FALKO DULAT

# ITERATED INTEGRALS IN MULTI-REGGE KINEMATICS

IN COLLABORATION WITH

VITTORIO DEL DUCA, STEFAN DRUC, JAMES DRUMMOND, CLAUDE DUHR, ROBIN MARZUCCA, GEORGIOS PAPATHANASIOU, BRAM VERBEEK

- Amazing progress in studying / computing multi loop amplitudes
- On the formal side (N=4 SYM):

  6-point at 4-loop, 7-point cluster bootstrap

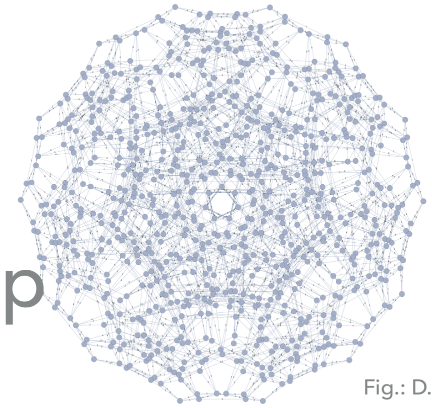
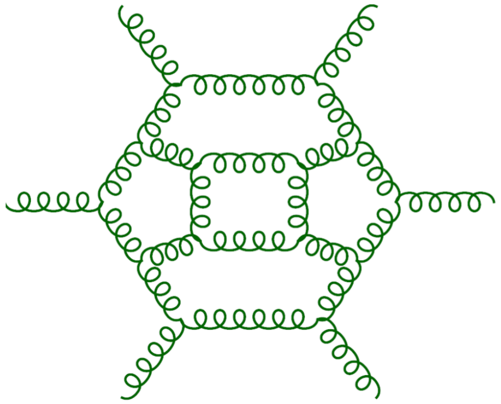


Fig.: D. Parker



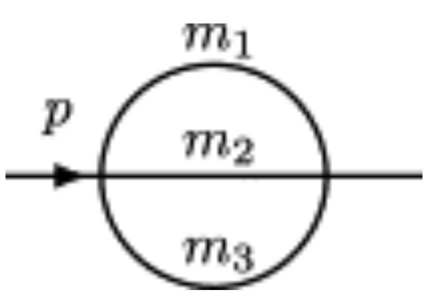
- On real-world side (QCD):

  virtually all 2⟶2 at NNLO processes, gluon fusion at N3LO

**MANY CALCULATIONS BENEFITED FROM IMPROVED UNDERSTANDING OF ITERATED INTEGRALS**

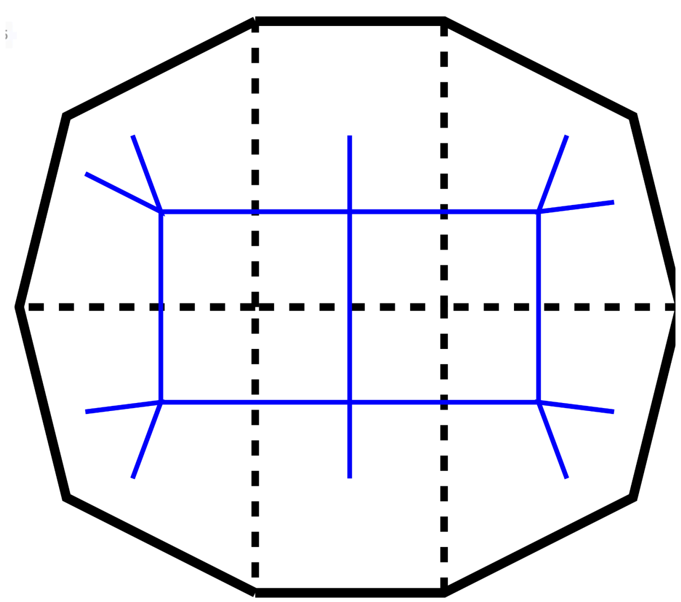
- Many different physical systems can be understood using iterated integrals (e.g. polylogarithms)
- Intense mathematical studies of iterated integrals
- Current knowledge runs out of steam at some point!

- How far can we go with our current technology
- Can we slowly approach the cases that do not work anymore
- Ideal playground: special / restricted kinematics → MRK

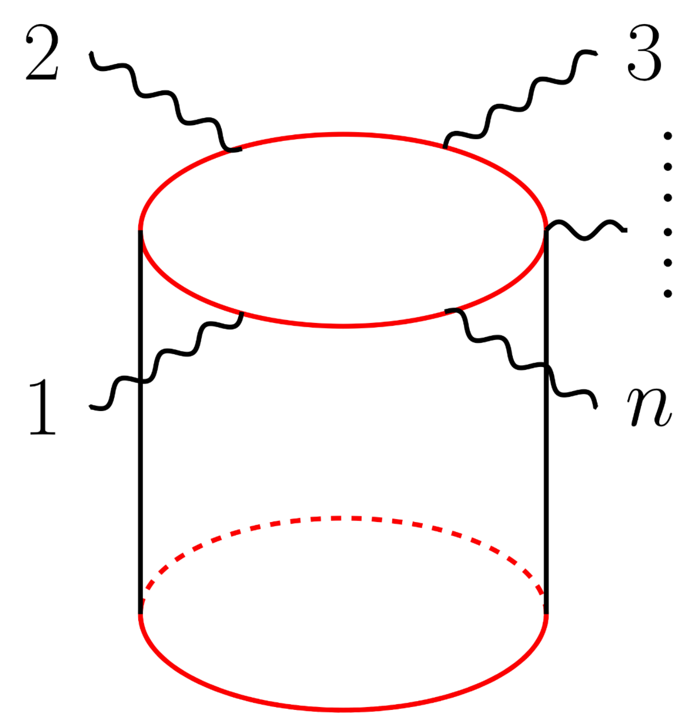



MASSIVE 2-LOOP SUNRISE

ELLIPTIC INTEGRALS

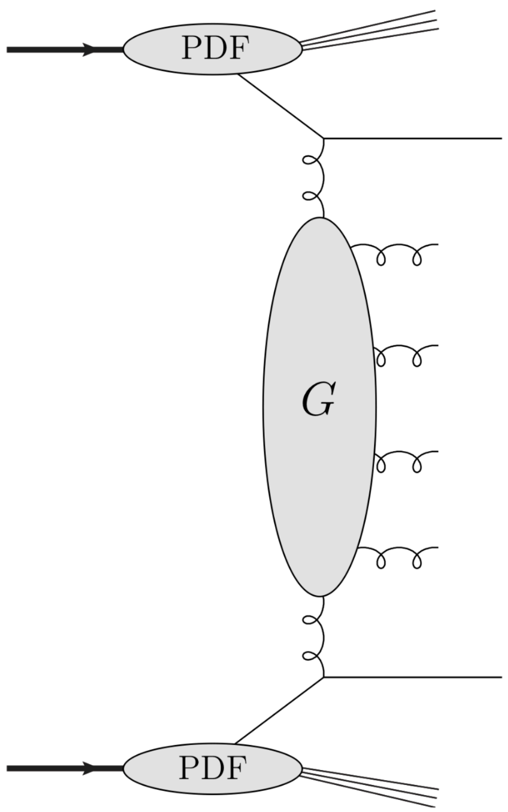


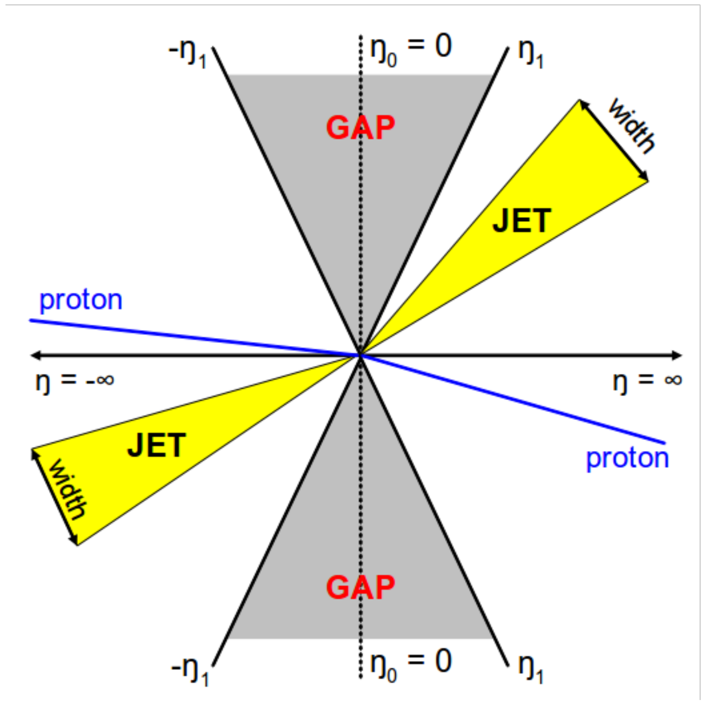
10-POINT 2-LOOP AMPLITUDE IN N=4




1-LOOP AMPLITUDES IN STRING THEORY

- Classical example in QCD: Mueller-Navelet jets




$$y_1 \gg y_2 \quad \text{or} \quad s \gg |t|$$

- Cross section in this limit is described by the BFKL equation

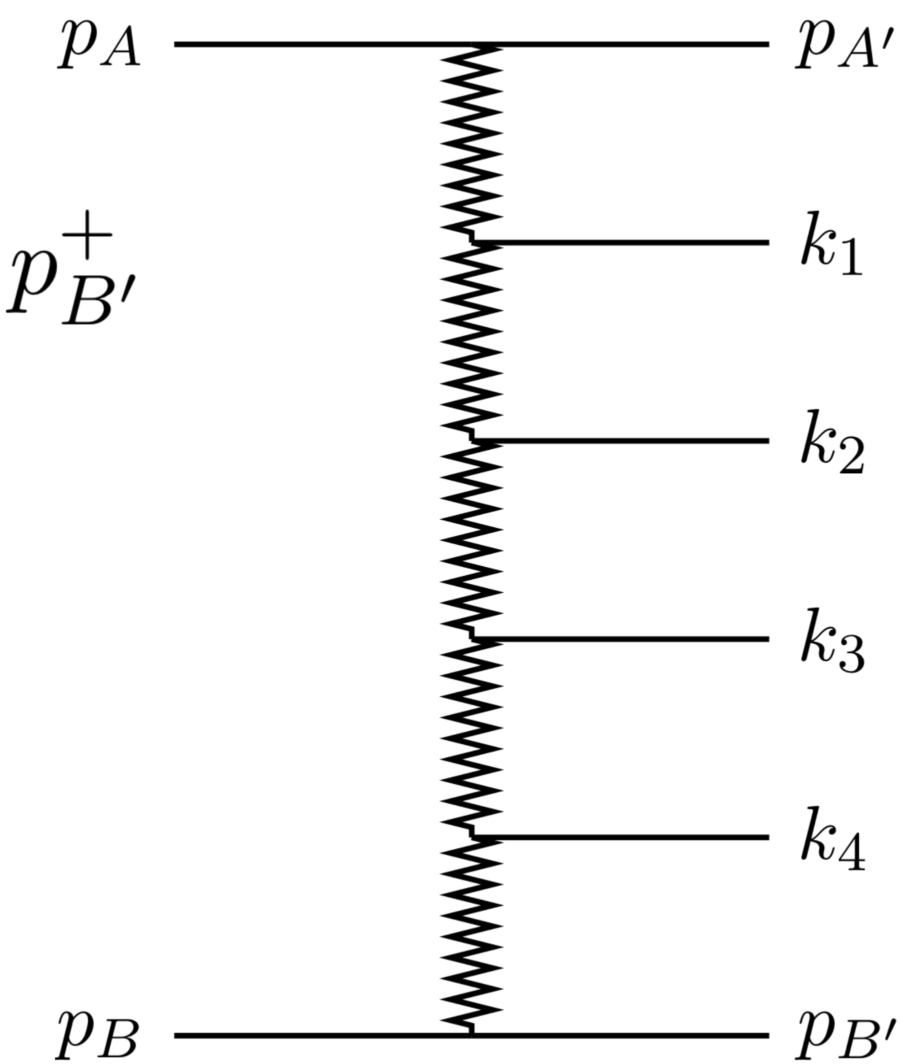
$$\frac{d\hat{\sigma}}{dp_{1\perp}^2\, dp_{2\perp}^2\, d\phi} \propto \sum_{n=-\infty}^{+\infty} e^{in\phi} \int_{-\infty}^{\infty} d\nu \left(\frac{p_{1\perp}^2}{p_{2\perp}^2}\right)^{i\nu} e^{\log(y_1-y_2)\omega(\nu,n,\alpha_s)}$$

- Resums large logarithms of the rapidity gap to any order

- MRK: Generalization of Regge Kinematics, more resolved jets with rapidity gaps between them

$$p^+_{A'} \gg k^+_1 \gg k^+_2 \gg \ldots \gg k^+_{N-4} \gg p^+_{B'}$$

$$|\mathbf{k}^\perp_1| \simeq |\mathbf{k}^\perp_2| \simeq \ldots \simeq |\mathbf{k}^\perp_{N-4}|$$




- Non-trivial kinematics only in 2d transverse subspace
- No collinear divergences, only soft

- To leading logarithmic accuracy at any loop order:

$$\mathcal{R}_{h_1,\dots,h_{N-4}} = 1 + ai\pi(\text{1-loop}) + ai\pi(-1)^{N-5}\left[\prod_{k=1}^{N-5}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}\right]$$

$$\times\left[-1+\prod_{k=1}^{N-5}\tau_k^{aE_{\nu_k,n_k}}\right]\chi^{h_1}(\nu_1,n_1)\left[\prod_{k=1}^{N-6}C^{h_{k+1}}(\nu_k,n_k,\nu_{k+1},n_{k+1})\right]\chi^{-h_{N-4}}(\nu_{N-5},n_{N-5})$$

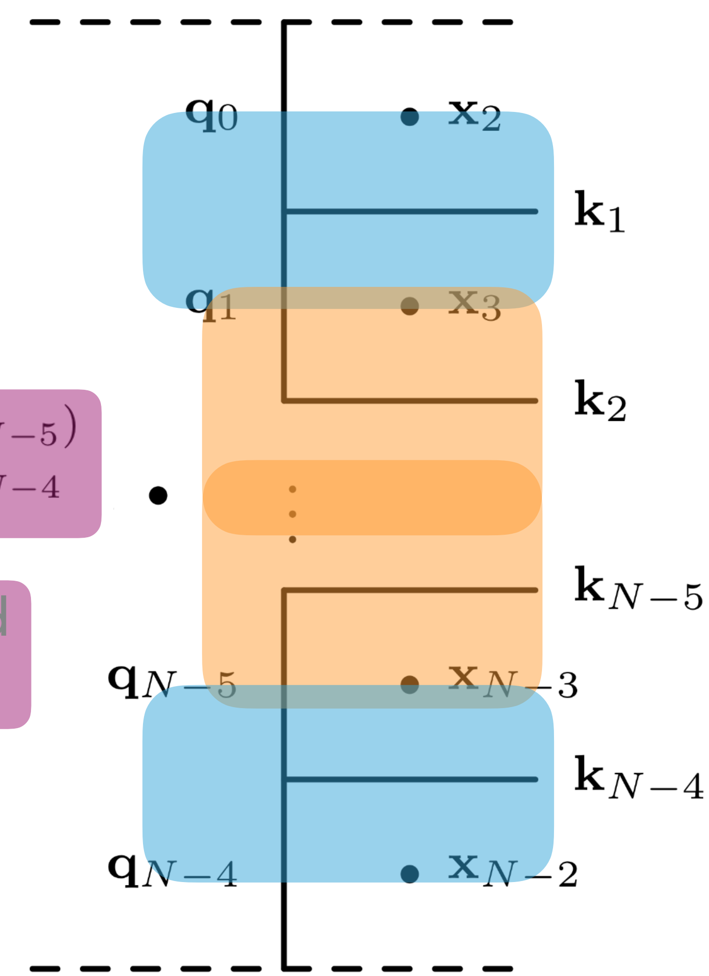
BFKL eigenvalue | Large logarithms | Impact factors | Central emission block

- Perturbatively we have:

$$= 1 + ai\pi(\text{1-loop}) + 2\pi i\sum_{i=2}^{\infty}\sum_{i_1+\cdots+i_{N-5}=i-1} a^i\left(\prod_{k=1}^{N-5}\frac{1}{i_k!}\log^{i_k}\tau_k\right) g^{(i_1,\dots,i_{N-5})}_{h_1,\dots,h_{N-4}}$$

Perturbative coefficients indexed by log power and helicity

| | |
|---|---|
| 2-loop | $\log^1(\tau)\times$ weight 2 |
| 3-loop | $\log^2(\tau)\times$ weight 3 |
| 4-loop | $\log^3(\tau)\times$ weight 4 |



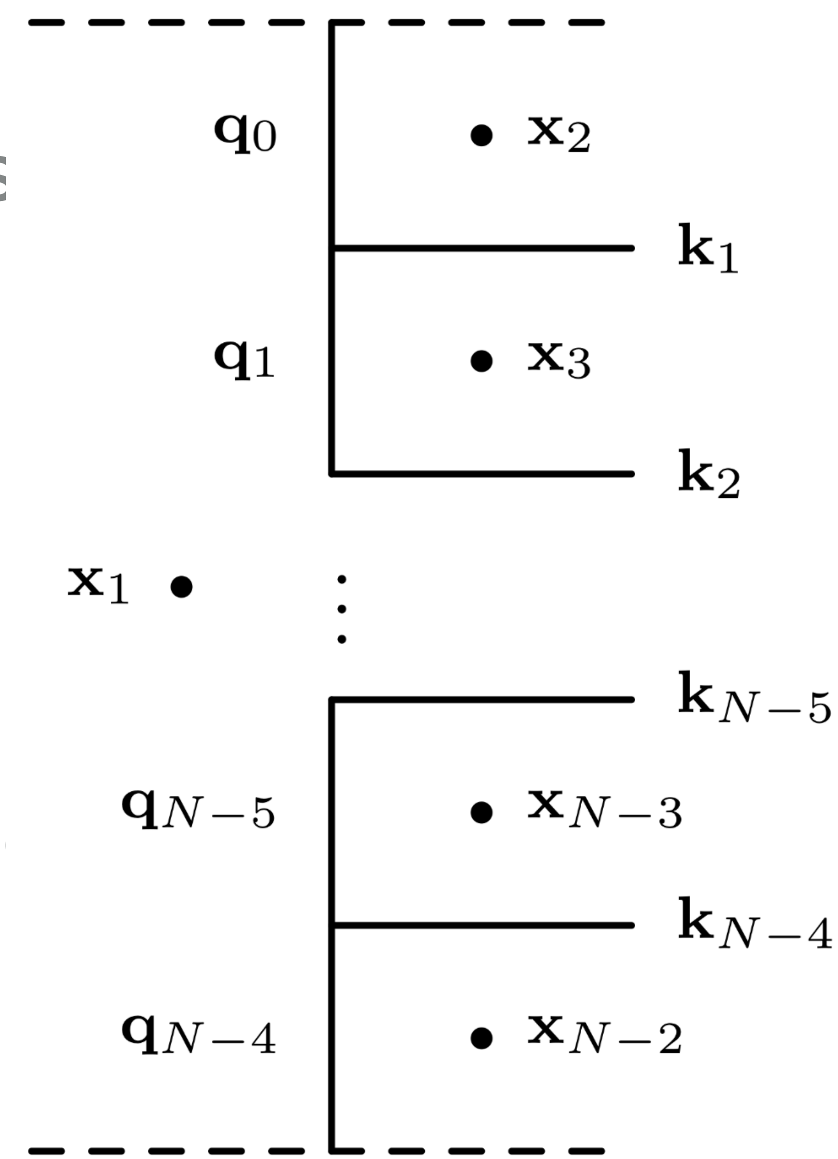
- MRK is defined in 2-dimensional transverse space, can be expressed as complex space using

$$\mathbf{x}_k = x_k^x + i x_k^y$$

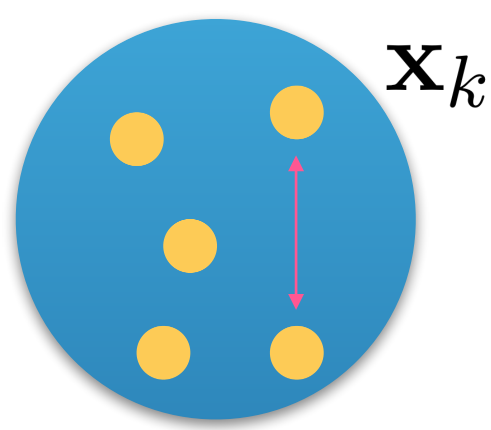
$$\mathbf{k}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}$$

$$\mathbf{q}_i = \mathbf{x}_{i+2} - \mathbf{x}_1$$

- N-2 points in complex space

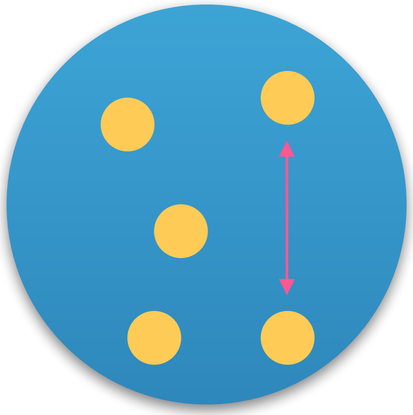



- Particular space: space of configurations of n points on a Riemann sphere




- Soft divergences when two points collid

$$\mathbf{k}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}$$



- This space has been studied extensively by mathematicians
- Natural iterated integrals on this space are a particular set of multiple polylogarithms

$$G(a_1,\ldots,a_n;z)=\int_0^z \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t)$$

$$\mathrm{Li}_n(z)=\int_0^z \frac{dt}{t}\mathrm{Li}_{n-1}(t)$$

$$G(a;z)=\log\left(1-\frac{z}{a}\right) \quad G(0;z)=\log z \quad G(0,1;z)=-\mathrm{Li}_2(z)$$

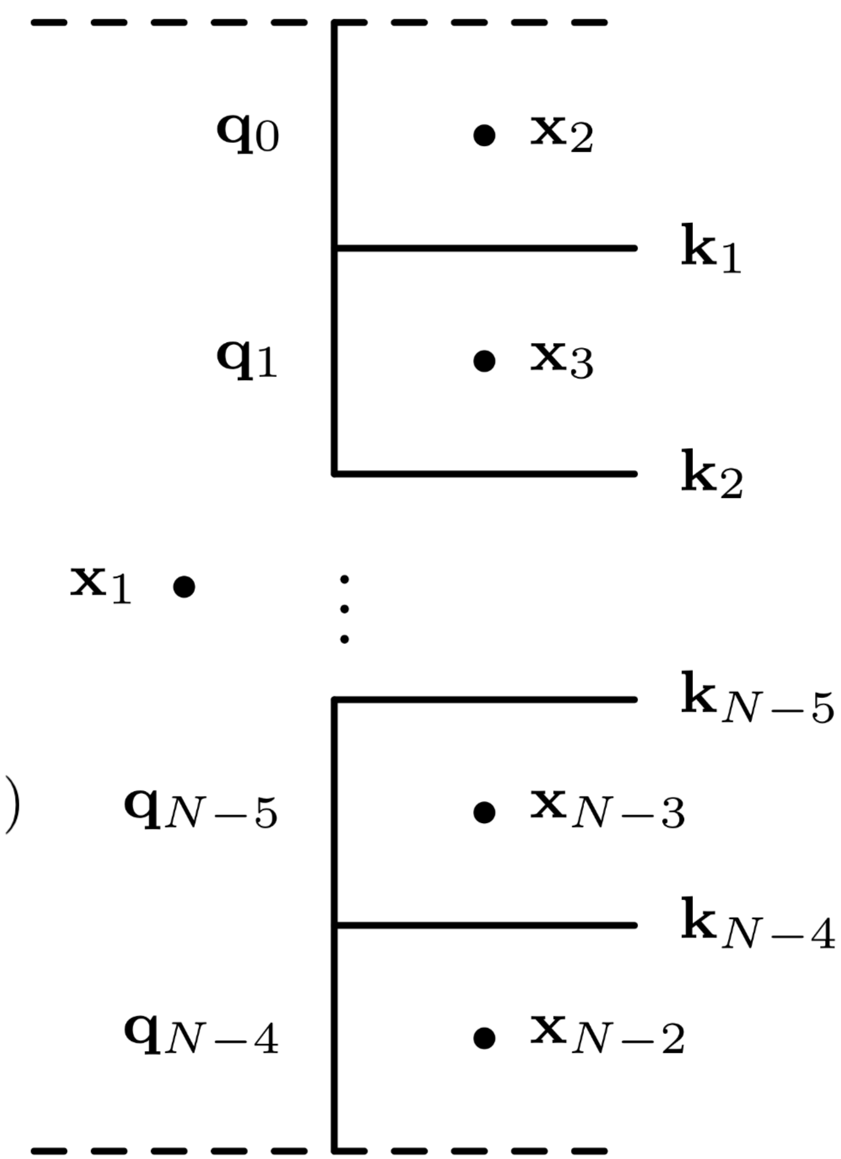
- Highly constrained set of possible integrands

$$\{d\log(t_i),\, d\log(1-t_i),\, d\log(t_i-t_j)\}$$

$$(\mathbf{x}_1,\ldots,\mathbf{x}_n)=(0,1,\infty,t_1,\ldots,t_{n-3})$$

- Simplest function in this space $\log \mathbf{x}_i$
- Physics: Only branch cuts when Mandelstam invariants vanish
- Build linear combinations of our functions so that the branch cuts cancel
  → Single Valued Multiple Polylogs

$$\mathbf{k}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}$$
$$\mathbf{q}_i = \mathbf{x}_{i+2} - \mathbf{x}_1$$

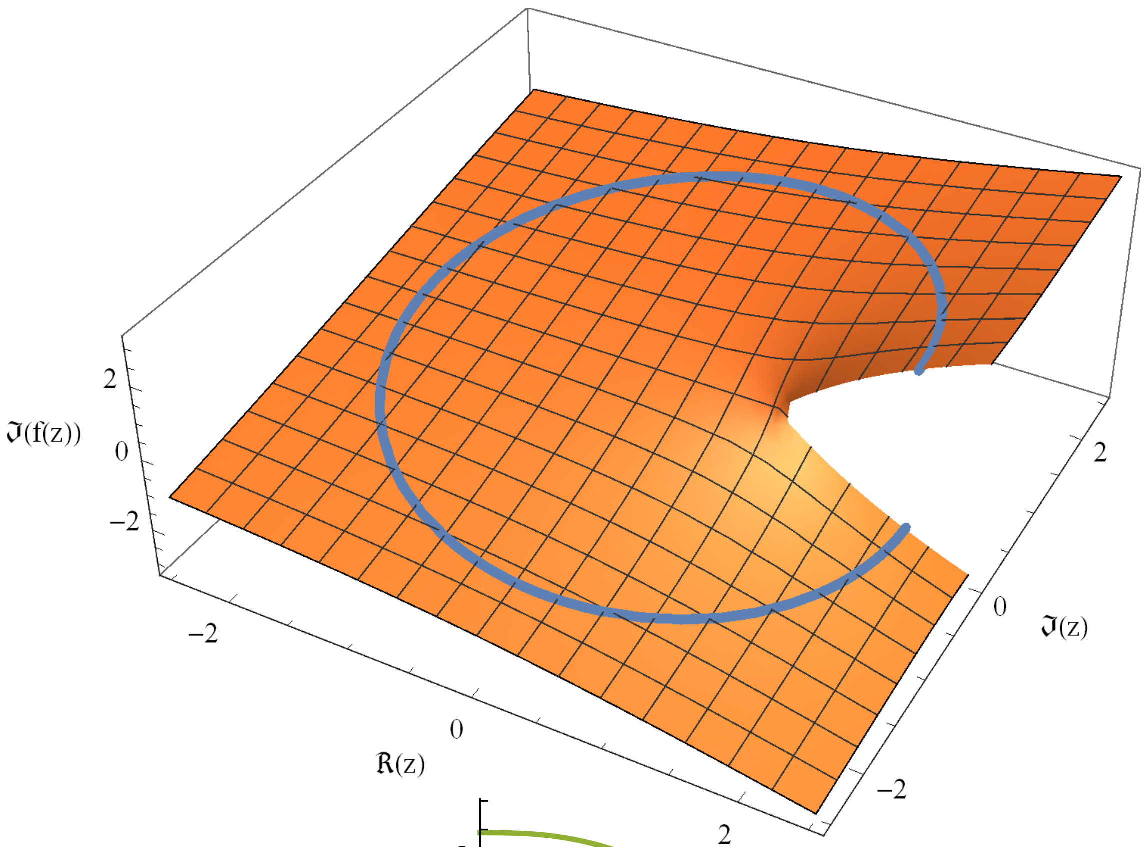


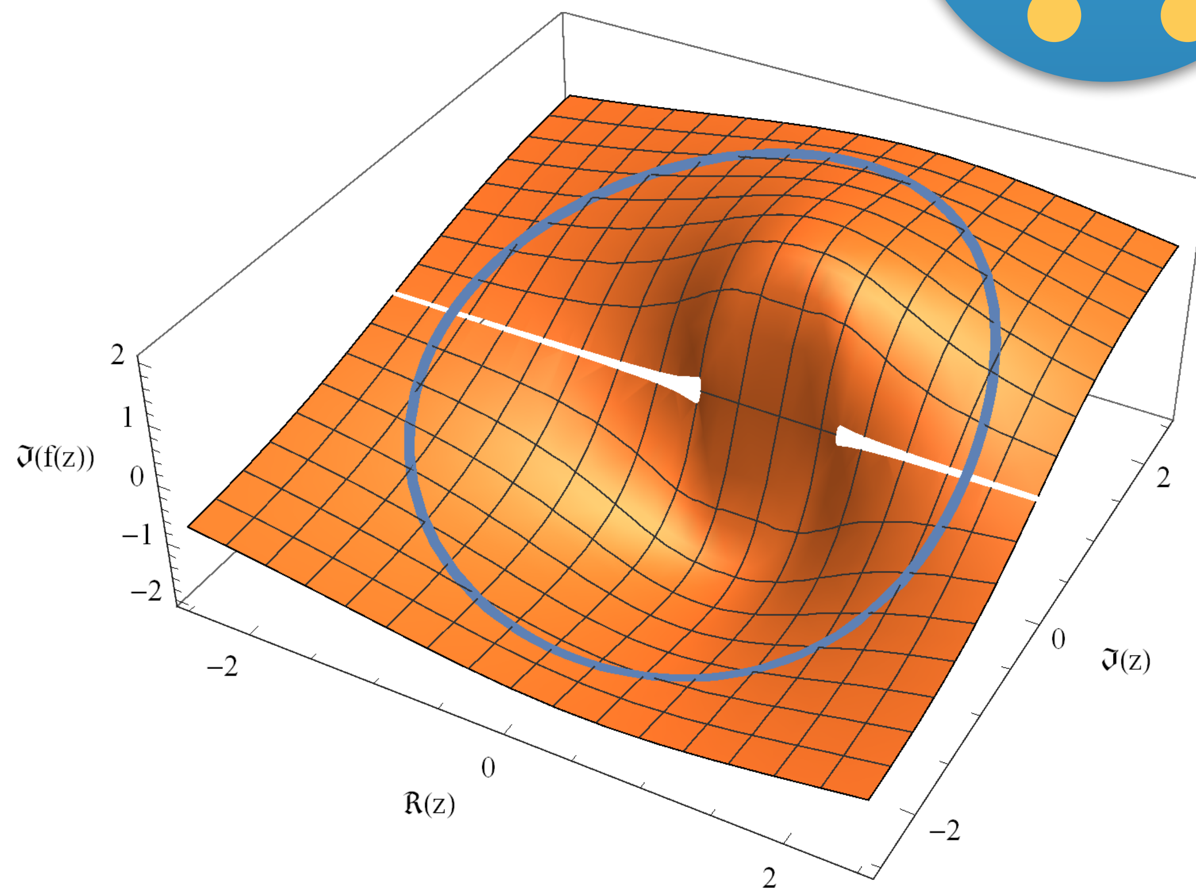

$$\log \mathbf{x}_i \to \log |\mathbf{x}_i|^2 = \log \mathbf{x}_i + \log \bar{\mathbf{x}}_i$$
$$\mathrm{Li}_2(\mathbf{x}_i) \to \mathrm{Li}_2(\mathbf{x}_i) - \mathrm{Li}_2(\bar{\mathbf{x}}_i) - \log(\mathbf{x}_i)\log(1-\bar{\mathbf{x}}_i) - \log(\bar{\mathbf{x}}_i)\log(1-\mathbf{x}_i)$$

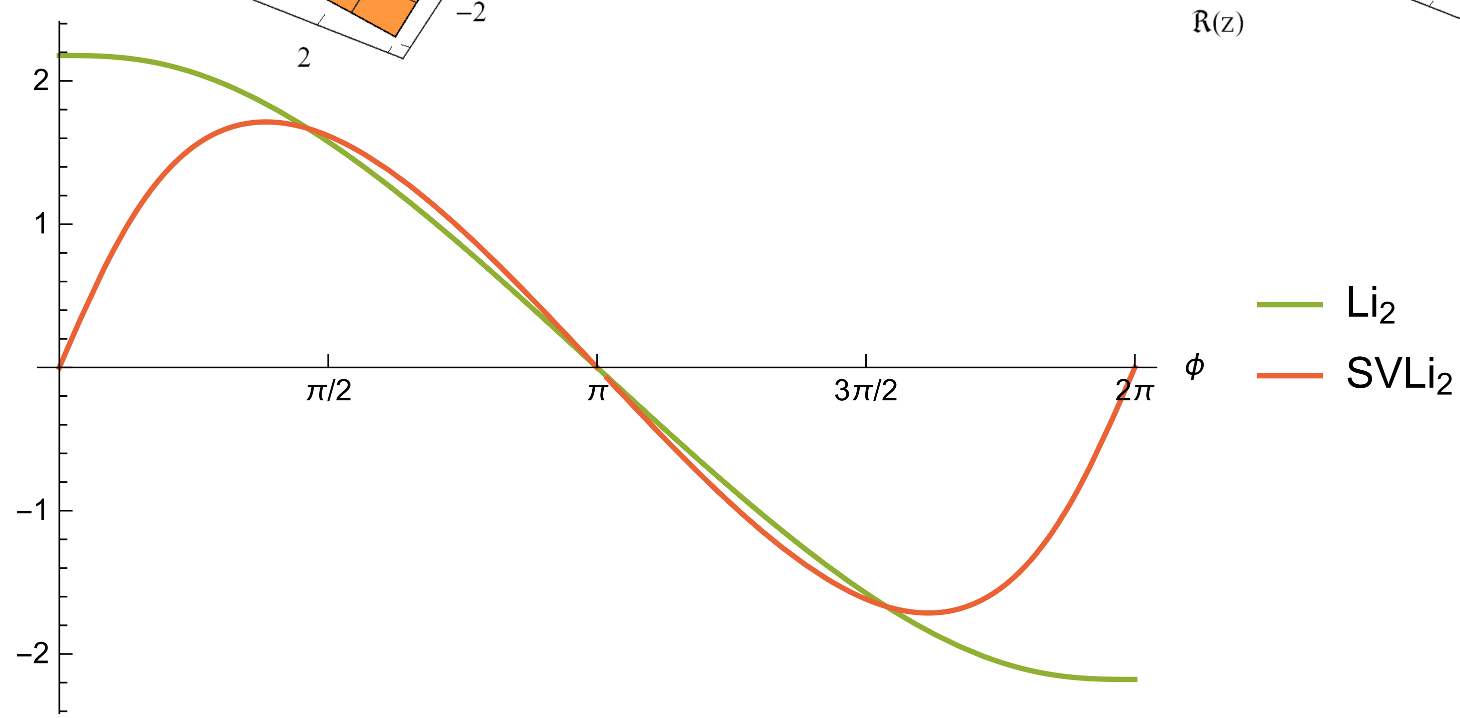
$$\mathbf{x}_k = x_k^x + i x_k^y$$

$$\log \mathbf{x}_i \to \log |\mathbf{x}_i|^2 = \log \mathbf{x}_i + \log \bar{\mathbf{x}}_i$$

$$\mathrm{Li}_2(\mathbf{x}_i) \to \mathrm{Li}_2(\mathbf{x}_i) - \mathrm{Li}_2(\bar{\mathbf{x}}_i) - \log(\mathbf{x}_i)\log(1-\bar{\mathbf{x}}_i) - \log(\bar{\mathbf{x}}_i)\log(1-\mathbf{x}_i)$$

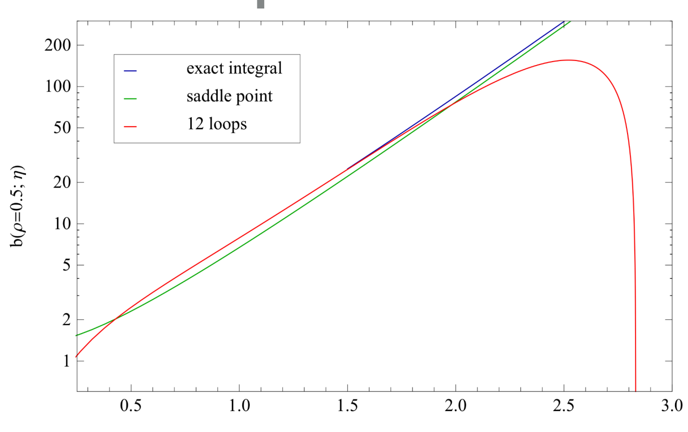


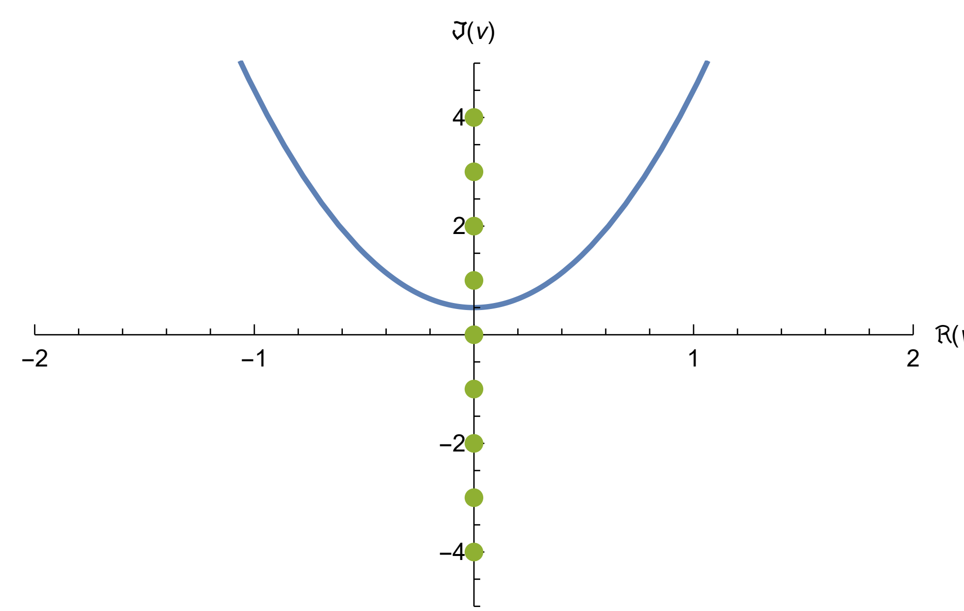



$$g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2}\left[\prod_{k=1}^{N-5}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar z_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}E^{i_k}_{\nu_k n_k}\right]$$
$$\times\,\chi^{h_1}(\nu_1,n_1)\left[\prod_{j=1}^{N-6}C^{h_j}(\nu_j,n_j,\nu_{j+1},n_{j+1})\right]\chi^{-h_{N-5}}(\nu_{N-5},n_{N-5})\,.$$

- Knowledge of function space can be exploited by making an ansatz and fixing the coefficients [Lipatov, Prygarin; Dixon, Drummond, Henn; Dixon, Duhr, Pennington; Pennington; Brödel, Sprenger; Bartels, Kormilitzin, Lipatov, Prygarin; Prygarin, Spradlin, Vergu, Volovich, Bargheer, Schomerus, Papathanasiou; Bargheer]
  - Successfully used at 6-point and for some 7-point amplitudes
  - Interesting QCD result: Dijet cross section in Regge kinematics at 12 loops [Del Duca, Duhr, Dixon, Pennington]




- We can do something more powerful

$$g^{(i_1,\ldots,i_{N-5})}_{h_1,\ldots,h_{N-4}}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2}\left[\prod_{k=1}^{N-5}\sum_{n_k=-\infty}^{+\infty}\left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2}\int_{-\infty}^{+\infty}\frac{d\nu_k}{2\pi}|z_k|^{2i\nu_k}E^{i_k}_{\nu_k n_k}\right]$$
$$\times\,\chi^{h_1}(\nu_1,n_1)\left[\prod_{j=1}^{N-6}C^{h_j}(\nu_j,n_j,\nu_{j+1},n_{j+1})\right]\chi^{-h_{N-5}}(\nu_{N-5},n_{N-5})\,.$$

- Fourier-Mellin integral factorizes convolutions

$$\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{+\infty}\left(\frac{z}{\bar{z}}\right)^{n/2}\int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\,|z|^{2i\nu}\,F(\nu,n)$$

$$\mathcal{F}[F\cdot G] = \mathcal{F}[F]*\mathcal{F}[G] = f*g = \frac{1}{\pi}\int\frac{d^2w}{|w|^2}\,f(w)\;g\left(\frac{z}{w}\right)$$

- What does this mean at e.g. 6 point?

| | | |
|---|---|---|
| 2-loop | $g^{(1)} \propto \mathcal{F}[\chi^+\chi^-]$ | |
| 3-loop | $g^{(2)} \propto \mathcal{F}[\chi^+E\chi^-]$ | $g^{(1)}*\mathcal{E}$ |
| 4-loop | $g^{(3)} \propto \mathcal{F}[\chi^+E^2\chi^-]$ | $g^{(2)}*\mathcal{E} = g^{(1)}*\mathcal{E}*\mathcal{E}$ |

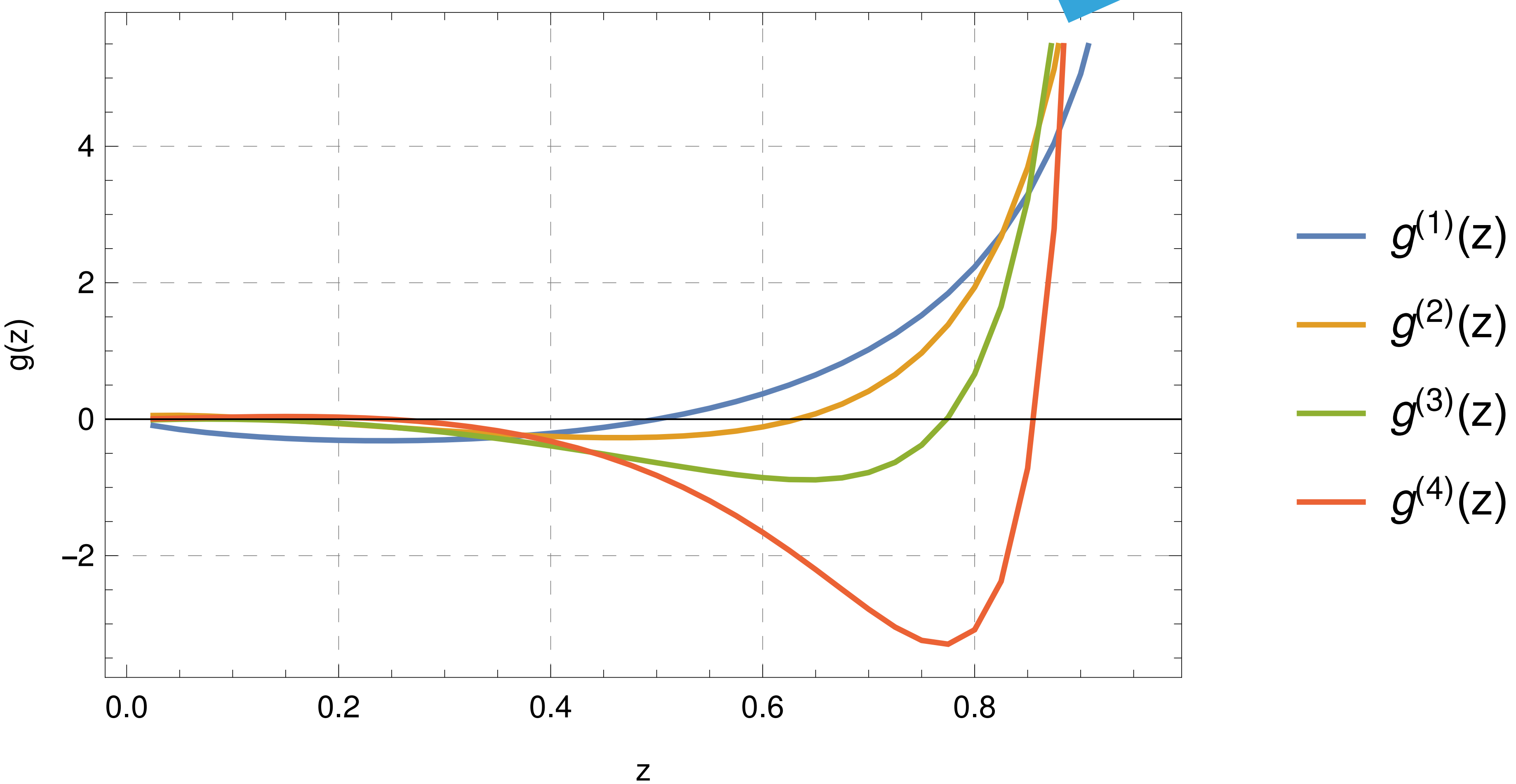
$$\mathcal{E}(z) = \mathcal{F}\left[E_{\nu n}\right] = -\frac{z+\bar{z}}{2|1-z|^2}$$

- In general not possible to compute these convolutions
- But: We know the function space: Single Valued Multiple Polylogarithms → here it is possible! [Schnetz]

$$\mathcal{E}(z) = \mathcal{F}\left[E_{\nu n}\right] = -\frac{z+\bar{z}}{2|1-z|^2}$$

INCREASES LOOP NUMBER BY ONE

- Start from two-loop amplitude and iterate convolutions

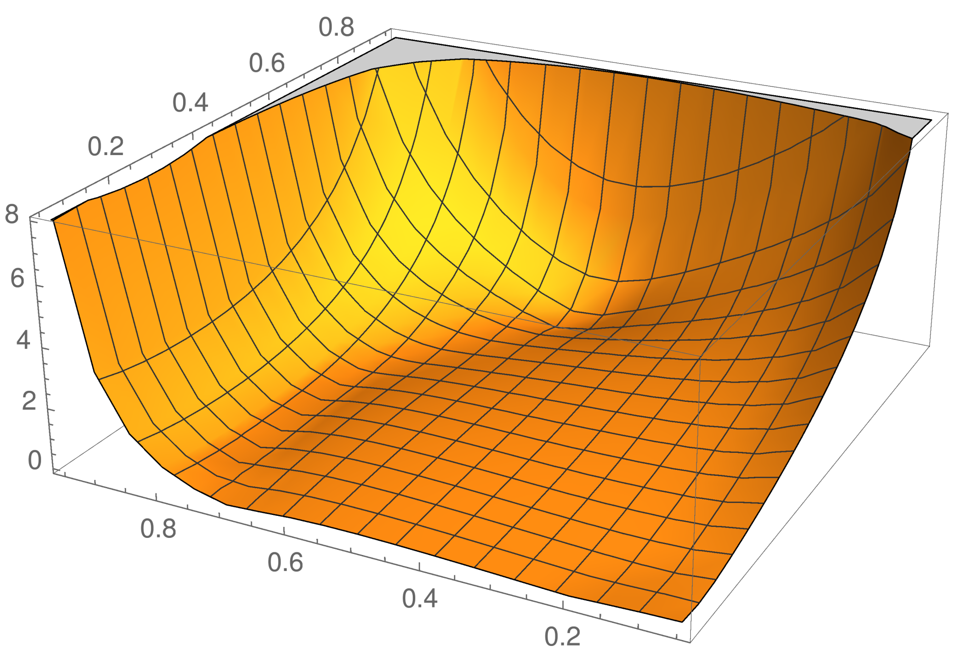


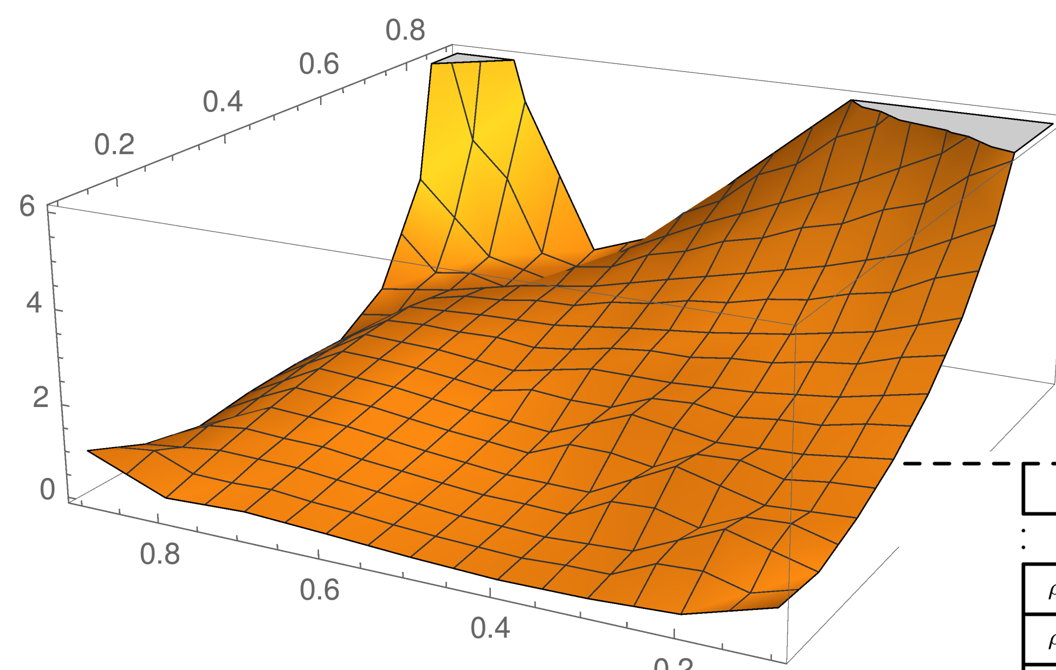
- At higher loops and legs new building blocks appear
- But in a systematic way → factorization structure for any number of loops and legs

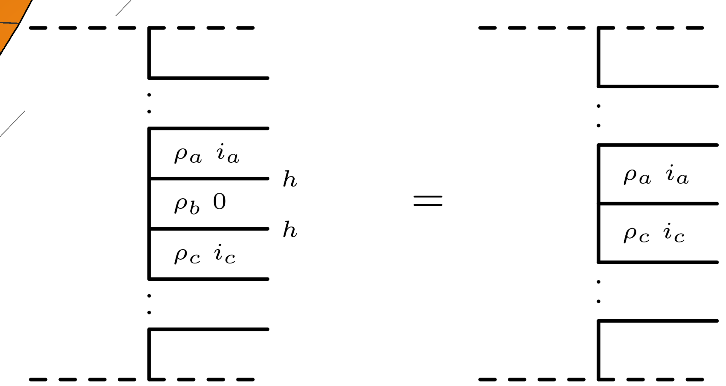
$$\mathcal{R}_6^{(3)}(\rho_1) = g^{(2)}(\rho_1)$$

$$\mathcal{R}_7^{(3)}(\rho_1,\rho_2) = g^{(2)}(\rho_1) + g^{(2)}(\rho_2) + g^{(1,1)}(\rho_1,\rho_2)$$

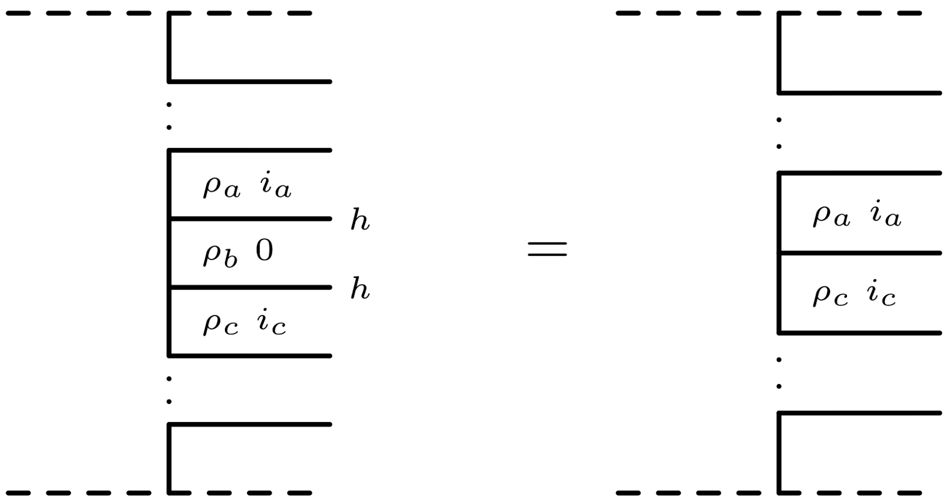
$$\mathcal{R}_8^{(3)}(\rho_1,\rho_2,\rho_3) = g^{(2)}(\rho_1)+g^{(2)}(\rho_2)+g^{(2)}(\rho_3)+g^{(1,1)}(\rho_1,\rho_2)+g^{(1,1)}(\rho_1,\rho_3)+g^{(1,1)}(\rho_2,\rho_3)$$



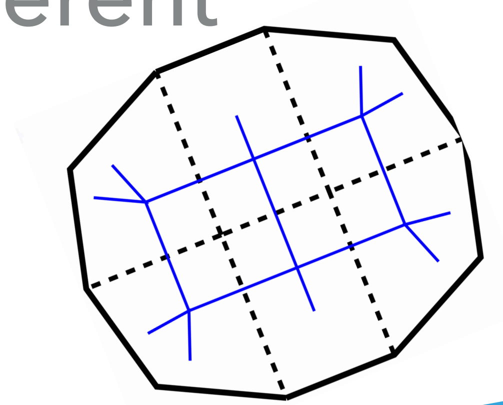



- Results so far: MHV, all outgoing particles have the same helicity




- Non-MHV amplitudes also possible
- Helicity flip kernel: $\mathcal{H}(z) = -\dfrac{z}{(1-z)^2}$
- Correctly produces the rational prefactors
- Factorization holds beyond MHV but infinitely many building blocks required to account for the different helicity structures



$$\mathcal{R}^{(2)}_{-+\ldots} = \log\tau_1\, g^{(1)}_{-+}(\rho_1) + \sum_{j=2}^{N-5} \log\tau_j\, g^{(0,1)}_{-++}(\rho_1,\rho_j)$$

$$\mathcal{R}^{(2)}_{+-+\ldots} = \log\tau_1\, g^{(1,0)}_{+-+}(\rho_1,\rho_2) + \log\tau_2\, g^{(0,1)}_{+-+}(\rho_1,\rho_2) + \sum_{j=3}^{N-5} \log\tau_j\, g^{(0,0,1)}_{+-++}(\rho_1,\rho_2,\rho_j)$$

NOT ELLIPTIC IN MRK

- General formalism for describing amplitudes in MRK at any loop order and for any number of legs
- Successful application of new mathematical results to physics
- Potential to apply modified versions to less symmetric problems like QCD
- Interesting questions beyond leading log: Central emission block, overlap with pentagon OPE
- Single valued multiple polylogarithms useful for other calculations

# BACKUP

- Parametrize transverse space

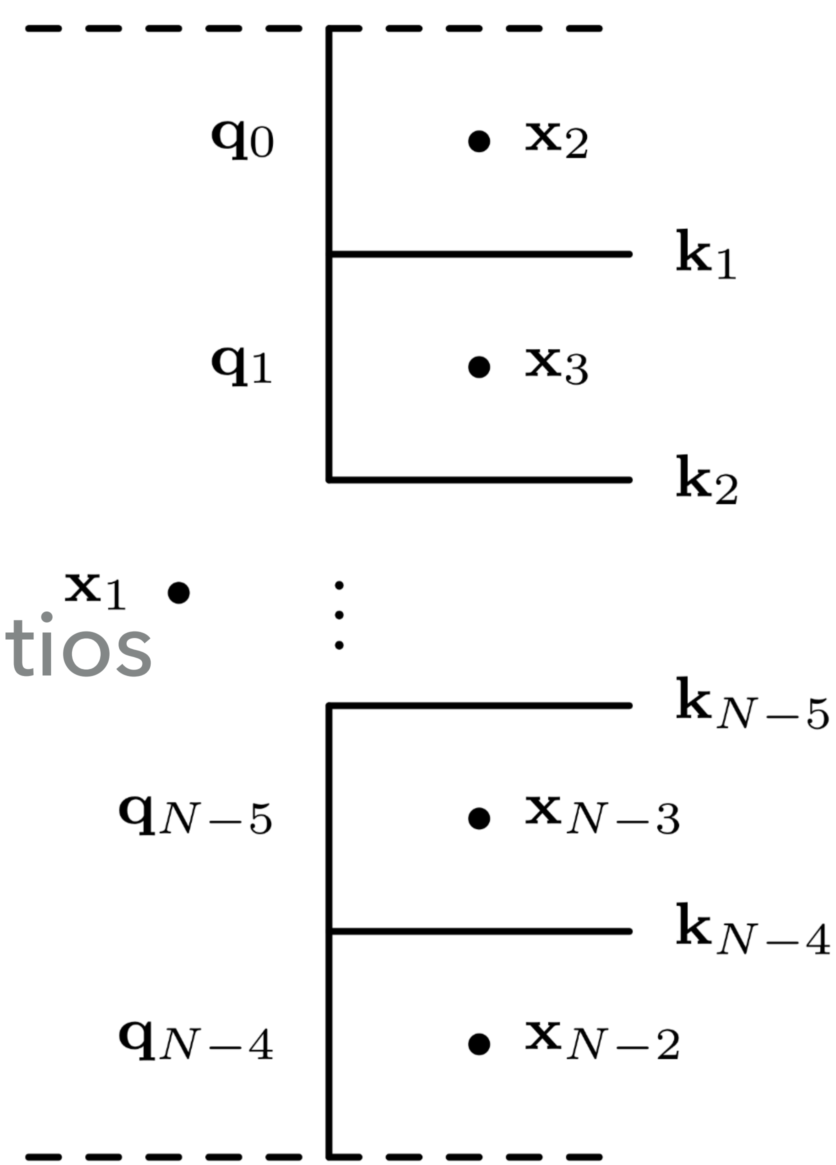
$$\mathbf{k}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}$$

$$\mathbf{q}_i = \mathbf{x}_{i+2} - \mathbf{x}_1$$

- Highly symmetric space
  - Dual conformal symmetry

  Conformal symmetry of the x variables

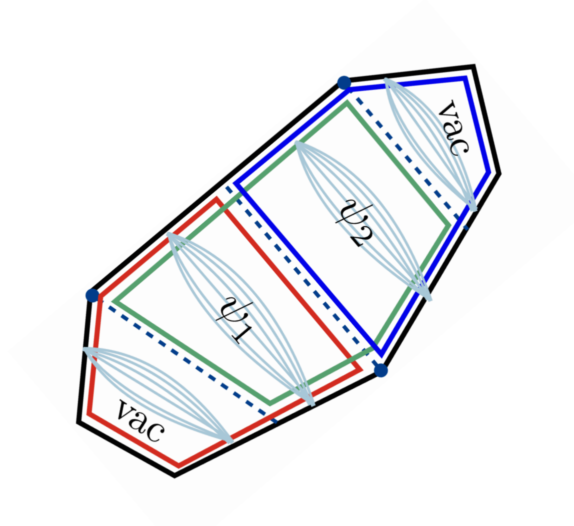
  - Target-projectile symmetry




- Describe amplitudes in terms of cross ratios

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$

- Why study restricted kinematics?
- Simplifications and maybe new structures in restricted kinematics
  - Collinear kinematics in N=4:
    Integrability, Pentagon OPE




  - Multi-Regge Kinematics: Integrable structures, BFKL resummation, N=4 and QCD!
- QCD and N=4 more similar in restricted kinematics than in general

- Building blocks have perturbative expansions and/or all order expressions, for LLA BFKL the leading term in the expansion suffices
- Goal is to perform the Fourier-Mellin integral to determine the remainder function at a given loop order in MRK
- At six point MHV and NMHV amplitudes known to any loop order in terms of single-valued harmonic polylogarithms

  [Lipatov, Prygarin; Dixon, Drummond, Henn; Dixon, Duhr, Pennington; Pennington; Brödel, Sprenger; Bartels, Kormilitzin, Lipatov, Prygarin]
- At two loop MHV factorizes into six point amplitudes

  [Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov]
- **Our result:** Arbitrary loop order for any number of legs

- How to use these functions to solve the BFKL equation?

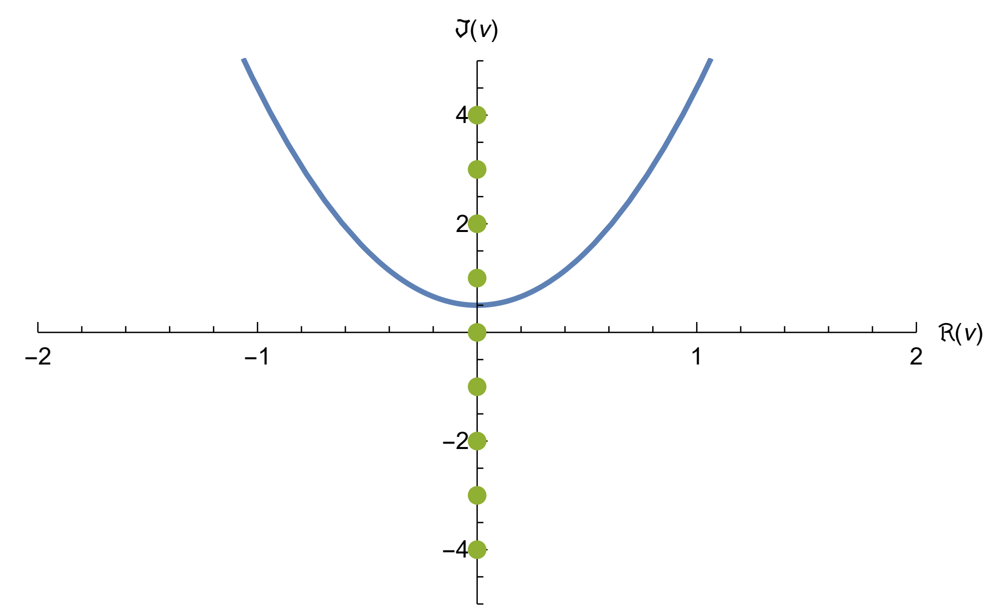
$$= 1 + ai\pi(1\text{-loop}) + 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\cdots+i_{N-5}=i-1} a^i \left( \prod_{k=1}^{N-5} \frac{1}{i_k!} \log^{i_k} \tau_k \right) g_{h_1,\ldots,h_{N-4}}^{(i_1,\ldots,i_{N-5})}$$

$$g_{h_1,\ldots,h_{N-4}}^{(i_1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2} \left[ \prod_{k=1}^{N-5} \sum_{n_k=-\infty}^{+\infty} \left(\frac{z_k}{\bar{z}_k}\right)^{n_k/2} \int_{-\infty}^{+\infty} \frac{d\nu_k}{2\pi} |z_k|^{2i\nu_k} E_{\nu_k n_k}^{i_k} \right]$$

$$\times \chi^{h_1}(\nu_1, n_1) \left[ \prod_{j=1}^{N-6} C^{h_j}(\nu_j, n_j, \nu_{j+1}, n_{j+1}) \right] \chi^{-h_{N-5}}(\nu_{N-5}, n_{N-5}) \,.$$

- Treat it like any Mellin integral

  (Just like e.g. in N3LO Higgs)

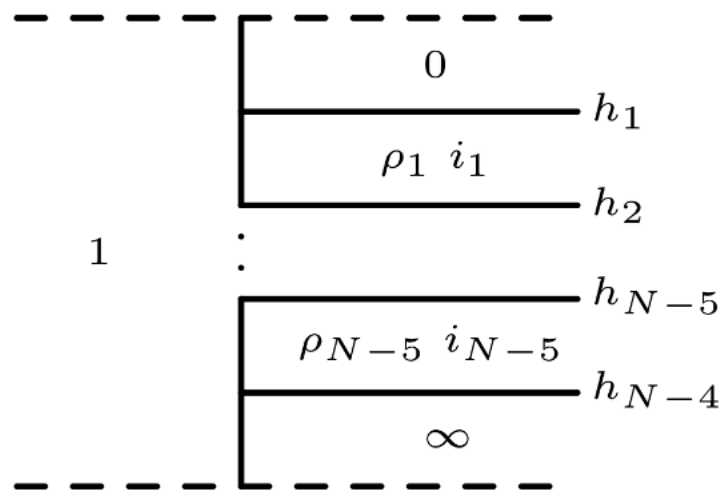



- Take residues and sum

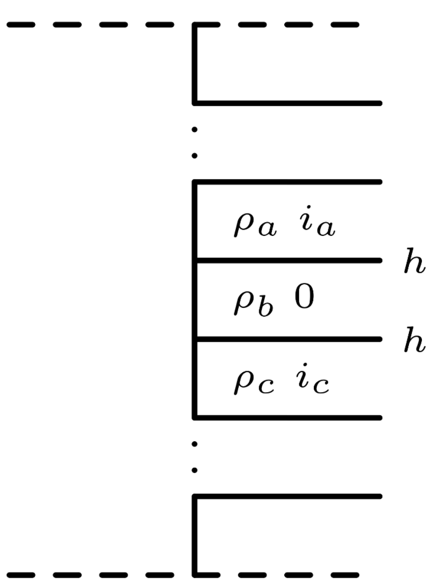
- Tedious, does not use knowledge of the function space, difficult at higher loops but provides a check

- Possibly many functions when going to many legs and loops
- Fortunately, the convolutions imply further factorization

$$g^{(i_1,\ldots,i_{N-5})}_{h_1\ldots h_{N-4}}(\rho_1,\ldots,\rho_{N-5}) = \quad 1$$




$$z_i = \frac{(\rho_i - \rho_{i-1})(\rho_{i+1}-1)}{(\rho_i - \rho_{i+1})(\rho_{i-1}-1)}$$

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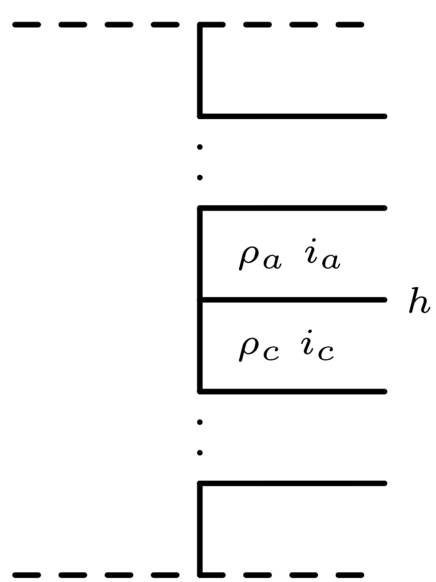


$$\mathcal{R}_7^{(2)}(\rho_1,\rho_2) = g^{(1,0)}(\rho_1,\rho_2) + g^{(0,1)}(\rho_1,\rho_2) = \mathcal{R}_6^{(2)}(\rho_1) + \mathcal{R}_6^{(2)}(\rho_2)$$

$$\mathcal{R}_8^{(2)}(\rho_1,\rho_2,\rho_3) = \mathcal{R}_6^{(2)}(\rho_1) + \mathcal{R}_6^{(2)}(\rho_2) + \mathcal{R}_6^{(2)}(\rho_3)$$

$$\mathcal{R}_N^{(2)}(\{\rho_i\}) = \sum_{n=1}^{N} \mathcal{R}_6^{(2)}(\rho_n)$$

[Prygarin, Spradlin, Vergu, Volovich; Bartels, Prygarin, Lipatov]