \rightarrow 5g$ in momentum twistor space
- Comparison of two calculations of recursion relations
- Even simpler case: polygon

(Picture by Stavros Garoufalis)
Point inside the polygon

Consider a point inside a polygon in projective plane

\[ \text{Space of all points inside convex polygon} \]

\[ Y = c_1 Z_1 + c_2 Z_2 + \ldots + c_n Z_n \]

Form with logarithmic singularities on boundaries

\[ \Omega(Y, Z_i) \]

\[ C = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \end{pmatrix} \in G_+(1, n) \]

\[ Z = \begin{pmatrix} Z_1 & Z_2 & Z_3 & \ldots & Z_n \end{pmatrix} \in M_+(3, n) \]
**Triangulation**

- **BCFW using on-shell diagrams is a triangulation**

\[ C = \begin{pmatrix} 1 & 0 & 0 & c_4 & c_5 & 0 \end{pmatrix} \in G_+(1, 6) \]

\[ \Omega = \frac{dc_4}{c_4} \frac{dc_5}{c_5} \]

Supersymmetry
- \( \rightarrow \) higher dimensional bosonic space
Road to Amplituhedron

Start:
Point inside a convex polygon
Road to Amplituhedron

Start:
Point inside a convex polygon
Road to Amplituhedron

Start:
Point inside a convex polygon
Road to Amplituhedron

Start:
Point inside a convex polygon

Amplituhedron $A_{n,k,\ell}$
A $k$-dim plane and $\ell$ lines inside a $(k + 4)$-dim convex space defined by $n$ vertices
Amplituhedron conjecture

- Volume of $A_{n,k,\ell}$: Loop integrand in maximally supersymmetric Yang-Mills theory

  $\ell = 0$: Amplitudes of gluons in QCD

- Consistency check: Locality and Unitarity

- Explicit checks against reference theoretical data
Positivity of amplitudes

(Arkani-Hamed, Hodges, JT, 2014)

- All terms combined in the amplitude
  \[ I = \frac{\text{(Numerator)}}{\text{(all poles)}} \]

- Illegal singularities in denominator

- Numerator fixed by zeroes
  - Points outside Amplituhedron
  - Canceling higher poles

- Amplitude **positive** (for points inside): volume interpretation
Singularities of non-planar amplitudes
Non-planar amplitudes in N=4 SYM

- No unique integrand, labeling problem
- No momentum twistors, no known symmetries
- On-shell diagrams for singularities
  - No recursion relations

What is \( \ell \) ?
Non-planar on-shell diagrams

Non-planar diagrams
(Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT, 2014)
(Franco, Galloni, Penante, Wen 2015)

Same logarithmic form

$$C = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} \in G(3, 6)$$

$$\Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \delta(C \cdot Z)$$

Conjecture: logarithmic singularities of the amplitude
MHV on-shell diagrams

- Planar sector: all are Parke-Taylor factors

\[
\begin{align*}
1 &= \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} = PT(12345) \\
\end{align*}
\]

required by superconformal symmetry

- Non-planar diagrams: holomorphic functions

\[
\begin{align*}
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 14 \rangle &= \frac{(\langle 34 \rangle \langle 51 \rangle \langle 62 \rangle + \langle 14 \rangle \langle 25 \rangle \langle 63 \rangle)^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 25 \rangle \langle 56 \rangle \langle 62 \rangle \langle 34 \rangle \langle 46 \rangle \langle 63 \rangle \langle 45 \rangle \langle 51 \rangle \langle 14 \rangle} \\
\end{align*}
\]
**MHV on-shell diagrams**

- **Planar sector:** all are Parke-Taylor factors

\[ \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} = PT(12345) \]

required by superconformal symmetry

- **Non-planar diagrams:** holomorphic functions

\[ = PT(123456) + PT(124563) + PT(142563) + PT(145623) + PT(146235) + PT(146253) + PT(162345) \]

Linear combination of Parke-Taylor factors
Dual conformal symmetry

- Conservative approach: amplitude as a sum of integrals

\[ A = \sum_i a_i \cdot C_i \cdot I_i \]

- Planar limit: integrals \( I_i \) dual conformal invariant (DCI)

- How to distinguish: DCI vs non-DCI? \( \rightarrow \) depends on \( I \)

Can we see it in the structure of singularities?
DCI in action

Unit leading singularities

Chiral vs scalar pentagon

\[
\frac{\langle ABd^2 A \rangle \langle ABd^2 B \rangle \langle AB13 \rangle \langle 2345 \rangle \langle 4512 \rangle}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB45 \rangle \langle AB51 \rangle}
\]

\[
\frac{\langle ABd^2 A \rangle \langle ABd^2 B \rangle \langle ABI \rangle}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB45 \rangle \langle AB51 \rangle}
\]

On all 4L-cut the residue is 1

For \( n > 6 \) can be cross ratio

Unit

Non-unit
DCI in action

No poles at infinity \( \ell \to \infty \)

\[
\frac{\langle ABd^2 A \rangle \langle ABd^2 B \rangle \langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB41 \rangle}
\]

No pole

\[
\frac{\langle ABd^2 A \rangle \langle ABd^2 B \rangle \langle 1234 \rangle \langle 23I \rangle}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle ABI \rangle}
\]

Pole

Cut this propagator
DCI in action

No poles at infinity \( \ell \to \infty \)

\[
\frac{\langle ABd^2A \rangle \langle ABd^2B \rangle \langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB41 \rangle}
\]

No pole

\[
\frac{d^4 \ell}{\ell^2(\ell + k_2)^2(\ell + k_2 + k_3)^2}
\]

Pole
DCI in action

No poles at infinity \( \ell \to \infty \)

\[
\frac{\langle ABd^2 A \rangle \langle ABd^2 B \rangle \langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB41 \rangle}
\]

No pole

\[
\frac{d^4 \ell}{\ell^2 (\ell + k_2)^2 (\ell + k_2 + k_3)^2}
\]

Pole

\[
\begin{align*}
\downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad 0 & \quad 0
\end{align*}
\]
DCI in action

No poles at infinity  \( \ell \to \infty \)

\[
\frac{\langle ABd^2A \rangle \langle ABd^2B \rangle \langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB41 \rangle}
\]

No pole

\[
\frac{d\alpha}{\alpha} \quad \ell + k_2 = \alpha \lambda_2 \tilde{\lambda}_3
\]

Pole

\[
\alpha \to \infty \quad \ell \to \infty
\]
Source of poles at infinity

- Planar: triangle sub-diagrams present
- Non-planar: more types of poles at infinity

![Diagram]

Only box subdiagrams
Has poles at infinity
Logarithmic singularities vs DCI

- Multiple poles in the cut structure
  
  \[ \text{Integral} \rightarrow \text{Cut} \rightarrow \text{Cut} \rightarrow \frac{d\alpha}{\alpha^2} \]

- At low loops saved by DCI

Not at higher loops: new condition

\[ N \sim \langle AB34 \rangle \]
Expansion of the amplitude


- Expansion for the (MHV) amplitude
  \[ A = \sum_i a_i \cdot C_i \cdot I_i \]
  - Parke-Taylor factors
  - Color factors
  - Basis of integrals

- Conjecture: term-by-term
  - Logarithmic singularities
  - No poles at infinity
  - Unit leading singularities

- Uniform transcendental non-planar integrals studied
  (Gehrmann, Henn, Huber 2011, Henn 2014, Henn, Smirnov, Smirnov 2015)

- After integration:
  - Uniform transcendentality
  - No prefactors
  - Special cross-ratios…
Explicit checks


- Construct basis for 4pt at 2-loop and 3-loop, 5pt 2-loop

Expand the amplitude:
Explicit checks

- Construct basis for 4pt at 2-loop and 3-loop, 5pt 2-loop

Expand the amplitude:
Coefficient fixed by standard unitarity methods
Coefficients from zeroes

(Bern, Herrmann, Litsey, Stankowicz, JT 2015)

- Go even further in the analogy with planar
- Use only equations of type $\text{Cut } I = 0$
  - Illegal cuts: non-MHV or spurious cuts
  - No target (product of trees) necessary
- In planar: conjecture, evidence for dual Amplituhedron

Conjecture: $A = \sum_{i} a_i \cdot C_i \cdot I_i$

Test for our three cases:

Fixed by vanishing cuts
Coefficients from zeroes

(Bern, Herrmann, Litsey, Stankowicz, JT 2015)

- Go even further in the analogy with planar
- Use only equations of type $\text{Cut } I = 0$
  - Illegal cuts: non-MHV or spurious cuts
  - No target (product of trees) necessary
- In planar: conjecture, evidence for dual Amplituhedron
- Conjecture: $A = \sum a_i \cdot C_i \cdot I_i$ Test for our three cases:
  - Fixed by vanishing cuts
Example of zero condition

- Expansion of the amplitude

\[ M_2 = \sum_\sigma a_1 + a_2 \]

Diagram: Two graphs with edges labeled \( \ell_1 \) and \( \ell_2 \) connecting vertices 1 to 2, 2 to 3, 3 to 4, and 1 to 4.
Example of zero condition

- Expansion of the amplitude

\[ \text{Cut } M_2 = \sum_{\sigma} a_1 + a_2 = 0 \]

Illegal 5-cut \[ k = 1 \]

Fixes relative coefficient \[ a_1 = a_2 \]
Non-planar N=4 conclusion

- Amplitudes (integrands) in complete N=4 SYM:
  - Analogue of dual conformal symmetry
  - On-shell diagrams / Amplituhedron insights

- No poles at infinity
- Special leading singularities
- Logarithmic singularities
- Zero conditions

- Homogeneous conditions define the amplitude
  This is begging for geometric/volume interpretation
  Role of color factors?
Few words about supergravity amplitudes
Similarities with Yang-Mills

- **BCJ relations**
  \[ A^{(YM)} = \sum_j \frac{n_j c_j}{s_j} \rightarrow A^{(GR)} = \sum_j \frac{n_j^2}{s_j} \]

- Squaring has dramatic consequences on singularities

- Loop amplitudes in N=8 SUGRA: poles at infinity

  Bad UV behavior of integrals in the amplitude
Gravity on-shell diagrams
(Herrmann, JT, in progress)

- Well defined on-shell objects
- No recursion relations, capture singularity structure
- MHV on-shell diagrams: not holomorphic in N=8

\[
\begin{align*}
&= \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} \\
&\text{in N=4 SYM}
\end{align*}
\]
Gravity on-shell diagrams
(Herrmann, JT, in progress)

- Well defined on-shell objects
- No recursion relations, capture singularity structure
- MHV on-shell diagrams: not holomorphic in N=8

\[
\begin{aligned}
\langle 5|1 + 6|2\rangle\langle 2|3 + 4|5\rangle [16]^2 [34]^2 \\
\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 25\rangle^2
\end{aligned}
\]

in N=8 SUGRA
Gravity on-shell diagrams
(Herrmann, JT, in progress)

- Default way to calculate: product of 3pt amplitudes
- Amalgamation of 3pt vertices: Grassmannian formula
  Does not preserve logarithmic or any other nice form
- Rules to define the form globally needed

\[ \Omega = F(\alpha) \delta(C \cdot Z) \]

We start to understand how it looks like

Dramatic implications for singularities of gravity amplitudes
Conclusion
Conclusion

- Planar N=4 SYM: on-shell diagrams, Amplituhedron
- Non-planar N=4 SYM: same properties seem to hold
  - Evidence for non-planar geometric construction
  - Good variables needed
- Gravity in progress
Thank you for your attention