Cuts and discontinuities

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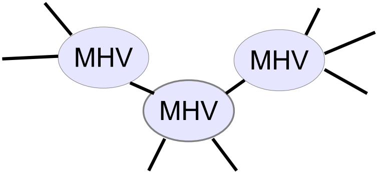
Amplitudes as building blocks

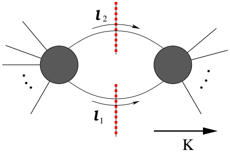
- MHV amplitudes, for other helicity amplitudes
  - Saw the Parke-Taylor formula in Witten's 2003 paper.
  - MHV diagrams build tree amplitudes on-shell [Cachazo, Svrcek, Witten]



Amplitudes as building blocks

- MHV amplitudes, for other helicity amplitudes

- Tree amplitudes, for loop amplitudes
  - Unitarity method: on-shell "cut" techniques with master integarals [Bern, Dixon, Kosower; with Dunbar, Weinzierl, Del Duca, ...]
  - Updated with MHV diagrams [Cachazo, Svrcek, Witten; Brandhuber, Spence, Travaglini]
  - Twistor geometry led to differential operators for NMHV in $\mathcal{N} = 4$ SYM; complex cuts for NNMHV [RB, Cachazo, Feng]
  - Tree-level recursion observed as a byproduct [Roiban, Spradlin, Volovich; RB, Cachazo, Feng]
  - Extended spinor integration and cuts for more realistic theories [collaborations with Anastasiou, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia, Mirabella, Ochirov, Yang]



Amplitudes as building blocks

- MHV amplitudes, for other helicity amplitudes
- Tree amplitudes, for loop amplitudes
- Cuts beyond one loop: what are we computing?

What are cuts?

What are generalized cuts?

Cuts and Hopf algebra of Feynman integrals

Cutkosky: *Cut diagrams compute discontinuities across branch cuts.*

We conjecture:

- For integrals in the class of multiple polylogarithms (MPL), the discontinuities described by cuts are naturally found within the Hopf algebra of MPL.
- There is a diagrammatic Hopf algebra that
  - involves cut diagrams, and
  - corresponds to the Hopf algebra of MPL.

[Based on: 1401.3546 with Abreu, Duhr, Gardi, and work in preparation ; 1504.00206 with Abreu and Grönqvist]

Cuts as discontinuities

From the Largest Time Equation [Veltman]:

$$F + F^* = -\sum_s \mathrm{Cut}_s\, F,$$

Hence:

$$\mathrm{Disc}_s\, F = -\,\mathrm{Cut}_s\, F.$$

Valid in a particular kinematic region: cut invariant s positive, others negative.

Generalize to:

$$\boxed{\mathrm{Cut}_{s_1,\ldots,s_k}\, F = (-1)^k\, \mathrm{Disc}_{s_1,\ldots,s_k}\, F.}$$

Multiple unitarity cuts

Cut propagators: put them on shell.

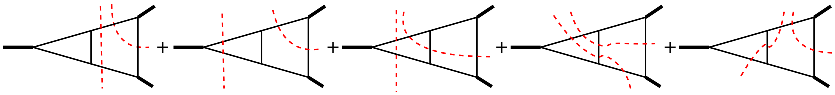
Choose propagators corresponding to a sequence of momentum channels.

$$\mathrm{Cut}_{s_1,\ldots,s_k} F$$

With *real* kinematics.

Defined by: cut propagators + consistent energy flow + corresponding kinematic region

Multiple cuts are taken simultaneously.



Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) \equiv \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t)$$

Examples:

$$I(0; 0; z) = \log z, \quad I(0; a; z) = \log\left(1 - \frac{z}{a}\right)$$
$$I(0; \vec{a}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{a}\right), \quad I(0; \vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

Harmonic polylog if all $a_i \in \{-1, 0, 1\}$.

n is the *transcendental weight*.

Observation: most known Feynman integrals can be written in terms of classical and harmonic polylogs.

Hopf algebra of MPL

Goncharov's coproduct formula for MPL (modulo π):

$$\Delta I(a_0; a_1, \dots, a_n; a_{n+1})$$
$$= \sum_{0=i_0<\cdots<i_k<i_{k+1}=n+1} I(a_0; a_{i_1}, \dots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

Examples:

$$\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log x \log y) &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1 \\
\Delta(\mathrm{Li}_n(z)) &= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}
\end{aligned}$$

Hopf algebra includes compatible counit and antipode.
Graded by transcendental weight.

Symbols of MPL

The "symbol" $\mathcal{S}$ is essentially the maximal iteration.

$$\mathcal{S}(F) \equiv \Delta_{1,\ldots,1}(F) \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1 \,.$$

$$\begin{aligned} \mathcal{S}\left(\frac{1}{n!}\log^n z\right) &= \underbrace{z \otimes \cdots \otimes z}_{n \text{ times}} \\ \mathcal{S}(\mathrm{Li}_n(z)) &= -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{(n-1) \text{ times}} \end{aligned}$$

Functions are all weight 1, i.e. log.

(The symbol was introduced to us for remainder functions [Goncharov, Spradlin, Vergu, Volovich] and applied widely since then.)

Coproducts of Feynman integrals

Observation: without internal masses, coproduct can be written such that

$$\Delta_{1,n-1} F = \sum_i \log(-s_i) \otimes f_{s_i}$$

- first entries are Mandelstam invariants,
- and each second entry f_{s_i} is the discontinuity of F in the channel s_i.

[Gaiotto, Maldacena, Sever, Viera]

Thus: the coproduct captures standard unitarity cuts.

What about generalized cuts?

Coproduct entries

If

$$\Delta_{\underbrace{1,1,\ldots,1}_{k\ \text{times}},n-k} F = \sum_{\{x_1,\ldots,x_k\}} \log x_1 \otimes \cdots \otimes \log x_k \otimes g_{x_1,\ldots,x_k}\,,$$

then

$$\delta_{x_1,\ldots,x_k} F \cong g_{x_1,\ldots,x_k}\,.$$

More precisely: match branch points. The "$\cong$" means modulo π.

Motivated by coproduct identity : $\Delta\ \text{Disc} = (\text{Disc} \otimes 1)\,\Delta$ [Duhr]
and first-entry condition.

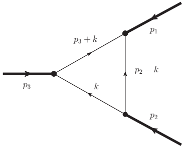
Coproduct and discontinuities for Feynman integrals

$$\mathrm{Disc}_{s_1} F = (-2\pi i)\, \delta_{s_1} F.$$

$$\boxed{\mathrm{Disc}_{s_1,\ldots,s_k} F = \sum_{x_1,\ldots,x_k} \pm(2\pi i)^k \; \delta_{x_1,\ldots,x_k} F.}$$

- Assume prior knowledge of alphabet (e.g. from cuts)
- Underlying claim: kinematics put us on the branch cuts, so that it is correct to use our definition of Disc.

Basic example: triangle



$$
\begin{aligned}
T &= -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right) \\
&\equiv -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\ \mathcal{P}_2
\end{aligned}
$$

where

$$
z\bar{z} = \frac{p_2^2}{p_1^2}, \qquad (1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}
$$

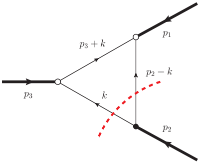
Coproduct of the triangle

$$\begin{aligned}\Delta\mathcal{P}_2 &= \mathcal{P}_2\otimes 1 + 1\otimes\mathcal{P}_2 + \frac{1}{2}\log(z\bar{z})\otimes\log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log[(1-z)(1-\bar{z})]\otimes\log\frac{\bar{z}}{z}\\ &= \mathcal{P}_2\otimes 1 + 1\otimes\mathcal{P}_2 + \frac{1}{2}\log\left(-p_2^2\right)\otimes\log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log\left(-p_3^2\right)\otimes\log\frac{\bar{z}}{z}\\ &\quad +\frac{1}{2}\log(-p_1^2)\otimes\log\frac{1-1/\bar{z}}{1-1/z}\end{aligned}$$

Alphabet: $\{z,\bar{z},1-z,1-\bar{z}\}$.

First cut of the triangle

Cut in the p_2^2 channel.



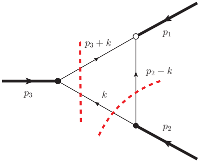
Kinematic region: $p_2^2 > 0; \quad p_1^2, p_3^2 < 0.$

$$\begin{aligned} \mathrm{Cut}_{p_2^2}\, T &= \frac{2\pi}{p_1^2(z-\bar{z})} \log \frac{1-z}{1-\bar{z}} \\ &= -\,\mathrm{Disc}_{p_2^2}\, T \end{aligned}$$

$$\delta_{p_2^2}\mathcal{P}_2 = \frac{1}{2}\log\frac{1-z}{1-\bar{z}}$$

$$\mathrm{Disc}_{p_2^2}\, T = (-2\pi i)\delta_{p_2^2} T.$$

Second cut of the triangle



$$\mathrm{Cut}_{p_3^2, p_2^2}\, T = \frac{4\pi^2 i}{p_1^2(z - \bar{z})}$$

Kinematic region: $p_3^2, p_2^2 > 0; \quad p_1^2 < 0$
Equivalently: $\bar{z} < 0, z > 1$.

Now we have to match the alphabet with Mandelstam invariants:

$$\mathrm{Disc}_{p_2^2, p_3^2}\, T = \mathrm{Cut}_{p_2^2, p_3^2}\, T.$$

$$\mathrm{Disc}_{p_2^2, p_3^2}\, T = 4\pi^2\, \delta_{p_2^2, 1-z}\, T$$

Reconstruction: from cut to symbol

Integrability condition on symbols: for each k,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} \, d\log a_{i_k} \wedge d\log a_{i_{k+1}} \left[a_{i_1} \otimes \cdots \otimes a_{i_{k-1}} \otimes a_{i_{k+2}} \otimes \cdots \otimes a_{i_n} \right] = 0\,.$$

Apparently related to exchanging order of cuts.

Combine with first entry condition (=Mandelstam invariant) and known cut(s).

Reconstruction of the symbol of the 2-loop 3-point ladder is unique from any of its single *or double* cuts. Various finite 1-loop examples also work.

Reconstruction: from symbol to full function

- In general, integrating a symbol is an unsolved problem.

- But in many cases we have enough information to constrain the function uniquely & algebraically.

- In the same example, from the p_2^2 cut of triangle, given that:

$$\mathcal{S}(\mathcal{T}) = \frac{1}{2}\, z\bar{z} \otimes \frac{1-z}{1-\bar{z}} + \frac{1}{2}\,(1-z)(1-\bar{z}) \otimes \frac{\bar{z}}{z}$$

  and antisymmetry under $z \leftrightarrow \bar{z}$, the solution

$$\begin{aligned} \mathcal{T} &= \mathcal{P}_2(z) \\ &= \left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2} \log(z\bar{z}) \log\left(\frac{1-z}{1-\bar{z}}\right)\right) \end{aligned}$$

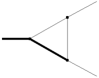
  is unique.

- Ladder & massive triangles are easy too. At most, fix a single free constant by numerical evaluation at a point.

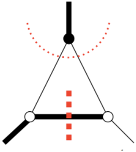
Generalizations for internal masses

[Abreu, RB, Grönqvist]

- First entries adjusted for thresholds

$$\mathcal{S}\Bigg($$  $$\Bigg) = \frac{1}{\epsilon}\frac{m^2-p^2}{m^2} + m^2 \otimes \frac{m^2\left(m^2-p^2\right)}{p^2} + \left(m^2 - p^2\right) \otimes \frac{p^2}{\left(m^2-p^2\right)^2} + \mathcal{O}\left(\epsilon\right)\ .$$

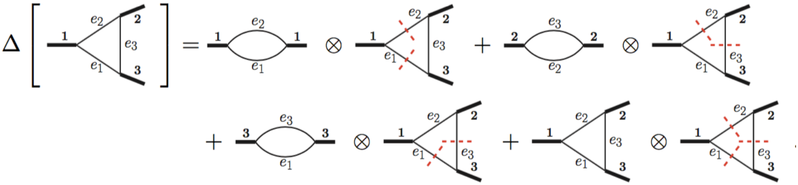
- Include cuts of massive propagators



 $=$  $= 2\pi\,\delta\left(p^2 - m^2\right)$

Coproducts of diagrams

[in preparation with Abreu, Duhr, Gardi]



Second entries are discontinuities; first entries have discontinuities.

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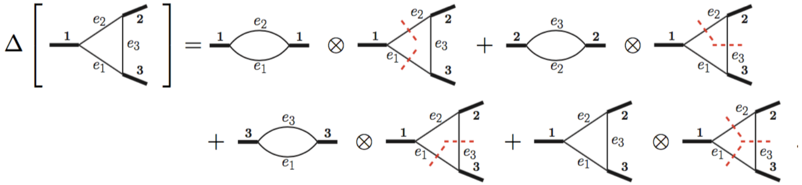


Second entries are discontinuities; first entries have discontinuities.

Motivated by the identity

$$\Delta\ \mathrm{Disc} = (\mathrm{Disc} \otimes 1)\, \Delta.$$

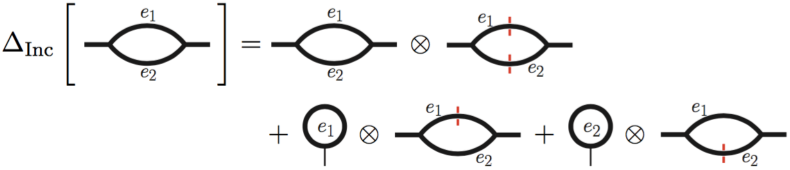
The companion relation

$$\Delta\, \partial = (1 \otimes \partial)\, \Delta$$

produces differential equations.

Diagram operations: pinch $\otimes$ cut

Pinch and cut *complementary* subsets of edges:



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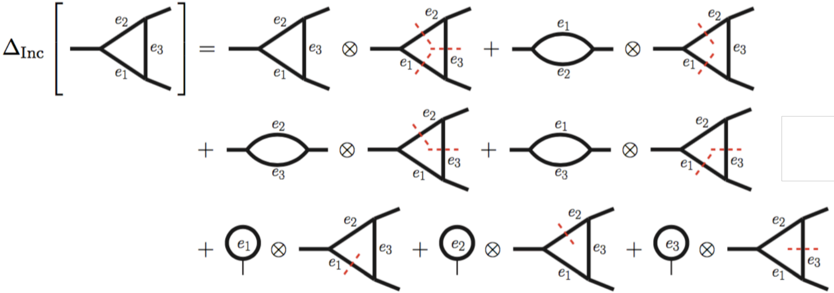

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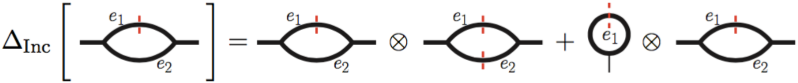
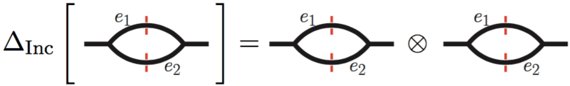




Can also start with a cut diagram.

Operation is purely combinatorial and is the coproduct of a full Hopf algebra with product, unit, counit, antipode. (*Incidence Hopf Algebra.*)

2 equivalent Hopf algebras

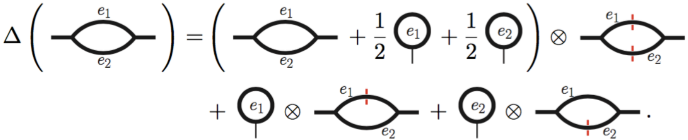
The combinatorial algebra agrees with the Hopf algebra on the MPL of evaluated diagrams!

But we have to make adjustments:

- Integrals live in different dimensions. With n propagators, $D = n - 2\epsilon$ for n even; $D = n + 1 - 2\epsilon$ for n odd. Conjecture: these integrals form a basis of 1-loop integrals.

  E.g. box and triangle in $D = 4 - 2\epsilon$, bubble and tadpole in $D = 2 - 2\epsilon$.

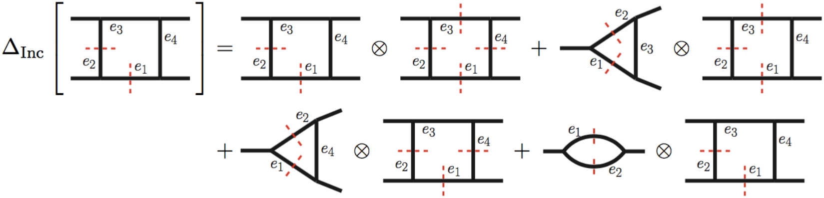
- Need to insert extra terms:



  Isomorphic to the more basic construction. (For any value of 1/2.)

What are these generalized cuts?

Generalized cuts are sensitive to kinematic regions. Our graphical relations make no reference to kinematics, and we need complex cuts like 4-cuts of boxes. Here, we define cuts as residues.



At least at 1-loop, differences are proportional to $i\pi$, which drops out of the coproduct.

The residues can be written beautifully in terms of determinants.
(Gram and modified Cayley)

Not very clear how to generalize to higher loops, but the nature of coproducts suggests it should be possible. [related work: Brown; Bloch and Kreimer]

Generalized cuts as residues and determinants

1-loop cuts defined as residues:

$$\mathcal{C}_S[I_n] = (2\pi)^{\lfloor |S|/2 \rfloor} \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_S} d^D k \prod_{j \notin S} \frac{1}{(k-q_j)^2 - m_j^2 + i0} \quad \text{mod } i\pi \,,$$

- S is the set of cut propagators
- Parametrize loop momentum in selected coordinates
- Contour encircles poles of cut propagators

Result:

$$\mathcal{C}_S[I_n] = \left(-\frac{1}{2}\right)^s \frac{(2\pi)^{\lfloor s/2 \rfloor} \, e^{\gamma_E \epsilon}}{\sqrt{Y_S}} \left(\frac{Y_S}{G_S}\right)^{(D-s)/2} \int \frac{d\Omega_{D-s+1}}{i\pi^{D/2}} \left[\prod_{j \notin S} \frac{1}{(k-q_j)^2 - m_j^2}\right]_S$$

where

$$\begin{aligned} G_S &= \det\left(q_i \cdot q_j\right)_{i,j \in S} \\ Y_S &= \det\left(\frac{1}{2}(m_i^2 + m_j^2 + (q_i - q_j)^2)\right)_{i,j \in S} \end{aligned}$$

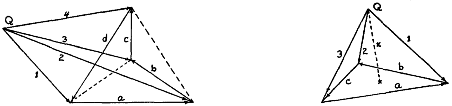
Maximal and next-to-maximal cuts

Some special cases:

$$\begin{aligned}
\mathcal{C}_{2k}[J_{2k}] &= 2^{1-k-2\epsilon}\,\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{e^{\gamma_E\epsilon}}{\sqrt{Y_{2k}}}\left(\frac{Y_{2k}}{G_{2k}}\right)^{-\epsilon} \\
\mathcal{C}_{2k+1}[J_{2k+1}] &= -2^{-k}\,\frac{e^{\gamma_E\epsilon}}{\Gamma(1-\epsilon)\sqrt{G_{2k+1}}}\left(\frac{Y_{2k+1}}{G_{2k+1}}\right)^{-\epsilon}.
\end{aligned}$$

$$\mathcal{C}_{2k-1}[J_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3}\,\frac{2^{-k+4\epsilon}e^{\gamma_E\epsilon}}{\sqrt{Y_{2k}}}\left(\frac{Y_{2k-1}}{G_{2k-1}}\right)^{-\epsilon}{}_2F_1\left(\frac{1}{2},-\epsilon;1-\epsilon;\frac{G_{2k}\,Y_{2k-1}}{Y_{2k}\,G_{2k-1}}\right).$$

Landau conditions are expressed in polytope geometry: these determinants are volumes of simplices determined by the external momenta.



Evidence for the graphical conjecture

- all tadpoles and bubbles
- triangles and boxes with several combinations of internal and external masses
- consistency checks for more complicated boxes, 0m pentagon, 0m hexagon

Checked to several orders in ϵ.

Summary

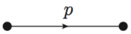
- The Parke-Taylor formula and helicity decompositions led to novel amplitude constructions.
- On-shell methods have been used to compute Feynman integrals and explain simplicity of amplitudes. For loops, this means cut diagrams.
- Cut diagrams are discontinuities of loop amplitudes.
- Hopf algebra coproduct identifies generalized discontinuities!
- Use Hopf algebra to interpret cut diagrams and reconstruct multi-loop integrals.
- The Hopf algebra can also be written diagrammatically.
- Similar relations at the amplitude level?

extra slides

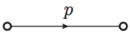
Cutting Rules

Traditional [Veltman]:

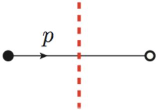
$$\bullet = i \qquad\qquad \circ = -i$$



$$= \frac{i}{p^2 + i\varepsilon}$$



$$= \frac{-i}{p^2 - i\varepsilon}$$



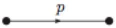
$$= 2\pi\,\delta\left(p^2\right)\theta\left(p_0\right)$$

for massless scalar theory.

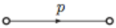
Cutting Rules

Generalized:

$$\bullet = i \qquad\qquad \circ = -i$$



$$= \frac{i}{p^2 + i\varepsilon}$$



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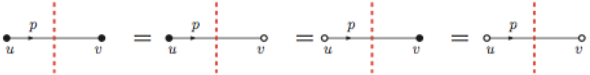


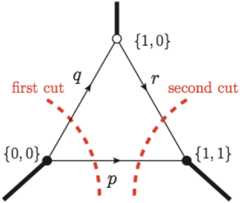
$$= 2\pi\,\delta\left(p^2\right) \prod_{i:c_i(u)\neq c_i(v)} \theta\left([c_i(v) - c_i(u)]p_0\right)$$

Colors are $c_i = 0, 1$ for each cut i: we overlay consistent energy flow across each cut.

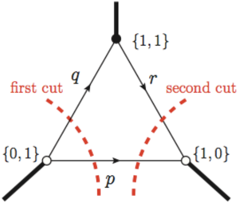
We allow repeated cuts of same propagator or same loop.

Cutting Rules

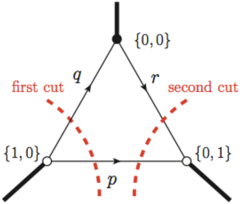
Example:



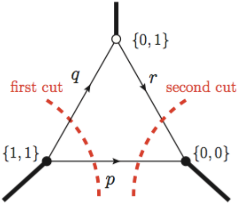
$$\theta(p_0)\theta(q_0)\theta(r_0)$$



$$\theta(p_0)\theta(-p_0) = 0$$



$$\theta(-p_0)\theta(p_0) = 0$$



$$\theta(-p_0)\theta(-q_0)\theta(-r_0)$$

Discontinuities across branch cuts

$$\text{Disc}_x \left[F(x \pm i0)\right] = F(x \pm i0) - F(x \mp i0),$$

Example:

$$\text{Disc}_x \log(x + i0) = 2\pi i\, \theta(-x).$$

Sequential:

$$\text{Disc}_{x_1,\ldots,x_k} F \quad \equiv \quad \text{Disc}_{x_k} \left(\text{Disc}_{x_1,\ldots,x_{k-1}} F\right).$$