Massive operator matrix elements at 3-loop order for deep-inelastic scattering

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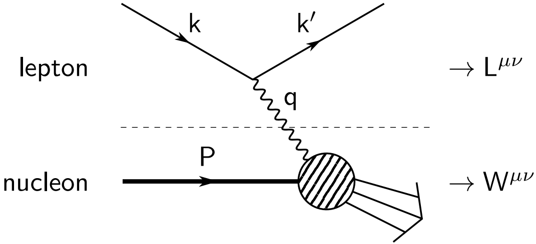
Heavy flavour contributions to deep-inelastic scattering



Kinematic variables: $Q^2 = -q^2$, $x = \frac{Q^2}{2P.q}$

Cross section: $$\frac{d\sigma}{dx dQ^2} \propto L_{\mu\nu}\, W^{\mu\nu}$$

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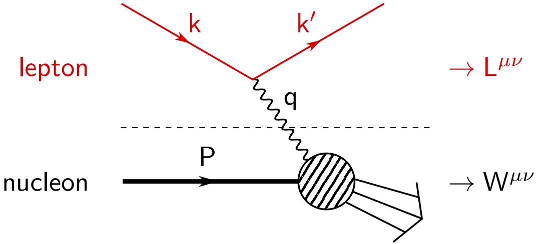

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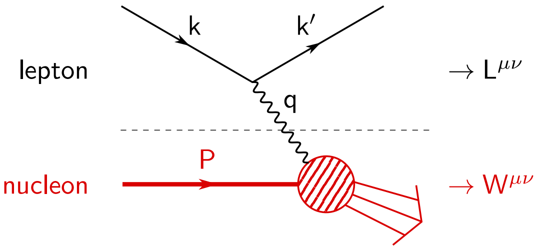

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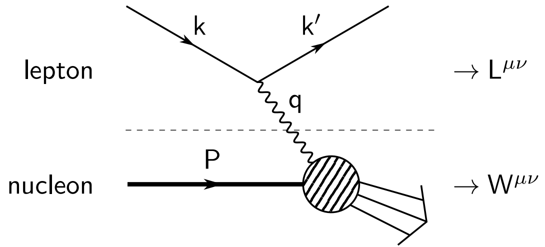

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Structure functions contain light and heavy quark contributions.

Motivation for NNLO heavy flavour corrections

- Precision of DIS world data: $\sim 1\%$ for F_2
  $\rightarrow$ requires $\mathcal{O}(\alpha_s^3)$ description
- Heavy quarks yield essential contributions to structure functions
  $\sim 20 - 30\%$ in the small x region
- Heavy quark contributions to the scaling violations
  have different shape than massless contributions

$\Rightarrow$ NNLO heavy quark contributions are important for precise measurement of the strong coupling constant

$$\delta\alpha_s(M_Z) \approx 1\%$$

and heavy quark masses [Alekhin et al. '12 (and updates)]

$$m_c(m_c) = 1.25 \pm 0.02(\mathrm{exp})^{+0.03}_{-0.02}(\mathrm{scale})^{+0.00}_{-0.07}(\mathrm{thy})\mathrm{GeV}$$

$$m_b(m_b) = 3.91 \pm 0.14(\mathrm{exp})^{+0.00}_{-0.11}(\mathrm{thy})\mathrm{GeV} \quad (\mathrm{preliminary})$$

($\overline{\mathrm{MS}}$ scheme)

Heavy flavour contributions to deep-inelastic scattering

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Wilson coefficients (perturbative)

PDFs (non-perturbative)

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Wilson coefficients (perturbative) — $\mathbb{C}_{2,j}$

PDFs (non-perturbative) — f_j

x- and N-space are connected by a Mellin transformation:

$$M[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

Representation simplifies in Mellin space.

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massless Wilson coefficients ($C_{2,j}$)
NNLO: [Moch, Vermaseren, Vogt '05]

heavy-flavor Wilson coefficients ($H_{2,j}$)

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For F_2 and $Q^2/m^2 \gtrsim 10$ the heavy flavor Wilson coefficients factorise:
[Buza, Matiounine, Smith, Migneron, van Neerven '96]

Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N)\, C_{2,i}(N)$

massive operator matrix elements (OMEs) (A_{ij})

LO: [Witten '76; Babcock, Sievers '78; Shifman, Vainshtein, Zakharov '78; Leveille, Weiler '79; Glück, Reya '79; Glück, Hoffmann, Reya '82]
NLO: [Laenen, van Neerven, Riemersma, Smith '93; Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07a, '07b, '08, '09a]

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Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

OMEs A_{ij} also essential to define the variable flavor number scheme
→ describe transition $N_F \to N_F + 1$ massless quarks
→ transitions relevant for the PDFs at the LHC

Massive operator matrix elements (OME)

Definition of the operator matrix elements

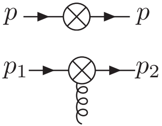
massless, on-shell parton states

$$A_{ij} := \langle j | O_i | j \rangle$$

Local operators from the light-cone expansion

e.g. $O^{\mathrm{NS}}_{q,a;\mu_1,\ldots,\mu_N} = i^{N-1} S[\overline{\Psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \frac{\lambda^a}{2} \Psi]$ − trace terms

Feynman rules for operators



$\propto (\Delta.p)^{N-1}$

$\propto \sum_{j=0}^{N-2} (\Delta.p_1)^j (\Delta.p_2)^{N-2-j}$

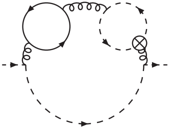
⋮

} Depend on integer variable N (Mellin variable)

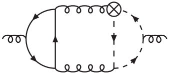
Massive operator matrix elements at NNLO

Fixed moments for OMEs: $N = 2 \ldots 10(14)$ ✓ [Bierenbaum, Blümlein, Klein, '09b]

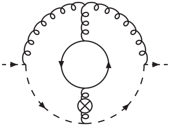
All logarithmic terms from renormalisation ✓ [Behring et al. '14]



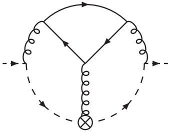
$A_{qq,Q}^{\mathrm{PS}}$
8 diagrams
✓ [Ablinger et al. '10]



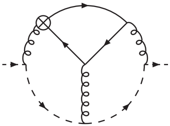
$A_{qg,Q}$
132 diagrams
✓ [Ablinger et al. '10]



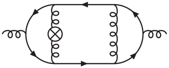
$A_{gq,Q}$
89 diagrams
✓ [Ablinger et al. '14a]



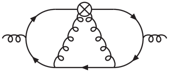
$A_{qq,Q}^{\mathrm{NS}}$ & $A_{qq,Q}^{\mathrm{TR}}$
112 diagrams
✓ [Ablinger et al. '14b]



A_{Qq}^{PS}
125 diagrams
✓ [Ablinger et al. '14c]



$A_{gg,Q}$
642 diagrams
✓

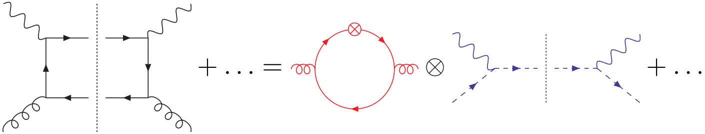


A_{Qg}
1233 diagrams
in progress
(1003 diags. done)

Feynman diagrams drawn using `axodraw` [Vermaseren '94]

Factorisation of Wilson coefficients for $Q^2 \gg m^2$

Factorisation into massive OMEs and massless Wilson coefficients



Example: [Buza, Matiounine, Smith, Migneron, van Neerven '96] [Bierenbaum, Blümlein, Klein, '09b]

$$
\begin{aligned}
H_{q,2}^{\mathrm{PS}}(N_F+1) = a_s^2 \Bigg[& A_{Qq}^{\mathrm{PS},(2)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(2)}(N_F+1)}{N_F+1} \Bigg] \\
+ a_s^3 \Bigg[& A_{Qq}^{\mathrm{PS},(3)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(3)}(N_F+1)}{N_F+1} \\
& + A_{gq,Q}^{(2)}(N_F+1) \frac{C_{g,2}^{(1)}(N_F+1)}{N_F+1} \\
& + A_{Qq}^{\mathrm{PS},(2)}(N_F+1) C_{q,2}^{\mathrm{NS},(1)}(N_F+1) \Bigg]
\end{aligned}
$$

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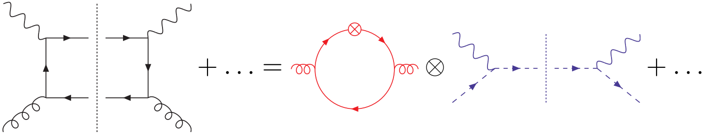
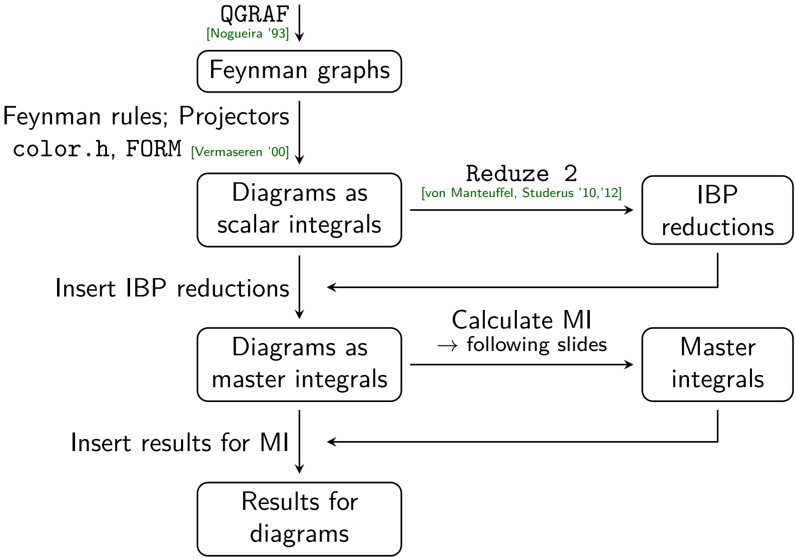


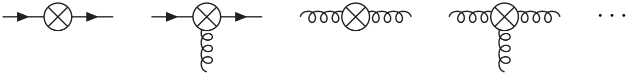
Status of heavy flavour Wilson coefficients at NNLO

$L_{q,2}^{\mathrm{PS}}$	$(\propto A_{qq,Q}^{\mathrm{PS},(3)})$	✓ [Ablinger et al. '10] [Behring et al. '14]
$L_{g,2}^{\mathrm{S}}$	$(\propto A_{qg,Q}^{(3)})$	✓ [Ablinger et al. '10] [Behring et al. '14]
$L_{q,2}^{\mathrm{NS}}$	$(\propto A_{qq,Q}^{\mathrm{NS},(3)})$	✓ [Ablinger et al. '14b]
$H_{q,2}^{\mathrm{PS}}$	$(\propto A_{Qq}^{\mathrm{PS},(3)})$	✓ [Ablinger et al. '14c]
$H_{g,2}^{\mathrm{S}}$	$(\propto A_{Qg}^{(3)})$	in progres

Outline of the calculation



Dealing with operator insertions



- Large number of scalar integrals ($\sim 10^5$) requires using integration-by-parts reductions to master integrals (474)
- Problem: Operators prevent straightforward application of Laporta's algorithm (N in exponents of scalar products)
- Solution: Introduce generating functions for operators

$$\sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t(\Delta.k)}$$

and similar expressions for more complex operators

$\Rightarrow$ treat them as linear propagators

- Allows to use `Reduze 2` to obtain IBP reductions
- Additional advantage: Allows to derive differential equations in t
- Result in N is recovered as Nth coefficient of expansion in t at the end of the calculation

Calculation of master integrals

Master integrals are calculated using a range of techniques:

- Hypergeometric function techniques
- Mellin-Barnes representations

⇒ Yields multi-sum representations

⇒ Simplify using summation algorithms based on $\Sigma\Pi$ fields/rings implemented in `Sigma` [Schneider '01-], `EvaluateMultiSums` and `SumProduction` [Ablinger, Blümlein, Hasselhuhn, Schneider'10-] and special function tools from `HarmonicSums` [Ablinger, Blümlein, Schneider '10,'13]

Moreover, we use

- Coupled systems of differential equations/difference equations [Ablinger et al. '15]
  `SolveCoupledSystem`
- Almkvist-Zeilberger algorithm [Almkvist, Zeilberger '90; Apagodu, Zeilberger '06]
  → `MultiIntegrate` [Ablinger '12]

⇒ Yields scalar recurrences for the integrals

⇒ Solve using the packages listed above

Nested sums and iterated integrals

Results require mathematical objects of increasing complexity:

$A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{gq,Q}$	Harmonic sums [Vermaseren '98] [Blümlein, Kurth '98] $\sum_{i=1}^{N} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}$	HPLs [Remiddi, Vermaseren '99] $\int_0^x \frac{\mathrm{d}y}{y} \int_0^y \frac{\mathrm{d}z}{1-z}$
A_{Qq}^{PS}	Generalised harmonic sums [Moch, Uwer, Weinzierl '01] [Ablinger, Blümlein, Schneider '13] $\sum_{i=1}^{N} \frac{2^{-i}}{i^2} \sum_{j=1}^{i} \frac{2^j}{j}$	(Here:) HPLs at $1-2x$ $\int_0^{1-2x} \frac{\mathrm{d}y}{y} \int_0^y \frac{\mathrm{d}z}{1-z}$
$A_{gg,Q}$, A_{Qg} (so far)	Cyclotomic & binomial sums [Ablinger, Blümlein, Schneider '11] [Ablinger, Blümlein, Raab, Schneider '14] $\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \binom{2j}{j} \frac{(-1)^j}{j^3}}{\binom{2i}{i}(2i+1)}$	Cyclotomic HPLs [Ablinger, Blümlein, Schneider '11] & iterated integrals over root-valued letters [Ablinger, Blümlein, Raab, Schneider '14] $\int_0^x \frac{\mathrm{d}y}{y\sqrt{y+\frac{1}{4}}} \int_0^y \frac{\mathrm{d}z}{z} \int_0^z \frac{\mathrm{d}w}{w}$

Anomalous dimensions

- Renormalisation of the OMEs [Bierenbaum, Blümlein, Klein, '09b] involves the NNLO anomalous dimensions [Moch, Vermaseren, Vogt '04a, '04b]
Example: $(\hat{\gamma}_{ij} = \gamma_{ij}(N_F + 1) - \gamma_{ij}(N_F))$

$$\hat{\hat{A}}_{qq,Q}^{\mathrm{NS},(3)} = \frac{1}{\varepsilon^3} \cdots + \frac{1}{\varepsilon^2} \cdots + \frac{1}{\varepsilon}\left[\frac{\hat{\gamma}_{qq}^{\mathrm{NS},(2)}}{3} - 4a_{qq,Q}^{\mathrm{NS},(2)}\left[\beta_0 + \beta_{0,Q}\right]\right.$$
$$\left. + \beta_{1,Q}^{(1)}\gamma_{qq}^{(0)} + \frac{\gamma_{qq}^{(0)}\beta_0\beta_{0,Q}\zeta_2}{2} - 2\delta m_1^{(0)}\beta_{0,Q}\gamma_{qq}^{(0)} - \delta m_1^{(-1)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)}\right] + \mathcal{O}\left(\varepsilon^0\right)$$

$\Rightarrow$ $\mathcal{O}(N_F)$ contributions to anomalous dimensions

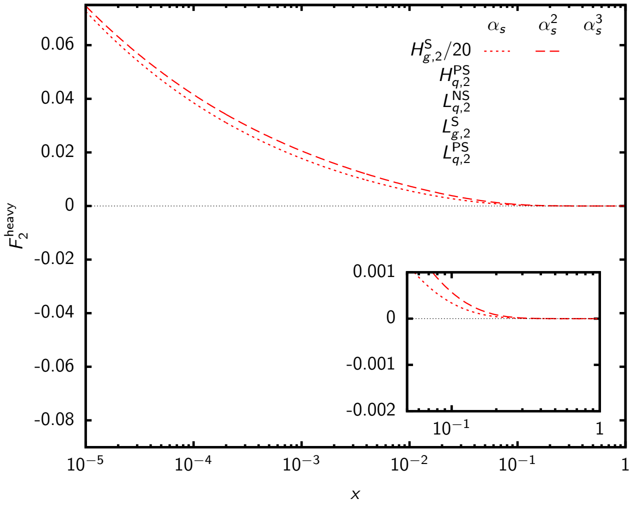
$A_{gq,Q} \rightarrow \gamma_{gq}^{(2)}$ [Ablinger et al. '14a] $A_{gg,Q} \rightarrow \gamma_{gg}^{(2)}$

$A_{qq,Q}^{\mathrm{NS}} \rightarrow \gamma_{qq}^{\mathrm{NS},(2)}$ [Ablinger et al. '14b] $A_{Qg} \rightarrow \gamma_{qg}^{(2)}$

$A_{Qq}^{\mathrm{PS}} \rightarrow \gamma_{qq}^{\mathrm{PS},(2)}$ [Ablinger et al. '14c] complete PS anom. dim.

- First independent calculation in a massive setting

Contributions to the structure function F_2



$H^{S}_{g,2}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ in progress

$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

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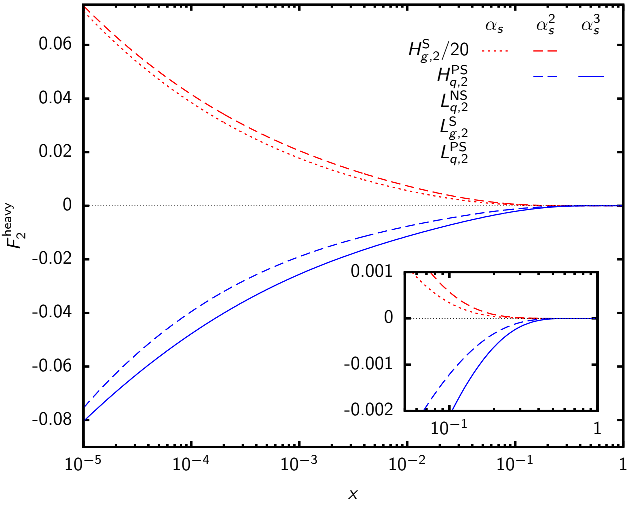

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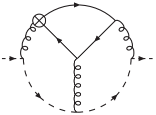
$H_{q,2}^{\mathrm{PS}}$ [Ablinger et al. '14c]



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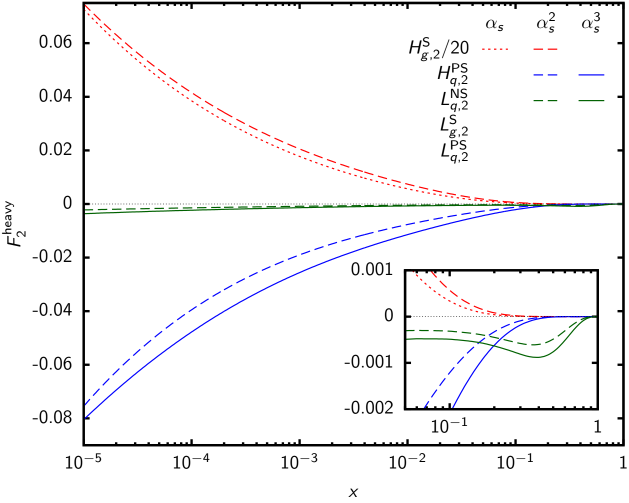

$H_{g,2}^{S}$ not yet known at $\mathcal{O}(\alpha_s^3)$
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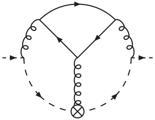
$H_{q,2}^{PS}$ [Ablinger et al. '14c]

$L_{q,2}^{NS}$ [Ablinger et al. '14b]



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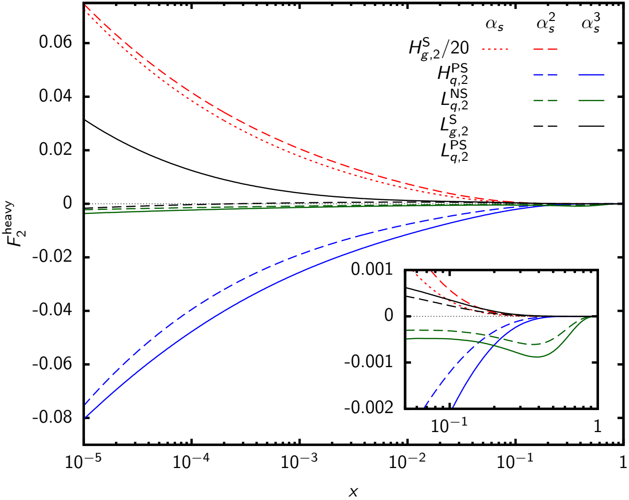

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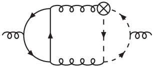
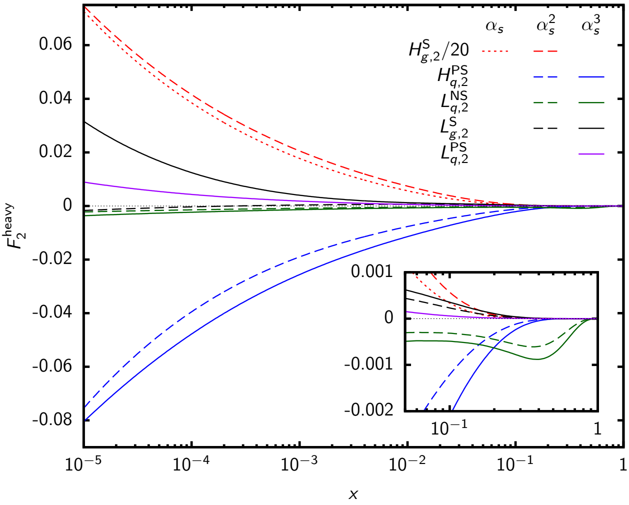
$L_{g,2}^{S}$ [Ablinger et al. '10] [Behring et al. '14]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs

Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Contributions to the structure function F_2



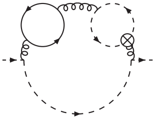
$H_{g,2}^{S}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ in progress

$H_{q,2}^{PS}$ [Ablinger et al. '14c]

$L_{q,2}^{NS}$ [Ablinger et al. '14b]

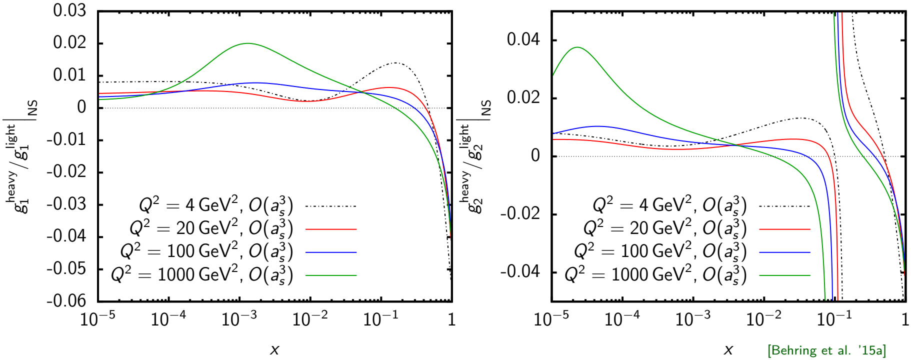
$L_{g,2}^{S}$ [Ablinger et al. '10] [Behring et al. '14]

$L_{q,2}^{PS}$ [Ablinger et al. '10] [Behring et al. '14]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Non-singlet part of polarised structure functions g_1 & g_2

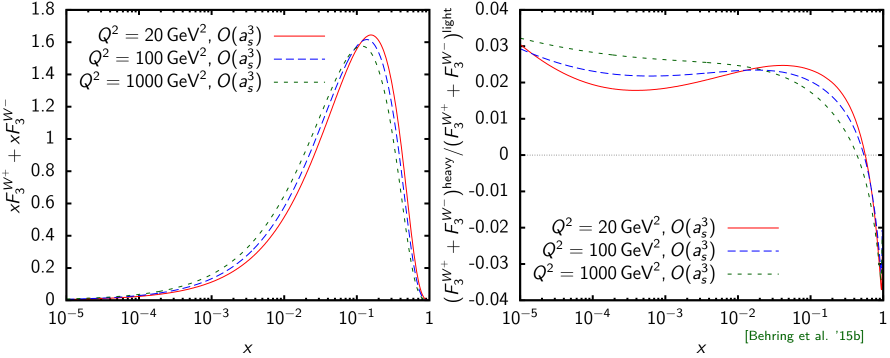


[Behring et al. '15a]

- Odd moments of $A_{qq,Q}^{\text{NS}}$ calculated as well [Ablinger et al. '14b]
- They enter the non-singlet contribution to g_1
- Twist-2 part of g_2 determined via Wandzura-Wilczek relation:

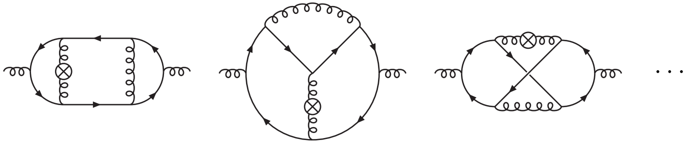
$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{\mathrm{d}y}{y} g_1(y, Q^2)$$

Charged current function xF_3



- Odd moments of $A_{qq,Q}^{\text{NS}}$ enter also $xF_3^{W^+} + xF_3^{W^-}$
- Two non-singlet Wilson coefficients:
  - $L_{q,3}^{\text{NS}}$: W couples to light quarks ($u \to d, \ldots$)
  - $H_{q,3}^{\text{NS}}$: W couples to heavy quark ($s \to c, \ldots$)

Gluonic operator matrix element $A_{gg,Q}$



- Important building block for the VFNS
  $\rightarrow$ enters the matching relation of the gluon PDF
  [Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$G(N_F+1,\mu^2) = A_{gq,Q} \otimes \Sigma(N_F,\mu^2) + A_{gg,Q} \otimes G(N_F,\mu^2)$$

- 642 diagrams $\rightarrow$ 67212 scalar integrals $\rightarrow$ 139 master integrals
- 2 crossed-box diagrams
- MI partly overlap with earlier calculations ($\sim$ 25%)
- Remaining MI calculated mainly via differential/difference equations

$\Rightarrow$ Diagrams are all done

$\Rightarrow$ Unrenormalised OME is known for all even N; vanishes for odd N

Constant term of the gluonic OME $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Bigg\{ & C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)}\left(\frac{128\,(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2})}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22)S_1^2S_2}{3N(N+1)(N+2)} + \cdots\right) + \cdots\Bigg] \\
& + C_A C_F T_F\Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} + \frac{32P_2S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23}S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63}S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \cdots\Bigg] \\
& + C_A^2 T_F\Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} + \frac{256P_5S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32P_{30}S_{-2,1,1}+16P_{35}S_{-3,1}+16P_{44}S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52}S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36}S_2^2}{9(N-1)N^2(N+1)^2} + \cdots\Bigg] \\
& + C_F T_F^2\Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{\binom{2i}{i}i^2}-7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86}S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)} + \cdots\Bigg] + \cdots\Bigg\}
\end{aligned}
$$

Constant term of the gluonic OME $A_{gg,Q}$

$$a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Big\{C_F^2 T_F\Big[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)}$$

$$+\tilde{\gamma}_{gq}^{(0)}\Big(\frac{128\,(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2})}{3N(N+1)\ldots} + \frac{4(5N^2+5N-22)S_1^2 S_2}{\ldots} + \cdots\Big) + \cdots\Big]$$

$$+ C_A C_F T_F\Big[\frac{1\ldots}{3(N-1)N^2(\ldots} \ldots \frac{32P_2S_{-2,2}}{\ldots(N-1)N^2(N+1)^2(N+2)}$$

$$-\frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(\ldots} \ldots \frac{4P_{63}S_4}{\ldots(N-1)N^2(N+1)^2(N+2)} + \cdots\Big]$$

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}$$

Binomial sums

- Two objects involving binomial weights appear
- One of them already occurred in the T_F^2 colour factor [Ablinger et al. '14d]

$$\ldots \cdots\Big] \qquad \ldots N-1)$$

$$+\frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)} + \cdots\Big] + \cdots\Big\}$$

Conclusions

- Heavy quark corrections yield important contributions to DIS
  → essential for precision measurements
  of α_s (1%) and m_c (3%). [Alekhin et al. '12]
- New mathematical and computer-algebraic methods required for analytic calculation of the 3-loop corrections
  → includes new classes of higher transcendental functions and function spaces
- Completed massive OMEs and Wilson coefficients:
  - $A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{qq,Q}^{\mathrm{TR}}$, A_{Qq}^{PS}, $A_{gq,Q}$, $A_{gg,Q}$,
  - $L_{q,2}^{\mathrm{PS}}$, $L_{g,2}^{\mathrm{S}}$, $L_{q,2}^{\mathrm{NS}}$, $H_{q,2}^{\mathrm{PS}}$, L_{q,g_1}^{NS}, $L_{q,3}^{\mathrm{NS}}$
- Calculation of the remaining massive OME A_{Qg} and Wilson coefficient $H_{g,2}^{\mathrm{S}}$ is in progress.

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