Adaptive Integrand Decomposition of multiloop scattering amplitudes

LoopFest XV
University at Buffalo,
North Campus, Amherst, NY

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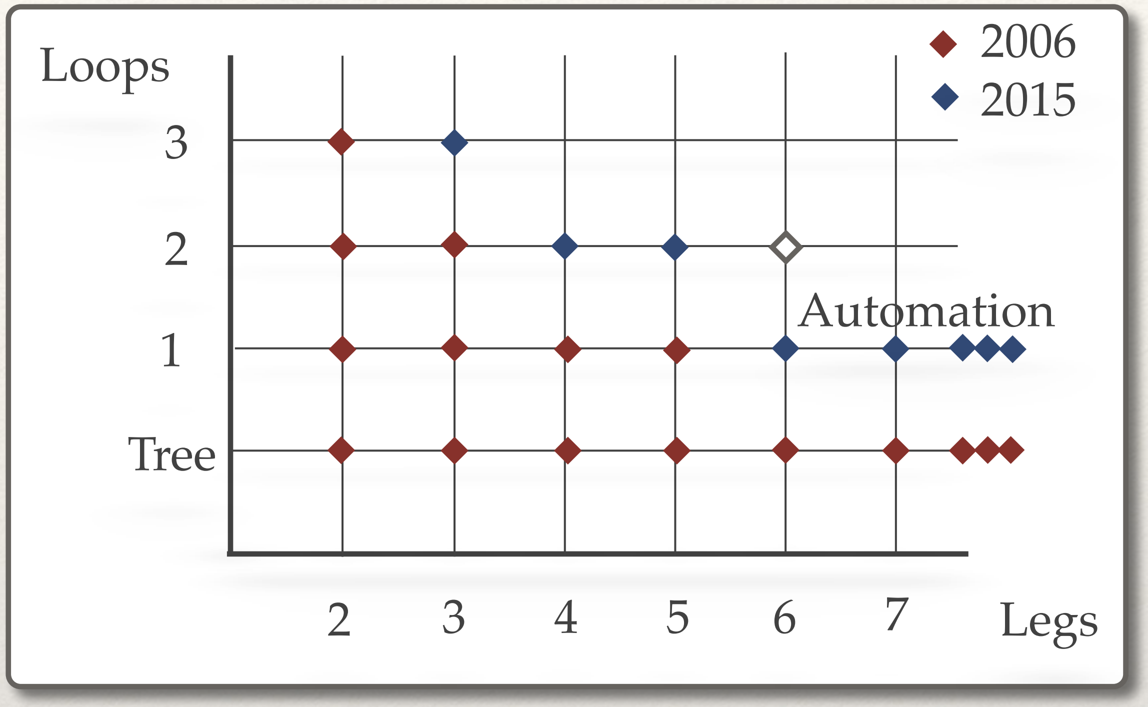
Based on **arXiv:1605.03157**
and on work in collaboration with P. Mastrolia and T. Peraro and W. J. Torres-Bobadilla

Motivation

- The long way towards multi-loop multi-scale processes
- In the last decade **automation** boosted **NLO** calculations
- Computation of virtual amplitudes allowed by **new techniques** :
  - **Generalised unitarity** (see **W. Torres**' talk)
  - **Integrand decomposition** method



Ossola, Papadopoulos, Pittau (07), Ellis, Giele Kunszt (08), Giele, Kunszt, Melnikov (08), Mastrolia Ossola, Papadopoulos, Pittau (08), Pittau, del Aguila (04), Mastrolia, Ossola, Reiter, Tramontano (10), Mastrolia, Mirabella, Peraro (12), …

- **Extension** to **NNLO** and **beyond** has been under intense investigation

Mastrolia, Ossola (11), Badger, Frellesvig, Zhang (12), Zhang (12), Mastrolia, Mirabella, Ossola, Peraro (12), Kleiss Malamos, Papadopoulos, Verheyen (12), Feng, Huang (13), Sogaard, Zhang (13), Feng, Zhen, Huang, Zhou (14), Badger Mogull, Ochirov, O'Connell (16), Badger, Mogull, Peraro (16), …

Outline

- Integrand Decomposition in $d = 4 - 2\epsilon$

  - Feynman integrals in $d = 4 - 2\epsilon$
  - Multivariate Polynomial Division and **Maximum-cut** Theorem

- **Adaptive** Integrand Decomposition in $d = d_{||} + d_{\perp}$

  - Feynman integrals in $d = d_{||} + d_{\perp}$
  - **Transverse space** and spurious directions
  - **Divide** and **Integrate** and **Divide** algorithm
  - 1-Loop decomposition **revisited**
  - **2-Loop** decomposition
  - Examples

- Summary and Conclusions

Integrand decomposition

Ossola, Papadopoulos, Pittau(2007)
Ellis, Giele, Kunszt, Melnikov (08)
Mastrolia,Ossola,
Papadopoulos,Pittau (08)

- **Goal** : decompose Feynman amplitudes in a minimal set of **integrals**

  e.g. Passarino-Veltman **decomposition** of **one-loop amplitudes**

$$\int d^4q \frac{\mathcal{N}(q)}{D_1 \cdots D_n} = \sum_{i \ll l} c_{ijkl} \int d^4q \frac{1}{D_i D_j D_k D_l} + \sum_{i \ll k} c_{ijk} \int d^4q \frac{1}{D_i D_j D_k} + \sum_{i \ll k} c_{ij} \int d^4q \frac{1}{D_i D_j} + \sum_i c_i \int d^4q \frac{1}{D_i}$$

- **Idea** : find a **decomposition** of the **integrand** first

$$\frac{\mathcal{N}(q)}{D_1 \cdots D_n} = \sum_{i \ll l} \widetilde{c}_{ijkl} \frac{\Delta_{ijkl}(q)}{D_i D_j D_k D_l} + \sum_{i \ll k} \widetilde{c}_{ijk} \frac{\Delta_{ijk}(q)}{D_i D_j D_k} + \sum_{i \ll k} \widetilde{c}_{ij} \frac{\Delta_{ij}(q)}{D_i D_j} + \sum_i \widetilde{c}_i \frac{\Delta_i(q)}{D_i}$$

The **residues** $\Delta_{i\cdots k}(q)$ are polynomials in q

- Monomials in $\Delta_{i\dots k}(q)$ which do **not vanish** upon integration, give a representation of the amplitude in terms of a (non-minimal) set of integrals

- If the **parametric expression** of the residue is known, coefficients can be fixed by sampling the numerator on cuts
- Is there a general way to obtain the residues? Does this hold in d dimensions?

Feynman Integrals in $d = 4 - 2\epsilon$

- Arbitrary ℓ-loop integral with n external legs

$$I_n^{d\,(\ell)}[\mathcal{N}] = \int \left(\prod_{i=1}^{\ell} \frac{d^d q_i}{\pi^{d/2}} \right) \frac{\mathcal{N}(q_i)}{\prod_j D_j(q_i)},$$

$$D_j = l_j^2 + m_j^2$$
$$l_j^\alpha = \sum_i \alpha_{ij} q_i^\alpha + \sum_i \beta_{ij} p_i^\alpha,$$

- If external states are in **four dimensions,** split d-dimensional loop momenta as

$$\overbrace{q_i^\alpha}^{d} = \overbrace{q_{[4]\,i}^\alpha}^{4} + \overbrace{\mu_i^\alpha}^{-2\epsilon}$$
$$q_i \cdot q_j = q_{[4]\,i} \cdot q_{[4]\,j} + \mu_{ij}$$

- Parametrise the integral as

$$I_n^{d\,(\ell)}[\mathcal{N}] = \Omega_d^{(l)} \int \prod_{i=1}^{\ell} d^4 q_{[4]\,i} \int \prod_{1 \le i \le j \le \ell} d\mu_{ij} \, [G(\mu_{ij})]^{\frac{d-5-\ell}{2}} \frac{\mathcal{N}(q_{[4]\,i}, \mu_{ij})}{\prod_m D_m(q_{[4]\,i}, \mu_{ij})}$$

Gram determinants

$$G^{(1)}[\mu^2] = \mu^2$$
$$G^{(2)}[\mu_{ij}] = \mu_{11}\mu_{22} - \mu_{12}^2$$

- Introduce a four-dimensional basis $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$

$$q_{[4]\,i}^\alpha = p_{0\,i}^\alpha + x_{1i} e_1^\alpha + x_{2i} e_2^\alpha + x_{3i} e_3^\alpha + x_{4i} e_4^\alpha$$

$$\mathbf{z} = \{x_{1i}, x_{2i}, x_{3i}, x_{4i}, \mu_{ij}\}$$

$$[\mathbf{z}] = \frac{\ell(\ell+9)}{2}$$

Multivariate Polynomial Division

Mastrolia, Ossola (11)
Zhang (12)
Badger, Frellesvig, Zhang (12),
Mastrolia, Mirabella,
Ossola, Peraro (12)

- Given an integrand, consider the **ideal** generated by the set of denominators

$$\boxed{\mathcal{I}_{1\cdots n}(\mathbf{z}) = \frac{\mathcal{N}_{1\cdots n}(\mathbf{z})}{D_1(\mathbf{z})\cdots D_k(\mathbf{z})\cdots D_n(\mathbf{z})}}$$

$$\mathcal{J}_{1\cdots n} \equiv \langle D_1, \cdots, D_n \rangle = \left\{ \sum_{k=1}^{n} h_k(\mathbf{z}) D_k(\mathbf{z}) \; : \; h_k(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- Choose a monomial order and build a **Gröbner basis** $\mathcal{G}_{1\cdots n}(\mathbf{z}) = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$

$$D_1(\mathbf{z}) = \cdots = D_n(\mathbf{z}) = 0 \Longleftrightarrow g_1(\mathbf{z}) = \cdots = g_m(\mathbf{z}) = 0$$

- Perform the multivariate polynomial **division** of $\mathcal{N}_{1\cdots n}(\mathbf{z})$ modulo $\mathcal{G}_{1\cdots n}(\mathbf{z})$

$$\mathcal{N}_{1\cdots n}(\mathbf{z}) = \underbrace{\sum_{k=1}^{m} \Gamma_{1\cdots k-1\, k+1\cdots m}(\mathbf{z}) g_k(\mathbf{z})}_{\text{Quotient}} + \underbrace{\Delta_{1\cdots n}(\mathbf{z})}_{\text{Remainder}} \quad \Rightarrow \quad \mathcal{N}_{1\cdots n}(\mathbf{z}) = \sum_{k=1}^{n} \underbrace{\mathcal{N}_{i\cdots k-1\, k+1\cdots n}(\mathbf{z})}_{\text{Subtopology}} D_k(\mathbf{z}) + \underbrace{\Delta_{1\cdots n}(\mathbf{z})}_{\text{Residue}}$$

- **Iterate** and read off the **decomposition**

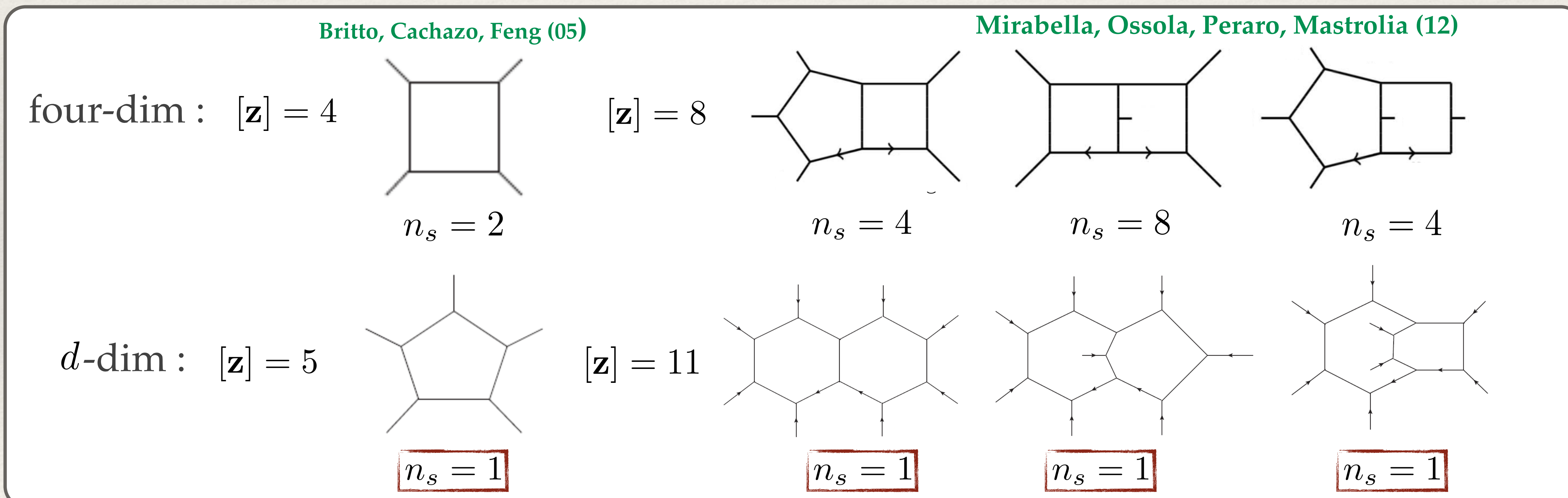
$$\mathcal{I}_{1\cdots n}(\mathbf{z}) = \sum_{k=0}^{n} \sum_{\{i_1\cdots i_k\}} \frac{\Delta_{i_1\cdots i_k}(\mathbf{z})}{D_{i_1}(\mathbf{z})\cdots D_{i_k}(\mathbf{z})} \quad \Rightarrow \quad \int d\mathbf{z}\, \mathcal{I}_{1\cdots n}(\mathbf{z}) = \sum_{k=0}^{n} \sum_{\{i_1\cdots i_k\}} \int d\mathbf{z}\, \frac{\Delta_{i_1\cdots i_k}(\mathbf{z})}{D_{i_1}(\mathbf{z})\cdots D_{i_k}(\mathbf{z})}$$

$$\boxed{\Delta_{i_1\cdots i_k} = \Delta_{i_1\cdots i_k} + \Delta^{\text{spurious}}_{i_1\cdots i_k}}$$

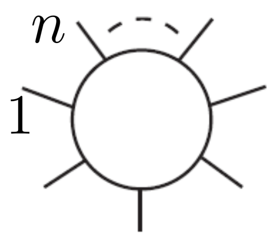
Maximum-cut Theorem

Mirabella, Ossola, Peraro, Mastrolia (12)

- **Maximum-cut theorem**: if the cut-conditions have n_s solutions, the residue is parametrised by n_s coefficients and admits a univariate representation of degree $(n_s - 1)$



Integrand decomposition @1Loop



$$= \int d^d q \frac{\mathcal{N}_{1\cdots n}(\mathbf{z})}{D_1(\mathbf{z})\cdots D_k(\mathbf{z})\cdots D_n(\mathbf{z})}$$

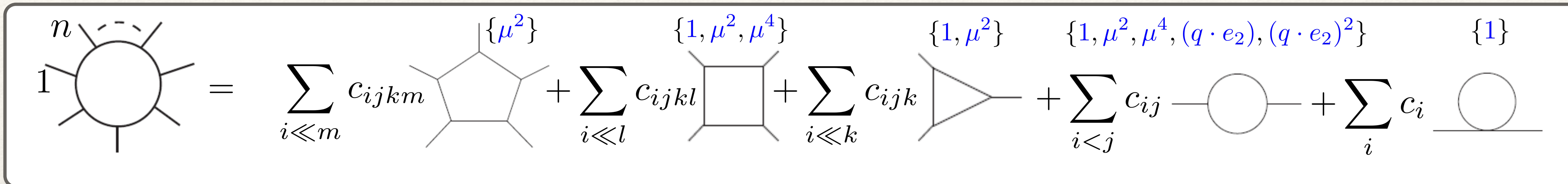
$$\mathbf{z} = \{x_1, x_2, x_3, x_4, \mu^2\}$$
$$\mathcal{N}_{1\cdots n}(\mathbf{z}) = \sum_{\vec{j}\in J_5(n)} \alpha_{\vec{j}} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} z_5^{j_5}$$

- Integrands with $n \geq 6$ are **reducible**. For $n \leq 5$ the **universal** residues are

$$\begin{aligned}
\Delta_{ijklm} &= c_0\mu^2 \\
\Delta_{ijkl} &= c_0 + c_1x_4 + c_2\mu^2 + c_3x_4\mu^2 + c_4\mu^4 \\
\Delta_{ijk} &= c_0 + c_1x_4 + c_2x_4^2 + c_3x_4^3 + c_4x_3 + c_5x_3^2 + c_6x_3^3 + c_7\mu^2 + c_8x_4\mu^2 + c_9x_3\mu^2 \\
\Delta_{ij} &= c_0 + c_1x_1 + c_2x_1^2 + c_3x_4 + c_4x_4^2 + c_5x_3 + c_6x_3^3 + c_7x_1x_4 + c_8x_1x_3 + c_9\mu^2 \\
\Delta i &= c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4
\end{aligned}$$

Ossola, Papadopoulos, Pittau (07)
Ellis, Giele, Kunszt, Melnikov(08),
Mirabella, Ossola, Peraro, Mastrolia (12)

Integrand decomposition @1Loop



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\Delta_{ij} &= c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_4 + c_4 x_4^2 + c_5 x_3 + c_6 x_3^2 + c_7 x_1 x_4 + c_8 x_1 x_3 + c_9\mu^2 \\
\Delta i &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}
$$

Ossola, Papadopoulos, Pittau (07)
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$$\boxed{\Delta_{i_1\cdots i_k} = \Delta_{i_1\cdots i_k} + \Delta^{\text{spurious}}_{i_1\cdots i_k}}$$

- The set of integrals in the decomposition is not minimal due to **integral relations**

$$
\begin{aligned}
I_n^{(1)\,d}[\mu^2] &= -\epsilon I_n^{(1)\,d+2}[1] \\
I_n^{(1)\,d}[\mu^4] &= -\epsilon(1-\epsilon) I_n^{(1)\,d+4}[1]
\end{aligned}
$$

Bern, Morgan (95)

$$I_n^{(1)\,d+2} = \frac{1}{(n-d-1)c_0}\left[I_n^{(1)\,d} - \sum_{i=1}^{n} c_i I_{n-1,i}^{(1)\,d}\right]$$

Tarasov (96), Lee (10)

- Pentagon residue fixed by the maximum-cut theorem. What about lower-point residues?
- Is there any **symmetry**? How to find spurious terms at **higher loops**?

see **M. Jaquier's** talk

Feynman Integrals in $d = d_{\parallel} + d_{\perp}$

- In an arbitrary ℓ-loop integral with $n \le 4$ legs external momenta span a **reduced** space

$$I_n^{d\,(\ell)}[\mathcal{N}] = \int \left(\prod_{i=1}^{\ell} \frac{d^d q_i}{\pi^{d/2}} \right) \frac{\mathcal{N}(q_i)}{\prod_j D_j(q_i)}$$

$$\boxed{d = 4 - 2\epsilon}$$

$$\overbrace{q_i^{\alpha}}^{d} = \overbrace{q_{[4]\,i}^{\alpha}}^{4} + \overbrace{\mu_i^{\alpha}}^{-2\epsilon}$$

$$q_i \cdot q_j = q_{[4]\,i} \cdot q_{[4]\,j} + \mu_{ij}$$

- **Split** space-time in **parallel** $d_{\parallel} = n - 1$ and **orthogonal** $d_{\perp} = 5 - n - 2\epsilon$ space

Collins(84), van Neerven and Vermaseren (84), Kreimer (92)

$$\mathcal{E} = \{e_1, e_2, e_3, e_4\}$$

$$e_i \cdot p_j = 0, \qquad i > d_{\parallel}$$

$$e_i \cdot e_j = \delta_{ij}, \qquad i, j > d_{\parallel}$$

$$\boxed{d = d_{\parallel} + d_{\perp}}$$

$$\overbrace{q_i^{\alpha}}^{d} = \overbrace{q_{\parallel\,i}^{\alpha}}^{d_{\parallel}} + \overbrace{\lambda_i^{\alpha}}^{d_{\perp}}$$

$$q_i \cdot q_j = q_{\parallel\,i} \cdot q_{\parallel\,j} + \lambda_{ij}$$

- The numerator and the denominators depend on **different variables**

$$\boxed{\lambda_i^{\alpha} = \mu_i^{\alpha} + \sum_{j=d_{\parallel}+1}^{4} x_{ji} e_j^{\alpha}}$$

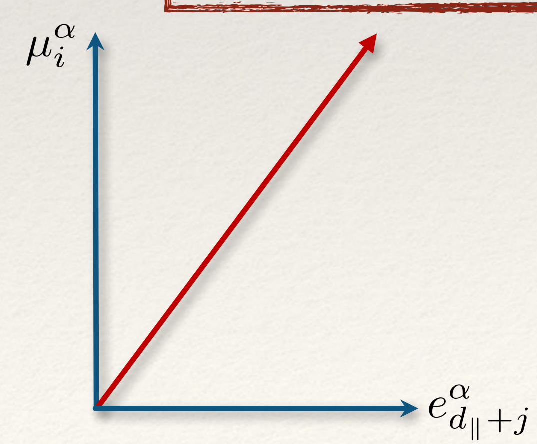
$$\mathcal{N}(q_i) = P[\mathbf{z}]$$

$$D_i(\boldsymbol{\tau}) = l_{\parallel\,i}^2 + \sum_{j,l} \alpha_{ij}\alpha_{il}\,\lambda_{jl} + m_i^2$$

$$[\mathbf{z}] = \frac{\ell(\ell+9)}{2}$$

$$[\boldsymbol{\tau}] = \ell(\ell + 2d_{\parallel} + 1)/2$$

$$\mathbf{z} = \boldsymbol{\tau} \cup \mathbf{x}_{\perp}$$



Feynman Integrals in $d = d_{\parallel} + d_{\perp}$

- Recursively define **orthonormal** basis for the transverse space of **each** loop momentum

$$\boxed{\lambda_i \; : \; \{e_{d_{\parallel}+1}, \ldots, e_4, \hat{\mu}_i\}} \quad \underset{\text{Gram-Schmidt}}{\Longrightarrow} \quad \boxed{\begin{array}{ccc} \lambda_1 & : & \{e_{d_{\parallel}+1}, \ldots, e_4, \hat{\mu}_i\} \\ \lambda_2 & : & \{e'_{d_{\parallel}+1}, \ldots, e'_4, \hat{\mu}'_i\} \\ \lambda_3 & : & \{e''_{d_{\parallel}+1}, \ldots, e''_4, \hat{\mu}''_i\} \\ \ldots & & \ldots \end{array}}$$

- Any ℓ-loop integral with $n \leq 4$ can be parametrised as

Mastrolia, Peraro, A.P. (16)

$$I_n^{d\,(\ell)}[\mathcal{N}] = \Omega_d^{(\ell)} \int \prod_{i=1}^{\ell} d^{n-1} q_{\parallel\, i} \int d^{\frac{\ell(\ell+1)}{2}} \mathbf{\Lambda} \int d^{(4-d_{\parallel})\ell} \mathbf{\Theta}_{\perp} \frac{\mathcal{N}(q_{i\,\parallel}, \mathbf{\Lambda}, \mathbf{\Theta}_{\perp})}{\prod_j D_j(q_{\parallel\, i}, \mathbf{\Lambda})}$$

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Mastrolia, Peraro, A.P. (16)

$$I_n^{d\,(\ell)}[\mathcal{N}] = \Omega_d^{(\ell)} \int \overbrace{\prod_{i=1}^{\ell} d^{n-1} q_{\parallel\, i}}^{d_{\parallel}\text{-space}} \overbrace{\int d^{\frac{\ell(\ell+1)}{2}} \boldsymbol{\Lambda} \int d^{(4-d_{\parallel})\ell} \boldsymbol{\Theta}_{\perp}}^{d_{\perp}\text{-space}} \frac{\mathcal{N}(q_{i\,\parallel}, \boldsymbol{\Lambda}, \boldsymbol{\Theta}_{\perp})}{\prod_j D_j(q_{\parallel\, i}, \boldsymbol{\Lambda})}$$

- Transverse space parametrised in terms of **radial variables** and **transverse angles**

$$\int d^{\frac{\ell(\ell+1)}{2}} \boldsymbol{\Lambda} = \int_0^{\infty} \prod_{i=1}^{\ell} d\lambda_{ii} (\lambda_{ii})^{\frac{d_{\perp}-2}{2}} \int_{-1}^{1} \prod_{1 \leq i < j \leq \ell} d\cos\theta_{ij} (\sin\theta_{ij})^{d_{\perp}-2-i}$$

$$\int d^{(4-d_{\parallel})\ell} \boldsymbol{\Theta}_{\perp} = \int_{-1}^{1} \prod_{i=1}^{4-d_{\parallel}} \prod_{j=1}^{\ell} d\cos\theta_{i+j-1\; j} (\sin\theta_{i+j-1\; j})^{d_{\perp}-i-j-1}$$

$$\boxed{\begin{array}{l} \lambda_{ij} \to P\left[\lambda_{kk}, \sin[\boldsymbol{\Theta}_{\Lambda}], \cos[\boldsymbol{\Theta}_{\Lambda}]\right] \\ x_{d_{\parallel}+j\; i} \to P\left[\lambda_{kk}, \sin[\boldsymbol{\Theta}_{\perp,\Lambda}], \cos[\boldsymbol{\Theta}_{\perp,\Lambda}]\right] \end{array}}$$

- All $\boldsymbol{\Theta}_{\perp}$ integrals reduced to orthogonality relations for **Gegenbauer polynomials**

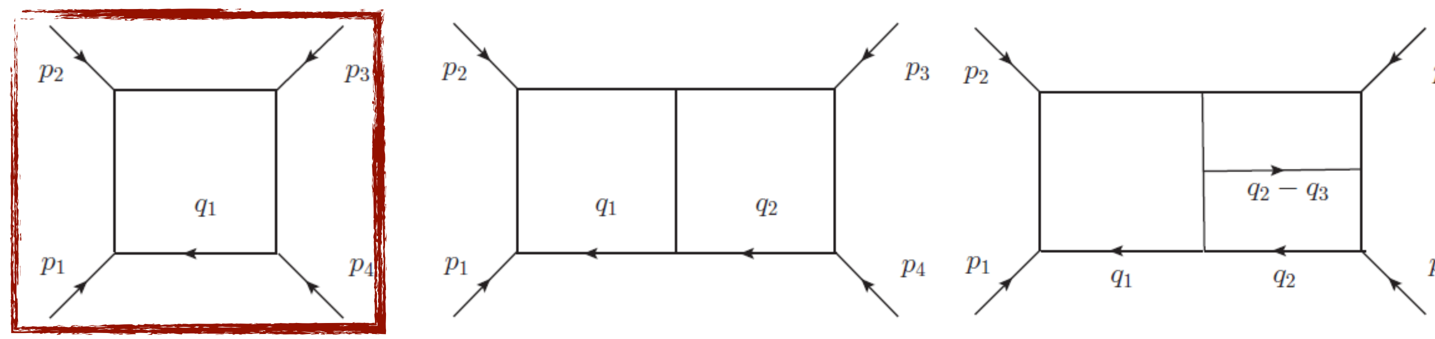
$$\int_{-1}^{1} d\cos\theta (\sin\theta)^{2\alpha-1} C_n^{(\alpha)}(\cos\theta) C_m^{(\alpha)}(\cos\theta) = \delta_{mn} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}$$

Examples

- Four-point integrals : $d_{\parallel} = 3$

$$q_i^\alpha = q_{[3]\,i}^\alpha + \lambda_i^\alpha$$

$$q_{[3]\,i}^\alpha = \sum_{j=1}^{3} x_{ji} e_j^\alpha \qquad \lambda_i^\alpha = x_{4i} e_4^\alpha + \mu_i^\alpha$$



$$I_4^{d\,(1)}[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int d^3 q_{[3]\,1} \int_0^\infty d\lambda_{11} (\lambda_{11})^{\frac{d-5}{2}} \int_{-1}^{1} d\cos\theta_{11} (\sin\theta_{11})^{d-6} \frac{\mathcal{N}(q_{[3]\,1}, \lambda_{11}, \cos\theta_1)}{\prod_{m=0}^{3} D_m(q_{[3]\,1}, \lambda_{11})}$$

$$I_4^{d\,(1)}[1] = \frac{1}{\pi^{3/2} \Gamma\left(\frac{d-3}{2}\right)} \int d^3 q_{[3]\,1} \int_0^\infty d\lambda_{11} (\lambda_{11})^{\frac{d-5}{2}} \frac{1}{\prod_{m=0}^{3} D_m(q_{[3]\,1}, \lambda_{11})}$$

scalar integral

Transverse variable :

$$x_{41} = \sqrt{\lambda_{11}} \cos\theta_{11}$$

$$\cos\theta_{11} = \frac{1}{(d-5)} C_0^{\left(\frac{d-5}{2}\right)}(\cos\theta_{11}) C_1^{\left(\frac{d-5}{2}\right)}(\cos\theta_{11})$$

$$\cos\theta_{11}^2 = \frac{1}{(d-5)^2} \left[C_1^{\left(\frac{d-5}{2}\right)}(\cos\theta_{11})\right]^2$$

Tensor integrals :

$$I_4^{d\,(1)}[x_{41}] = I_4^{d\,(1)}[x_{41}^3] = 0$$

$$I_4^{d\,(1)}[x_{41}^2] = \frac{1}{d-3} I_4^{d\,(1)}[\lambda_{11}] = \frac{1}{2} I_4^{d+2\,(1)}[1]$$

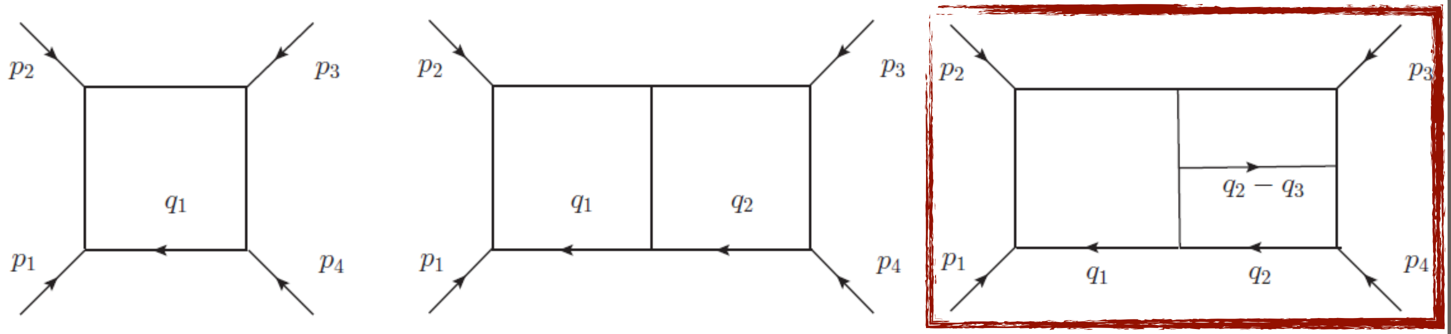
$$I_4^{d\,(1)}[x_{41}^4] = \frac{3}{(d-3)(d-1)} I_4^{d\,(1)}[\lambda_{11}^2] = \frac{3}{4} I_4^{d+4\,(1)}[1]$$

Examples

- Four-point integrals : $d_{\parallel} = 3$

$$q_i^\alpha = q_{[3]\,i}^\alpha + \lambda_i^\alpha$$

$$q_{[3]\,i}^\alpha = \sum_{j=1}^{3} x_{ji} e_j^\alpha \qquad \lambda_i^\alpha = x_{4i} e_4^\alpha + \mu_i^\alpha$$



$$I_4^{d\,(3)}[\mathcal{N}] = \frac{2^{d-7}}{\pi^8 \Gamma(d-6)\Gamma\left(\frac{d-4}{2}\right)} \int \prod_{i=1}^{3} d^3 q_{[3]\,i} \int_0^\infty \prod_{i=1}^{3} d\lambda_{ii} (\lambda_{ii})^{\frac{d-5}{2}} \int_{-1}^{1} \prod_{1\le i<j\le 3} d\cos\theta_{ij} (\sin\theta_{ij})^{d-5-i}$$

$$\int_{-1}^{1} \prod_{j=1}^{3} d\cos\theta_{j\,j} (\sin\theta_{j\,j})^{d-5j} \frac{\mathcal{N}(q_{[3]\,i}, \lambda_{ii}, \cos\theta_{ij}, \sin\theta_{ij})}{\prod_{m=0}^{9} D_m(q_{[3],i}, \lambda_{ii}, \cos\theta_{12}, \cos\theta_{13}, \cos\theta_{23})}.$$

Transverse variables :

$$\begin{cases} \lambda_{12} = & \sqrt{\lambda_{11}\lambda_{22}} \cos\theta_{12} \\ \lambda_{23} = & \sqrt{\lambda_{22}\lambda_{33}} \cos\theta_{13} \\ \lambda_{13} = & \sqrt{\lambda_{11}\lambda_{33}} (\cos\theta_{12}\cos\theta_{13} + \sin\theta_{12}\sin\theta_{13}\cos\theta_{23}) \\ x_{41} = & \sqrt{\lambda_{11}} \cos\theta_{11} \\ x_{42} = & \sqrt{\lambda_{22}} (\cos\theta_{11}\cos\theta_{12} + \sin\theta_{11}\sin\theta_{12}\cos\theta_{22}) \\ x_{43} = & \sqrt{\lambda_{33}} (\cos\theta_{11}\cos\theta_{12}\cos\theta_{13} + \sin\theta_{11}\sin\theta_{12}\cos\theta_{22}\cos\theta_{13} \\ & -\sin\theta_{11}\sin\theta_{13}\cos\theta_{12}\cos\theta_{22}\cos\theta_{23} + \sin\theta_{12}\sin\theta_{13}\cos\theta_{11}\cos\theta_{23} \\ & +\sin\theta_{11}\sin\theta_{13}\sin\theta_{22}\sin\theta_{23}\cos\theta_{33}) \end{cases}$$

Tensor integrals :

$$I_4^{d\,(3)}[x_{41}^{\alpha_4} x_{42}^{\beta_4} x_{43}^{\gamma_4}] = 0, \quad \alpha_4 + \beta_4 + \gamma_4 = 2n+1$$

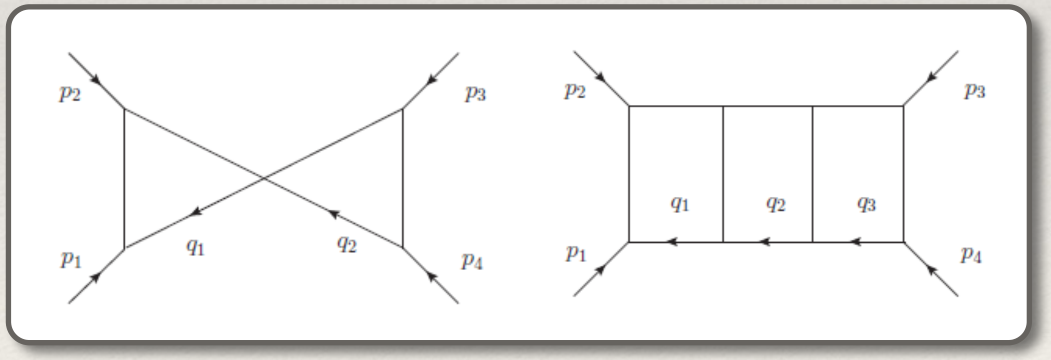
$$I_4^{d\,(3)}[x_{4i} x_{4j}] = \frac{1}{d-3} I_4^{d\,(3)}[\lambda_{ij}]$$

Feynman Integrals in $d = d_\parallel + d_\perp$

- Any ℓ-loop integral with $n \leq 4$ can be parametrised as

$$I_n^{d\,(\ell)}[\mathcal{N}] = \Omega_d^{(\ell)} \overbrace{\int \prod_{i=1}^{\ell} d^{n-1} q_{\parallel\, i}}^{d_\parallel\text{ -space}} \overbrace{\int d^{\frac{\ell(\ell+1)}{2}} \boldsymbol{\Lambda} \int d^{(4-d_\parallel)\ell} \boldsymbol{\Theta}_\perp}^{d_\perp\text{ -space}} \frac{\mathcal{N}(q_{i\,\parallel}, \boldsymbol{\Lambda}, \boldsymbol{\Theta}_\perp)}{\prod_j D_j(q_{\parallel\, i}, \boldsymbol{\Lambda})}$$

- Polynomial dependence on **transverse directions** is exposed
- **Integration** over transverse directions through **Gegenbauer polynomials**
  - All **spurious** contributions detected
  - Alternative to Passarino-Veltman reduction
  - Holds for all variables not appearing in the denominators (e.g. in **factorised** and **ladder** integrals)



- What happens if **combined** with integrand decomposition?

Adaptive Integrand Decomposition

- In $d = d_{\parallel} + d_{\perp}$ denominators depend on a **reduced** set of variables

$$\boxed{d = 4 - 2\epsilon}$$

$$\mathcal{I}_{i_1 \ldots i_r}(\mathbf{z}) \equiv \frac{\mathcal{N}_{i_1 \ldots i_r}(\mathbf{z})}{D_{i_1}(\mathbf{z}) \cdots D_{i_r}(\mathbf{z})}$$
$$\mathbf{z} = \{\mathbf{x}, \mu_{ij}\}$$

$$\boxed{d = d_{\parallel} + d_{\perp}}$$

$$\mathcal{I}_{i_1 \ldots i_r}(\boldsymbol{\tau}, \mathbf{x}_{\perp}) \equiv \frac{\mathcal{N}_{i_1 \ldots i_r}(\boldsymbol{\tau}, \mathbf{x}_{\perp})}{D_{i_1}(\boldsymbol{\tau}) \cdots D_{i_r}(\boldsymbol{\tau})}$$
$$\boldsymbol{\tau} = \{\mathbf{x}_{\parallel}, \lambda_{ij}\}$$

- Cuts are **adaptive,** the dimension of the cut-solution space depends on $d_{\perp}$

- In $d = d_{\parallel} + d_{\perp}$ **on-shell** conditions $\Leftrightarrow$ **linear equations** for the (reducible) variables

E.g. 1-loop :

$$\boxed{D_1 = \cdots = D_n = 0}$$

$$D_i(\boldsymbol{\tau}) = \Big(q_{\parallel} + \sum_{j=0}^{i} p_j\Big)^2 + \lambda^2 + m_i^2 \qquad i = 1, \ldots, n$$
$$\Leftrightarrow \begin{cases} D_i(\boldsymbol{\tau}) - D_1(\boldsymbol{\tau}) = q_{\parallel} \cdot v_i + c_i \\ D_1(\boldsymbol{\tau}) = q_{\parallel}^2 + \lambda^2 + m_1^2 \end{cases} \quad i = 2, \ldots, n$$
$$\Leftrightarrow \begin{cases} \boldsymbol{\tau}_1 = \kappa_1 \\ \cdots \\ \boldsymbol{\tau}_n = \kappa_n \end{cases}$$

- Polynomial division reduced to a **substitution rule** (of reducible variables in terms of **denominators** and physical **ISP**)

Divide and Integrate and Divide

Mastrolia, Peraro, A.P. (2016)

- Residues are determined in **three steps:**

1) **Divide**

$$\mathcal{N}_{i_1\ldots i_r}(\boldsymbol{\tau},\mathbf{x}_\perp) = \sum_{k=1}^{r} \underbrace{\mathcal{N}_{i_1\ldots i_{k-1} i_{k+1}\ldots i_r}(\boldsymbol{\tau},\mathbf{x}_\perp)}_{\text{Subtopology \#1}} D_{i_k}(\boldsymbol{\tau}) + \Delta_{i_1\ldots i_r}(\mathbf{x}_\parallel,\mathbf{x}_\perp)$$

Monomial order $\lambda_{ij} \prec \mathbf{x}_\parallel$

λ_{ij} are **reducible**

2) **Integrate**

$$\int \prod_{1=j}^{\ell} \frac{d^d q_j}{\pi^{d/2}} \frac{\Delta_{i_1\ldots i_r}(\mathbf{x}_\parallel,\mathbf{x}_\perp)}{D_{i_1}(\boldsymbol{\tau})\ldots D_{i_r}(\boldsymbol{\tau})} = \Omega_d^{(\ell)} \int \prod_{i=1}^{\ell} d^{n-1} q_{\parallel\, i} \int d^{\frac{\ell(\ell+1)}{2}} \boldsymbol{\Lambda} \frac{\Delta^{\text{int}}_{i_1\ldots i_n}(\boldsymbol{\tau})}{D_{i_1}(\boldsymbol{\tau})\ldots D_{i_r}(\boldsymbol{\tau})}$$

$$\Delta^{\text{int}}_{i_1\ldots i_r}(\boldsymbol{\tau}) = \int d^{(4-d_\parallel)\ell} \boldsymbol{\Theta}_\perp \Delta_{i_1\ldots i_r}(\boldsymbol{\tau},\boldsymbol{\Theta}_\perp)$$

$$\boxed{\mathbf{x}_{\perp i} \to P[\boldsymbol{\tau}, \sin[\boldsymbol{\Theta}_\perp], \cos[\boldsymbol{\Theta}_\perp]]}$$

Integrate over $\boldsymbol{\Theta}_\perp$

3) **Divide**

$$\Delta^{\text{int}}_{i_1\ldots i_r}(\boldsymbol{\tau}) = \sum_{k=1}^{r} \underbrace{\mathcal{N}^{\text{int}}_{i_1\ldots i_{k-1} i_{k+1}\ldots i_r}(\boldsymbol{\tau})}_{\text{Subtopology \#2}} D_{i_k}(\boldsymbol{\tau}) + \Delta'_{i_1\ldots i_r}(\mathbf{x}_\parallel)$$

physical ISP

monomials only

- The final residue is **free** from **spurious** terms and suitable for **integral reduction**

Adaptive Integrand Decomposition @1Loop

- @1Loop : $\boxed{[\boldsymbol{\tau}] = n \Rightarrow D_{i_1}(\boldsymbol{\tau}) = \cdots = D_{i_n}(\boldsymbol{\tau}) = 0}$ **all** cuts are **zero-dimensional** (No ISP)

  1) **Divide**

$$
\begin{aligned}
\Delta_{ijklm} =& c_0 \mu^2 \\
\Delta_{ijkl} =& c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_4^4 \\
\Delta_{ijk} =& c_0 + c_1 x_3 + c_2 x_4 + c_3 x_3^2 + c_4 x_3 x_4 + c_5 x_4^2 + c_6 x_3^3 + c_7 x_3^2 x_4 + c_8 x_3 x_4^2 + c_9 x_4^3 \\
\Delta_{ij} =& c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_2 x_3 + c_6 x_2 x_4 + c_7 x_3^2 + c_8 x_3 x_4 + c_9 x_4^2 \\
\Delta_{ij}|_{p^2=0} =& c_0 + c_1 x_1 + c_2 x_3 + c_3 x_4 + c_4 x_1^2 + c_5 x_1 x_3 + c_6 x_1 x_4 + c_7 x_3^2 + c_8 x_3 x_4 + c_9 x_4^2 \\
\Delta i =& c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}
$$

All residues fixed by the **Maximum-cut** theorem

Adaptive Integrand Decomposition @1Loop

- @1Loop : $\boxed{[\boldsymbol{\tau}] = n \Rightarrow D_{i_1}(\boldsymbol{\tau}) = \cdots = D_{i_n}(\boldsymbol{\tau}) = 0}$ **all** cuts are **zero-dimensional** (No ISP)

1) **Divide**

$$
\begin{aligned}
\Delta_{ijklm} &= c_0\mu^2 \\
\Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_4^4 \\
\Delta_{ijk} &= c_0 + c_1 x_3 + c_2 x_4 + c_3 x_3^2 + c_4 x_3 x_4 + c_5 x_4^2 + c_6 x_3^3 + c_7 x_3^2 x_4 + c_8 x_3 x_4^2 + c_9 x_4^3 \\
\Delta_{ij} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_2 x_3 + c_6 x_2 x_4 + c_7 x_3^2 + c_8 x_3 x_4 + c_9 x_4^2 \\
\Delta_{ij}|_{p^2=0} &= c_0 + c_1 x_1 + c_2 x_3 + c_3 x_4 + c_4 x_1^2 + c_5 x_1 x_3 + c_6 x_1 x_4 + c_7 x_3^2 + c_8 x_3 x_4 + c_9 x_4^2 \\
\Delta i &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}
$$

All residues fixed by the **Maximum-cut** theorem

2) **Integrate**

$$
\begin{aligned}
\Delta^{\text{int}}_{ijklm} &= c_0\mu^2 \\
\Delta^{\text{int}}_{ijkl} &= c_0 + \frac{1}{d-3} c_2 \lambda^2 + c_4 \frac{1}{(d-1)(d-3)} \lambda^4 \\
\Delta^{\text{int}}_{ijk} &= c_0 + \frac{1}{d-2}(c_3 + c_4)\lambda^2 \\
\Delta^{\text{int}}_{ij} &= c_0 + \frac{1}{d-1}(c_4 + c_7 + c_9)\lambda^2 \\
\Delta^{\text{int}}_{ij}|_{p^2=0} &= c_0 + c_1 x_1 + c_4 x_1^2 + \frac{1}{d-2}(c_7 + c_9)\lambda^2 \\
\Delta i^{\text{int}} &= c_0
\end{aligned}
$$

Spurious terms drop out

Dim-shifted integrals (but λ^2 reducible)

Adaptive Integrand Decomposition @1Loop

- @1Loop : $\boxed{[\boldsymbol{\tau}] = n \Rightarrow D_{i_1}(\boldsymbol{\tau}) = \cdots = D_{i_n}(\boldsymbol{\tau}) = 0}$ **all** cuts are **zero-dimensional** (No ISP)

1) **Divide**

$$
\begin{aligned}
\Delta_{ijklm} &= c_0 \mu^2 \\
\Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_4^4 \\
\Delta_{ijk} &= c_0 + c_1 x_3 + c_2 x_4 + c_3 x_3^2 + c_4 x_3 x_4 + c_5 x_4^2 + c_6 x_3^3 + c_7 x_3^2 x_4 + c_8 x_3 x_4^2 + c_9 x_4^3 \\
\Delta_{ij} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_2 x_3 + c_6 x_2 x_4 + c_7 x_3^2 + c_8 x_3 x_4 + c_9 x_4^2 \\
\Delta_{ij}|_{p^2=0} &= c_0 + c_1 x_1 + c_2 x_3 + c_3 x_4 + c_4 x_1^2 + c_5 x_1 x_3 + c_6 x_1 x_4 + c_7 x_3^2 + c_8 x_3 x_4 + c_9 x_4^2 \\
\Delta i &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}
$$

All residues fixed by the **Maximum-cut** theorem

2) **Integrate**

$$
\begin{aligned}
\Delta^{\text{int}}_{ijklm} &= c_0 \mu^2 \\
\Delta^{\text{int}}_{ijkl} &= c_0 + \frac{1}{d-3} c_2 \lambda^2 + c_4 \frac{1}{(d-1)(d-3)} \lambda^4 \\
\Delta^{\text{int}}_{ijk} &= c_0 + \frac{1}{d-2}(c_3 + c_4)\lambda^2 \\
\Delta^{\text{int}}_{ij} &= c_0 + \frac{1}{d-1}(c_4 + c_7 + c_9)\lambda^2 \\
\Delta^{\text{int}}_{ij}|_{p^2=0} &= c_0 + c_1 x_1 + c_4 x_1^2 + \frac{1}{d-2}(c_7 + c_9)\lambda^2 \\
\Delta i^{\text{int}} &= c_0
\end{aligned}
$$

3) **Divide**

$$
\begin{aligned}
\Delta'_{ijklm} &= c_0 \mu^2 \\
\Delta'_{ijkl} &= c_0(d) \\
\Delta'_{ijk} &= c_0(d) \\
\Delta'_{ij} &= c_0(d) \\
\Delta'_{ij}|_{p^2=0} &= c_0(d) + c_1 x_1 + c_4 x_1^2 \\
\Delta i' &= c_0
\end{aligned}
$$

Spurious terms drop out
Dim-shifted integrals (but λ^2 reducible)

Dim-recurrence
@integrand level

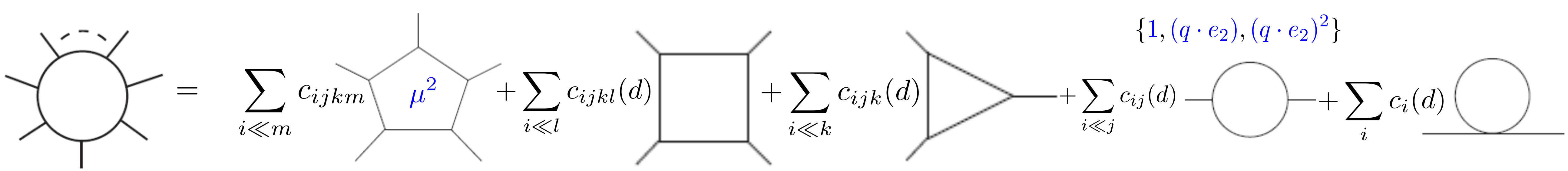
Adaptive Integrand Decomposition @1Loop

- @1Loop : $\boxed{[\boldsymbol{\tau}] = n \Rightarrow D_{i_1}(\boldsymbol{\tau}) = \cdots = D_{i_n}(\boldsymbol{\tau}) = 0}$ **all** cuts are **zero-dimensional** (No ISP)

  1) **Divide**

$$
\begin{aligned}
\Delta_{ijklm} &= c_0 \mu^2 \\
\Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_4^4 \\
\Delta_{ijk} &= c_0 + c_1 x_3 + c_2 x_4 + c_3 x_3^2 + c_4 x_3 x_4 + c_5 x_4^2 + c_6 x_3^3 + c_7 x_3^2 x_4 + c_8 x_3 x_4^2 + c_9 x_4^3
\end{aligned}
$$

All residues fixed by the



$$
\begin{aligned}
\Delta^{\text{int}}_{ijk} &= c_0 + \frac{1}{d-2}(c_3 + c_4)\lambda^2 \\
\Delta^{\text{int}}_{ij} &= c_0 + \frac{1}{d-1}(c_4 + c_7 + c_9)\lambda^2 \\
\Delta^{\text{int}}_{ij}|_{p^2=0} &= c_0 + c_1 x_1 + c_4 x_1^2 + \frac{1}{d-2}(c_7 + c_9)\lambda^2 \\
\Delta i^{\text{int}} &= c_0
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{ijkl} &= c_0(d) \\
\Delta'_{ijk} &= c_0(d) \\
\Delta'_{ij} &= c_0(d) \\
\Delta'_{ij}|_{p^2=0} &= c_0(d) + c_1 x_1 + c_4 x_1^2 \\
\Delta i' &= c_0
\end{aligned}
$$

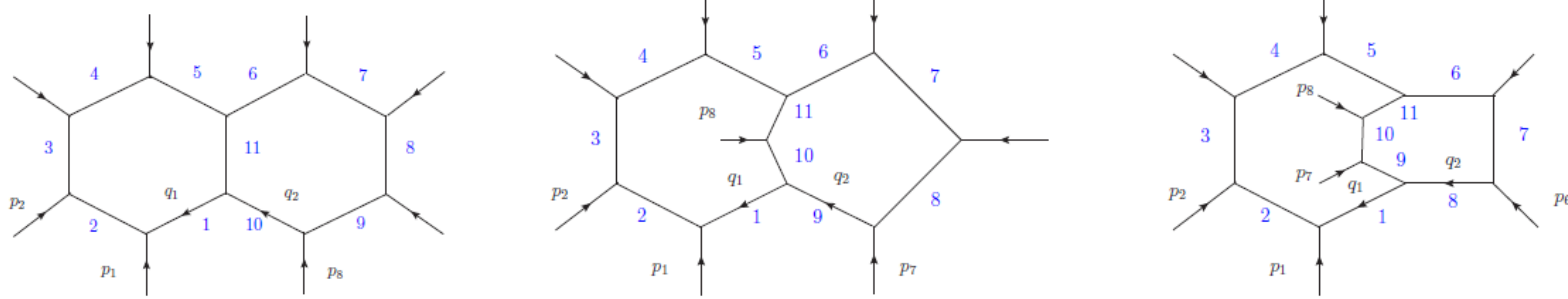
Spurious terms drop out
Dim-shifted integrals (but λ^2reducible)

Dim-recurrence
@integrand level

Mastrolia, Peraro, **A.P.** (16)

Adaptive Integrand Decomposition @2Loops

$r_1 = r_2 = 5,\ r_{12} = 1$ $\qquad$ $r_1 = 5,\ r_2 = 4,\ r_{12} = 2$ $\qquad$ $r_1 = 5,\ r_2 = 3,\ r_{12} = 3$



(a) $\mathcal{I}^{\mathrm{P}}_{12345678910\,11}$ $\qquad$ (b) $\mathcal{I}^{\mathrm{NP1}}_{12345678910\,11}$ $\qquad$ (c) $\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$

- Three maximum-cut topologies $\boxed{[\mathbf{z}] = \dfrac{2(2+9)}{2} = 11}$, in **arbitrary kinematics**
- Universal parametrisation of the residues in renormalisable theories

$$\mathcal{N}_{i_1\cdots i_r}(\mathbf{z}) = \sum_{\vec{j}\in J_{11}(s_1,s_2,s_{\mathrm{tot}})} \alpha_{\vec{j}}\, z_1^{j_1} z_2^{j_2} \ldots z_{11}^{j_{11}}$$

$$\begin{cases} \sum_{i=1}^{4} j_i + 2j_9 + j_{11} \le s_1, & s_1 = r_1 + r_{12} \\ \sum_{i=5}^{8} j_i + 2j_{10} + j_{11} \le s_2, & s_2 = r_2 + r_{12} \\ \sum_{i=1}^{8} j_i + 2(j_9 + j_{10} + j_{11}) \le s_{12}, & s_{\mathrm{tot}} = r_1 + r_2 + r_{12} - 1 \end{cases}$$

Mastrolia, Peraro, A.P. (16)

Adaptive Integrand Decomposition @2Loops

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{12345678910\,1}$	1 $\{1\}$
$\mathcal{I}^{\mathrm{NP1}}_{12345678910\,1}$	1 $\{1\}$
$\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$	1 $\{1\}$
$\mathcal{I}^{\mathrm{P}}_{2345678910\,11}$	6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2345678910\,11}$	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1234578910\,11}$	6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1234678910\,11}$	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{234678910\,11}$	15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{234578910\,11}$	33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{234578910\,11}$	39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456910\,11}$	15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{234678910\,11}$	45 $\{1, x_{41}, x_{42}\}$

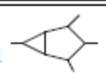
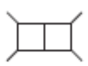
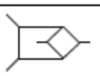
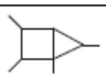
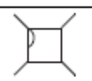
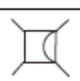
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{1245678910\,11}$	6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1245678910\,11}$	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1234568910\,11}$	6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1245678910\,11}$	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{245678910\,11}$	15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{123478910\,11}$	33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{124568910\,11}$	39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456810\,11}$	15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{124678910\,11}$	45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2478910\,11}$	20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{23478910\,11}$	76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24578910\,11}$	116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12457810\,11}$	80 $\{1, x_{31}, x_{41}, x_{42}\}$

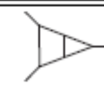
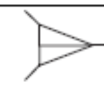
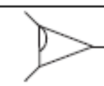
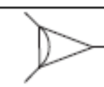
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{135678910\,11}$	15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{124567910\,11}$	62 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{23568910\,11}$	39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456910\,11}$	15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{135678910\,11}$	45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{25678910\,11}$	20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{23568910\,11}$	76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{25678910\,11}$	80 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24568910\,11}$	116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{3678910\,11}$	15 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{2578910\,11}$	94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{2357910\,11}$	160 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2457910\,11}$	185 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$

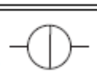
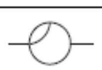
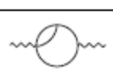
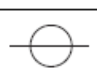
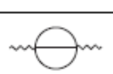
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{15678910\,11}$	20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{13567910\,11}$	76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{15678910\,11}$	80 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1678910\,11}$	15 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{13568910\,11}$	116 $\{1, x_{31}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1467910\,11}$	94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1678911}$	66 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1256910\,11}$	160 $\{1, x_{31}, x_{41}, y_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1357910\,11}$	185 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1256911}$	180 $\{1, x_{11}, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{246910\,11}$	246 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$

Mastrolia, Peraro, **A.P.** (16)

Adaptive Integrand Decomposition @2Loops

$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\mathrm{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{1567910\,11}$ 	94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$	53 $\{1, x_{21}, x_{31}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	10 $\{1, x_{21}, x_{31}\}$
$\mathcal{I}^{\mathrm{P}}_{12256910\,11}$ 	160 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$	93 $\{1, x_{31}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	22 $\{1, x_{31}, x_{32}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1356910\,11}$ 	184 $\{1, x_{31}, x_{42}, x_{32}, x_{42}\}$	105 $\{1, x_{31}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	25 $\{1, x_{31}, x_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{1356811}$ 	180 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	101 $\{1, x_{31}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	39 $\{1, x_{31}, x_{22}, y_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{168910\,11}$ 	66 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$	35 $\{1, x_{11}, x_{21}, x_{31}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	10 $\{1, x_{11}, x_{21}, x_{31}\}$
$\mathcal{I}^{\mathrm{NP1}}_{246910\,11}$ 	245 $\{1, x_{31}, x_{41}, x_{21}, x_{32}, x_{42}\}$	137 $\{1, x_{31}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	55 $\{1, x_{31}, x_{22}, y_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{36810\,11}$ 	115 $\{1, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	66 $\{1, x_{31}, x_{12}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	35 $\{1, x_{31}, x_{12}, x_{22}, x_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{136811}$ 	180 $\{1, x_{11}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	103 $\{1, x_{11}, x_{31}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	60 $\{1, x_{11}, x_{31}, x_{22}, x_{32}\}$

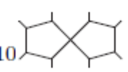
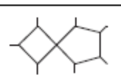
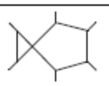
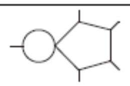
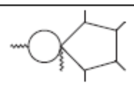
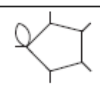
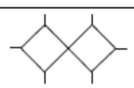
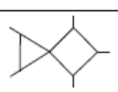
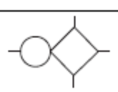
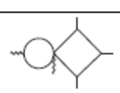
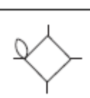
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\mathrm{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{1356911}$ 	180 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	22 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	4 $\{1, x_{22}\}$
$\mathcal{I}^{\mathrm{NP1}}_{156910\,11}$ 	240 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	30 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{22}\}$
$\mathcal{I}^{\mathrm{P}}_{15710\,11}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	33 $\{1, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	13 $\{1, x_{21}, x_{12}\}$
$\mathcal{I}^{\mathrm{P}}_{16910\,11}$ 	115 $\{1, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	20 $\{1, x_{11}, x_{22}\lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{12}, x_{22}\}$
$\mathcal{I}^{\mathrm{P}}_{3610\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{x_{11}, x_{21}, x_{22}\}$

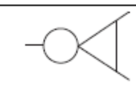
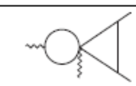
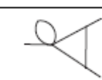
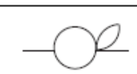
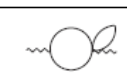
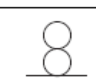
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\mathrm{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{15610\,11}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	8 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$
$\mathcal{I}^{\mathrm{P}}_{1610\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{4}, x_{22}, y_{3}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\mathrm{P}}_{1310\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{1, x_{11}, x_{21}, x_{12}\}$
$\mathcal{I}^{\mathrm{P}}_{210\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	9 $\{1, x_{11}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{11}, x_{12}\}$
$\mathcal{I}^{\mathrm{P}}_{210\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	18 $\{1, x_{11}, x_{21}, x_{12}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	15 $\{1, x_{11}, x_{22}, x_{21}, x_{22}\}$

$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\mathrm{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{110\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$

Mastrolia, Peraro, A.P. (16)

Adaptive Integrand Decomposition @2Loops

$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\rm int}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\rm P}_{12345678910}$ 	1 $\{1\}$	– –	– –
$\mathcal{I}^{\rm P}_{1245678910}$ 	5 $\{1, x_{41}\}$	3 $\{1, \lambda_{11}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{125678910}$ 	10 $\{1, x_{31}, x_{41}\}$	2 $\{1, \lambda_{11}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{15678910}$ 	10 $\{1, x_{21}, x_{31}, x_{41}\}$	2 $\{1, \lambda_{11}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{12678910}$ 	10 $\{1, x_{11}, x_{31}, x_{41}\}$	4 $\{1, x_{11}, \lambda_{11}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\rm P}_{1678910}$ 	5 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$	1 $\{1\}$	– –
$\mathcal{I}^{\rm P}_{23456789}$ 	25 $\{1, x_{41}, x_{42}\}$	9 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{2356789}$ 	50 $\{1, x_{31}, x_{41}, x_{42}\}$	6 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{256789}$ 	50 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$	6 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{236789}$ 	50 $\{1, x_{11}, x_{31}, x_{41}, x_{42}\}$	12 $\{1, x_{11}, \lambda_{11}, \lambda_{22}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\rm P}_{26789}$ 	25 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$	3 $\{1, \lambda_{22}\}$	1 $\{1\}$

$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\rm int}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\rm P}_{245689}$ 	100 $\{1, x_{31}, x_{42}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{24689}$ 	100 $\{1, x_{21}, x_{31}, x_{41}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{45689}$ 	100 $\{1, x_{11}, x_{31}, x_{41}, x_{32}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\rm P}_{2689}$ 	50 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{32}, x_{42}\}$	2 $\{1, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{2569}$ 	100 $\{1, x_{11}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{4569}$ 	100 $\{1, x_{11}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\rm P}_{4568}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{32}, x_{42}\}$	16 $\{1, x_{11}, x_{12}, \lambda_{11}, \lambda_{22}\}$	9 $\{1, x_{11}, x_{12}\}$
$\mathcal{I}^{\rm P}_{269}$ 	50 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	2 $\{1, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{268}$ 	50 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	4 $\{1, x_{12}, \lambda_{22}\}$	3 $\{x_{12}\}$
$\mathcal{I}^{\rm P}_{29}$ 	25 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	1 $\{1\}$	– –

D&I&D : $A^{2-\mathrm{loop}}(p_1^+, p_2^-, p_3^+, p_4^-)$

Mastrolia, Peraro, A.P. (16)

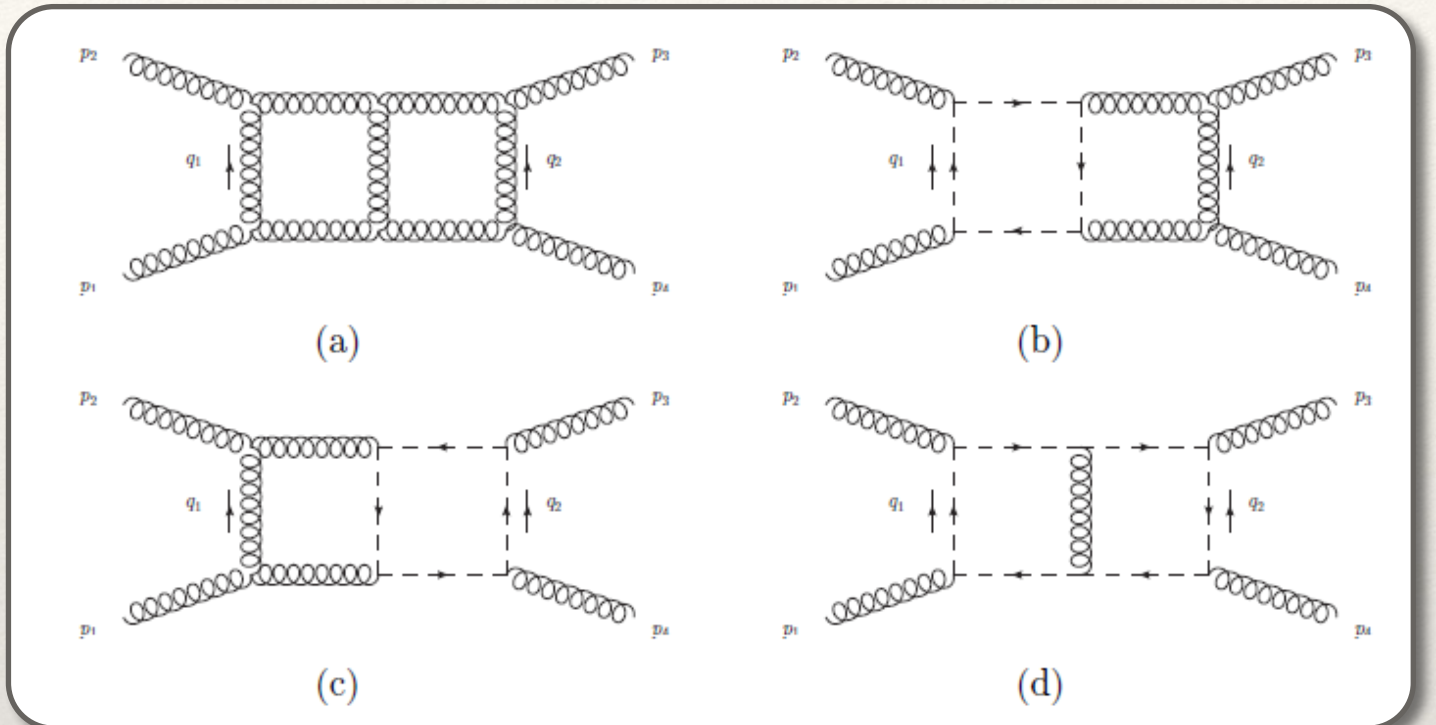
- Four-point kinematics : $d_{\|} = 3$

$$\mathbf{x}_\perp = \{x_{41}, x_{42}\}$$
$$\mathbf{x}_{\|} = \{x_{11}, x_{21}, x_{31}, x_{12}, x_{32}, x_{42}\}$$

- **Rank-six** numerator with **2025** terms in

$$\mathbf{z} = \{\mathbf{x}_{\|}, \mathbf{x}_\perp \lambda_{11}, \lambda_{22}, \lambda_{12}\}, \ [\mathbf{z}] = 11$$

$$D_1(\boldsymbol{\tau}) = \cdots = D_7(\boldsymbol{\tau}) = 0$$
$$\boldsymbol{\tau} = \{\mathbf{x}_{\|}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}, \qquad [\boldsymbol{\tau}] = 9$$



1) **Divide :** $\Delta_{1\ldots7}(x_{31}, x_{32}, x_{41}, x_{42})$ contains **70** terms

2) **Integrate :** $\Delta^{\mathrm{int}}_{1\ldots7}(x_{31}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12})$ contains **39** terms

3) **Divide :** $\Delta'_{1\ldots7}(x_{31}, x_{32})$ contains **15** terms

D&I&D: $A^{2-\text{loop}}(p_1^+, p_2^-, p_3^+, p_4^-)$

Mastrolia, Peraro, A.P. (16)

$$A^{2-\text{loop}}(p_1^+, p_2^-, p_3^+, p_4^-)\Big|_{cut} = i\frac{\langle 2\,4\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\Big(\sum_{\alpha,\beta} c_{\alpha,\beta}\, I_4^{d\,(2)}[(q_1\cdot p_4)^\alpha\,(q_2\cdot p_1)^\beta]\Big)$$

$$c_{4,0} = -\frac{(d_s-2)(2t+1)^2}{2t(t+1)^4} - \frac{(d_s-2)\left(2t^2-2t-1\right)}{(d-3)t(t+1)^4} - \frac{3(d_s-2)}{2(d-1)(d-3)t(t+1)^4},$$

$$c_{3,1} = -\frac{3(d_s-2)(2t+1)}{(d-1)(d-3)t(t+1)^4} - \frac{(d_s-2)(2t+1)}{t(t+1)^4} + \frac{2(d_s-2)\left(4t^2+2t+1\right)}{(d-3)t(t+1)^4},$$

$$c_{3,0} = -\frac{(2t+1)(d_s-2)}{(t+1)^3} + \frac{2(d_s-2)}{(d-3)(t+1)^2} - \frac{3(d_s-2)}{(d-1)(d-3)(t+1)^3},$$

$$c_{2,2} = -\frac{3(d_s-2)\left(8t^2+8t+3\right)}{2(d-1)(d-3)t(t+1)^4} - \frac{32t^2+32t+3(d_s-2)}{2t(t+1)^4} + \frac{32t^3+16t^2+12(d_s-2)t-16t+3(d_s-2)}{(d-3)t(t+1)^4},$$

$$c_{2,1} = -\frac{3(d_s-2)(4t+3)}{(d-1)(d-3)(t+1)^3} - \frac{(d_s-2)+8t+4}{(t+1)^3} + \frac{4\left(8t^2+2(d_s-2)t+2t+2(d_s-2)-3\right)}{(d-3)(t+1)^3},$$

$$c_{2,0} = -\frac{3(d_s-2)t(2t+3)}{2(d-1)(d-3)(t+1)^3} - \frac{(d_s-2)t+8t+4}{2(t+1)^2} + \frac{16t^3+7(d_s-2)t^2+16t^2+4(d_s-2)t+4t+4}{2(d-3)(t+1)^3},$$

$$c_{1,3} = -\frac{3(d_s-2)(2t+1)}{(d-1)(d-3)t(t+1)^4} - \frac{(d_s-2)(2t+1)}{t(t+1)^4} + \frac{2(d_s-2)\left(4t^2+2t+1\right)}{(d-3)t(t+1)^4},$$

$$c_{1,2} = -\frac{3(d_s-2)(4t+3)}{(d-1)(d-3)(t+1)^3} - \frac{(d_s-2)+8t+4}{(t+1)^3}$$

$$c_{1,1} = -\frac{2(2t+1)}{(t+1)^2} - \frac{3(d_s-2)t(4t+3)}{(d-1)(d-3)(t+1)^3} + \frac{32t^3+4(d_s-2)t^2+32t^2+7(d_s-2)t+2t+2}{(d-3)(t+1)^3},$$

$$c_{1,0} = -\frac{3(d_s-2)t^2}{(d-1)(d-3)(t+1)^2} + \frac{\left(8t^2+(d_s-2)t+6t+2\right)t}{(d-3)(t+1)^2} - \frac{2t}{t+1},$$

$$c_{0,4} = -\frac{(d_s-2)(2t+1)^2}{2t(t+1)^4} - \frac{(d_s-2)\left(2t^2-2t-1\right)}{(d-3)t(t+1)^4} - \frac{3(d_s-2)}{2(d-1)(d-3)t(t+1)^4}$$

$$c_{0,3} = -\frac{(2t+1)(d_s-2)}{(t+1)^3} + \frac{2(d_s-2)}{(d-3)(t+1)^2} - \frac{3(d_s-2)}{(d-1)(d-3)(t+1)^3},$$

$$c_{0,2} = -\frac{3(d_s-2)t(2t+3)}{2(d-1)(d-3)(t+1)^3} - \frac{(d_s-2)t+8t+4}{2(t+1)^2} + \frac{16t^3+7(d_s-2)t^2+16t^2+4(d_s-2)t+4t+4}{2(d-3)(t+1)^3},$$

$$c_{0,1} = -\frac{3(d_s-2)t^2}{(d-1)(d-3)(t+1)^2} + \frac{\left(8t^2+(d_s-2)t+6t+2\right)t}{(d-3)(t+1)^2} - \frac{2t}{t+1},$$

$$c_{0,0} = -\frac{3(d_s-2)t^3}{4(d-1)(d-3)(t+1)^2} + \frac{(2t+1)t^2}{(d-3)(t+1)} - \frac{t}{2},$$

Divide : $A^{2-\text{loop}}(p_1^+, p_2^+, p_3^+, p_4^+, p_5^+)$

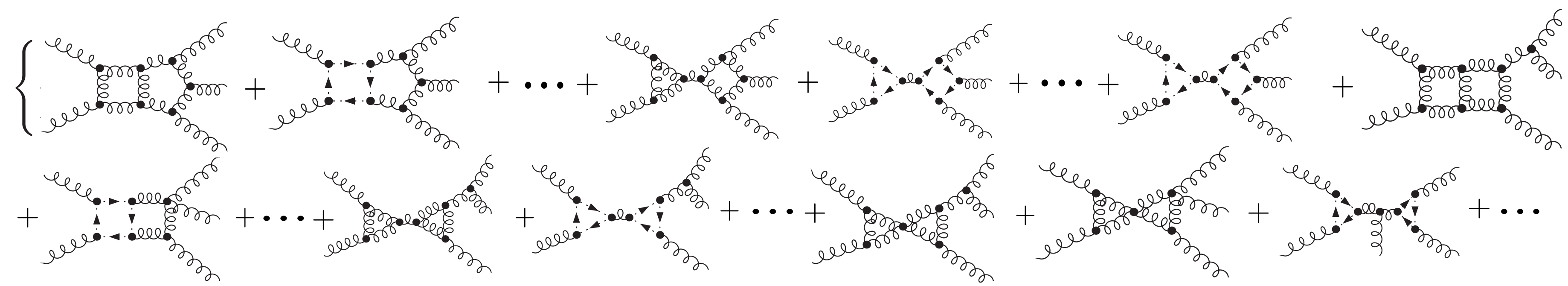
Mastrolia, Peraro, A.P, Torres-Bobadilla (16)

- Recent developments in the computation of higher multiplicity processes ad NNLO

Badger, Frellesvig, Zhang (13)
Badger, Mogull, Ochirov et al (15),
Papadopoulos, Tommasini, Wever (16)
Gehrmann, Henn Lo Presti (16)
Dunbar, Perkins (16)
Dunbar, Jehu, Perkins (16)
Badger, Mogull, Perabo (16)

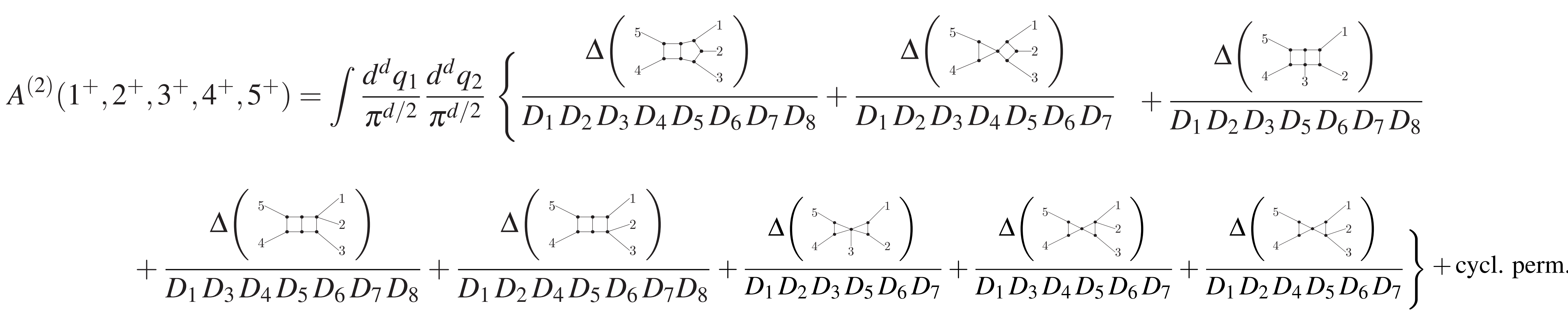
- Integrand built from **diagrams** in Feynman gauge

$$A^{(2)}(1^+,2^+,3^+,4^+,5^+) = \int \frac{d^d q_1}{\pi^{d/2}} \frac{d^d q_2}{\pi^{d/2}} \Bigg\{$$



- **Leading-colour** contribution recovered through AID

Badger, Frellesvig, Zhang (13)



Summary and Outlook

- **Algebraic** analysis of **integrands** is an efficient tool for the computation of multi-leg/scale amplitudes
  - Integrand decomposition fully automated @1-Loop (Cutools, Samurai, Ninja …)
- We proposed an **adaptive** version of the algorithm, based on the **splitting** of the space-time **dimensions** according to the **kinematics** of each integrand
  - **Polynomial division** modulo Gröbner basis **trivialised** @all-Loops
  - Detection of **spurious** terms via **Gegenbauer polynomials** @all-Loops
  - Transverse space symmetries of the residues exposed (e.g. maximum-cut @1-Loop)
- Integral basis still **non-minimal** (IBP, LI identities) but in a **suitable form** for further integral reduction
  - On the way to the **translate** integral properties at the **integrand level** (e.g. dim-recurrence @1-Loop)

Thank you!