d-Dimensional Generalised Unitarity and Colour-Kinematics duality

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LoopFest XV

15-17 August 2016 University at Buffalo,

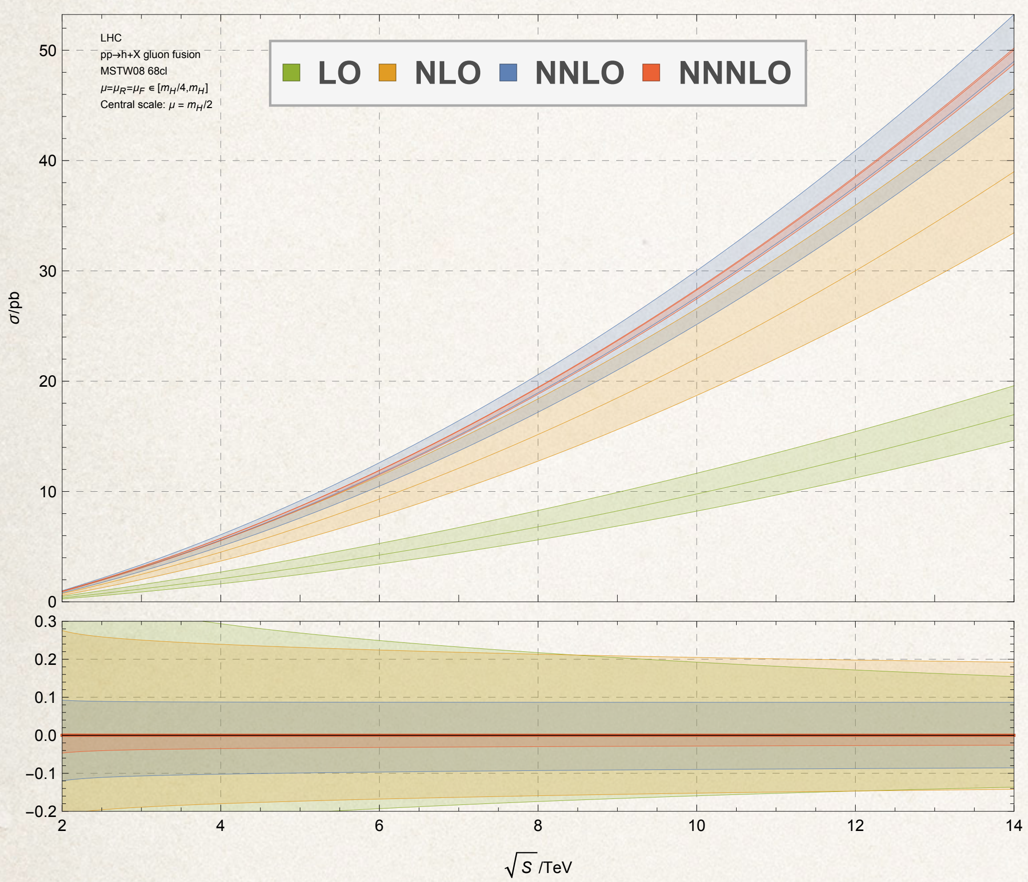
North Campus, Amherst, NY

Based on the collaborations with
P. Mastrolia and A. Primo

Introduction

- Scattering amplitudes are necessary to test our theoretical models by comparing their predictions against the experiments.

[Anastasiou, Duhr, Dulat, Herzog, Mistlberger (2015)]



- Tree-level (LO) predictions are qualitative due to the poor convergence of the truncated expansion at strong coupling.

$$\alpha_S\,(100\mathrm{GeV}) \sim 0.12$$

- K factors

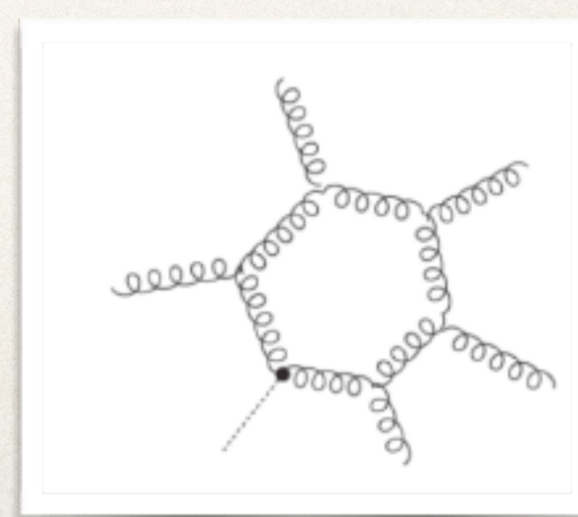
$$K = \frac{\mathrm{NLO}}{\mathrm{LO}} \sim 30\% \div 80\%$$

- Feynman diagrams, based on the Lagrangian, are not optimised for these processes.

- On-shell methods are based on amplitudes and take full advantage of the analyticity of the S-matrix.

See J.Huston's Talk

Motivation



- Simplify the calculations in High-Energy Physics.
- Discover hidden properties of Quantum Field Theories
- Towards NNLO is the **Present Frontier.**

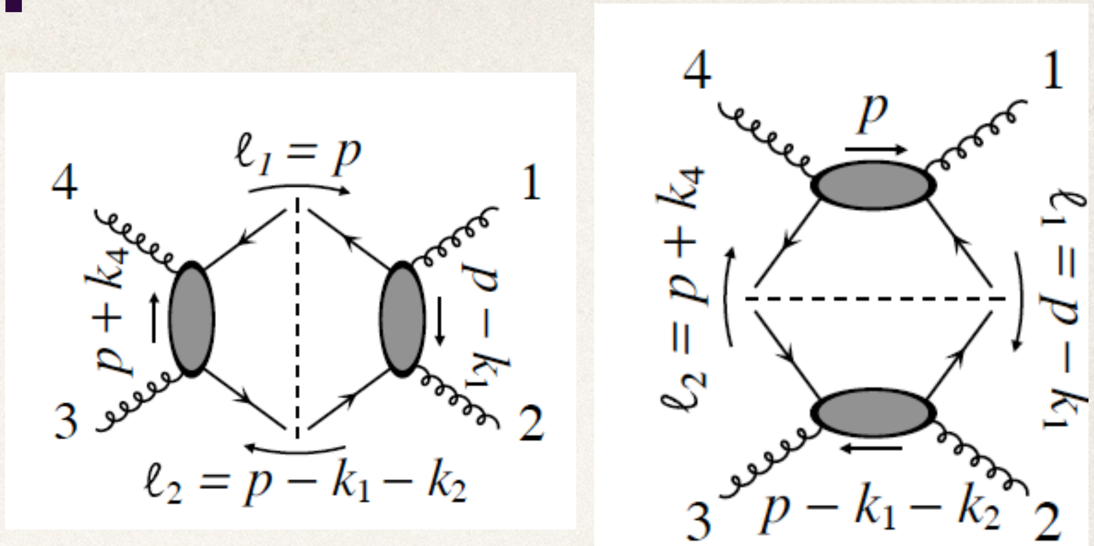
Outline

- Analytic one-loop amplitudes
  — d dimensional generalised unitarity
  — Four dimensional formulation of dimensional regularisation
  — Results

- Further simplifications from colour/kinematics duality
  — C/K relations @ tree-level in dimensional regularisation
  — C/K relations @ one-loop
  — Unitarity + C/K-relations @ work

- Summary and Outlook

Analytic one-loop scattering amplitudes



[Bern, Dixon, Dunbar, Kosower (1994)]

Standard Unitarity in 4D

Glue together the two amplitudes and uplift the integral with

$$2\pi\delta^{(+)}\left(p^2 - m^2\right) \to \frac{i}{p^2 - m^2 - i\epsilon}$$

Generalised Unitarity in 4D

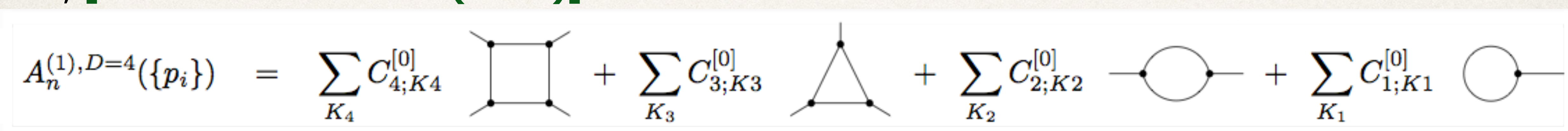
[Bern, Dixon, Kosower (1998)]
[Britto, Cachazo, Feng (2004)]

Isolate the leading discontinuity

$$\mathcal{A}^{(L)} = \sum_i c_i \mathcal{I}_i^{(L)}$$

Known basis of L-loop scalar integrals

For L=1, **[Passarino - Veltman (1979)]**



$$A_n^{(1),D=4}(\{p_i\}) = \sum_{K_4} C_{4;K4}^{[0]} + \sum_{K_3} C_{3;K3}^{[0]} + \sum_{K_2} C_{2;K2}^{[0]} + \sum_{K_1} C_{1;K1}^{[0]}$$

Scalar Master Integrals: Made of polylogarithmic functions

- If an amplitude is determined by its branch cuts, it is said to be cut-constructible.
- All one-loop amplitudes are cut-constructible in dimensional regularisation.

Analytic one-loop scattering amplitudes

In D=4-2ε we can do the decomposition

$$\bar{\ell}^\nu = \ell^\nu + \tilde{\ell}^\nu$$

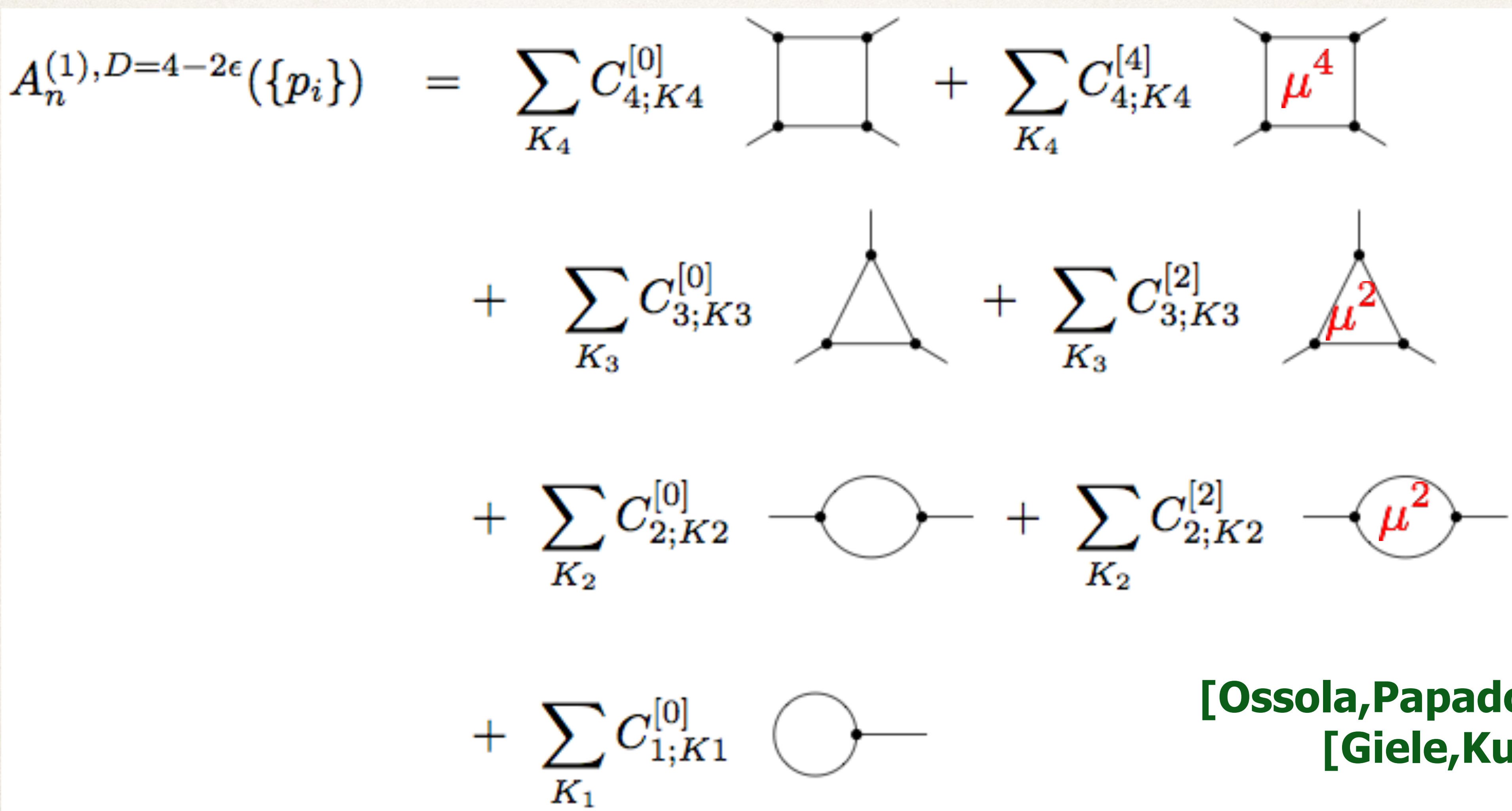
D=4 D=-2ε

The on-shell condition

$$\bar{\ell}^2 = \ell^2 - \mu^2 = 0 \longrightarrow \ell^2 = \mu^2$$

Mass term

Any massless one-loop becomes



[Ossola,Papadopoulos,Pittau (2006)]
[Giele,Kunszt,Melnikov (2008)]
[Badger (2008)]
[Mastrolia, Mirabella, Peraro (2012)]

How to compute those coefficients?

D-dimensional unitarity offers the determination of all pieces together

Four Dimensional Formulation of Dimensional Regularisation (FDF)

Live in 4 dimensions! [Fazio, Mastrolia, Mirabella, W.J.T (2014)]

— Explicit 4D representation of polarisation and. spinors
— 4D representation of D-reg loop propagators
— 4D Feynman rules + (-2ε)-Selection Rules
— Easy to implement in existing generators

FDH: 4D helicity scheme

[Bern and Kosower (1992)]

The d-dimensional metric tensor can be split as

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$$

d-dimensional 4-dimensional -2ε-dimensional

where

$$\tilde{g}^{\mu\nu} g_{\mu\nu} = 0, \qquad \tilde{g}^{\mu}_{\mu} = -2\epsilon \xrightarrow[d\to 4]{} 0, \qquad g^{\mu}_{\mu} = 4 \qquad \tilde{q}^2 = \tilde{g}^{\mu\nu}\bar{q}_\mu \bar{q}_\nu = -\mu^2$$

and the Clifford Algebra

$$[\tilde{\gamma}^\alpha, \gamma^5] = 0, \qquad \{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\,\tilde{g}^{\alpha\beta}, \qquad \{\tilde{\gamma}^\alpha, \gamma^\mu\} = 0.$$

Extend FDH by using 4D-objects only

In 4-dimensions, one can infer: $\tilde{\gamma} \sim \gamma^5$

And the Clifford algebra $\tilde{\gamma}^\mu \tilde{\gamma}_\mu \xrightarrow[d\to 4]{} 0 \quad \text{while} \quad \gamma^5\gamma^5 = 1$

Excludes any four-dimensional representation of the —2ε-subspace

—2ε-subspace $\longrightarrow$ **—2ε-Selection Rules (—2ε)-SRs**

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

FDH: 4D helicity scheme

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-2ε-Selection Rules

The Clifford algebra conditions are satisfied by imposing

$$\tilde{g}^{\alpha\beta} \to G^{AB}, \qquad \tilde{\ell}^\alpha \to i\,\mu\,Q^A\,, \qquad \tilde{\gamma}^\alpha \to \gamma^5\,\Gamma^A\,.$$

A,B := —2ε-dimensional vectorial indices traded for (—2ε)-SRs

$$\begin{aligned} G^{AB}G^{BC} &= G^{AC}, & G^{AA} &= 0, & G^{AB} &= G^{BA}, \\ \Gamma^A G^{AB} &= \Gamma^B, & \Gamma^A\Gamma^A &= 0, & Q^A G^A &= 1, \\ Q^A G^{AB} &= Q^B, & Q^A Q^A &= 1. \end{aligned}$$

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

Completeness relations within FDF

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

Gluon propagator

The helicity sum of the transverse polarisation vector is

$$\sum_{i=1}^{d-2} \varepsilon^{\mu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right)\varepsilon^{*\nu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right) = \underbrace{\left(-g^{\mu\nu}+\frac{\ell^{\mu}\ell^{\nu}}{\mu^2}\right)}_{d=4} - \underbrace{\left(\tilde{g}^{\mu\nu}+\frac{\tilde{\ell}^{\mu}\tilde{\ell}^{\nu}}{\mu^2}\right)}_{d=-2\varepsilon}.$$

d = 4 **d = -2ε**

massive gluon

$$\left(-g^{\mu\nu}+\frac{l^{\mu}l^{\nu}}{\mu^2}\right) = \sum_{\lambda=\pm,0} \varepsilon^{\mu}_{\lambda}(l)\,\varepsilon^{*\nu}_{\lambda}(l)$$

$$\left(\tilde{g}^{\mu\nu}+\frac{\tilde{l}^{\mu}\tilde{l}^{\nu}}{\mu^2}\right) \longrightarrow \hat{G}^{AB} = G^{AB} - Q^A Q^B$$

Fermion propagator

$$\sum_{\lambda=\pm} u_{\lambda}(l)\,\bar{u}_{\lambda}(l) = \not{l} + i\mu\gamma^5 + m$$

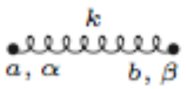
$$\sum_{\lambda=\pm} v_{\lambda}(l)\,\bar{v}_{\lambda}(l) = \not{l} + i\mu\gamma^5 - m$$

Allows to generalise the Dirac Equation

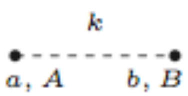
$$\left(\not{\ell} + i\mu\gamma^5 + m\right) u_{\lambda}(\ell) = 0\,, \quad \ell^2 = m^2+\mu^2\,, \quad \ell = \ell^{\flat} + \frac{m^2+\mu^2}{2\,\ell\cdot q_{\ell}} q_{\ell}\,, \quad (\ell^{\flat})^2 = (q_{\ell})^2 = 0\,.$$

Feynman Rules in FDF

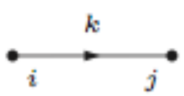
[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]



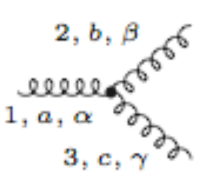
$$= -i\,\delta^{ab}\,\frac{1}{k^2-\mu^2+i0}\left[g^{\alpha\beta}-\frac{k^\alpha k^\beta}{\mu^2}\right] \qquad \text{(gluon)},$$



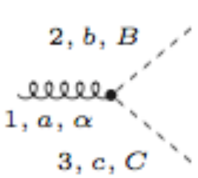
$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k^2-\mu^2+i0}, \qquad \text{(scalar)},$$



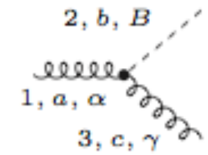
$$= i\,\delta^{ij}\,\frac{\not{k}+i\mu\gamma^5+m}{k^2-m^2-\mu^2+i0}, \qquad \text{(fermion)},$$



$$= -g\,f^{abc}\left[(k_1-k_2)^\gamma g^{\alpha\beta} + (k_2-k_3)^\alpha g^{\beta\gamma} + (k_3-k_1)^\beta g^{\gamma\alpha}\right],$$

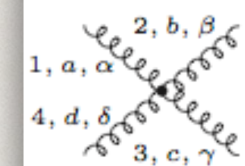


$$= -g\,f^{abc}\,(k_2-k_3)^\alpha\,G^{BC},$$

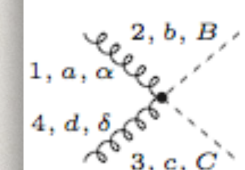


$$= \mp g\,f^{abc}\,(i\mu)\,g^{\gamma\alpha}\,Q^B$$

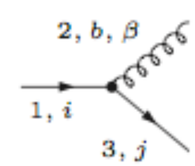
$$(\tilde{k}_1 = 0, \quad \tilde{k}_3 = \pm\tilde{\ell}),$$



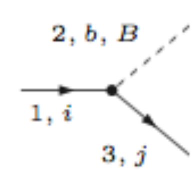
$$= -ig^2\Big[f^{xad}\,f^{xbc}\,\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\beta\delta}\right) + f^{xac}\,f^{xbd}\,\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\delta}g^{\beta\gamma}\right) + f^{xab}\,f^{xdc}\,\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}\right)\Big],$$



$$= 2ig^2\,g^{\alpha\delta}\,\left(f^{xab}\,f^{xcd}+f^{xac}\,f^{xbd}\right)\,G^{BC},$$



$$= -ig\,\left(t^b\right)_{ji}\,\gamma^\beta,$$

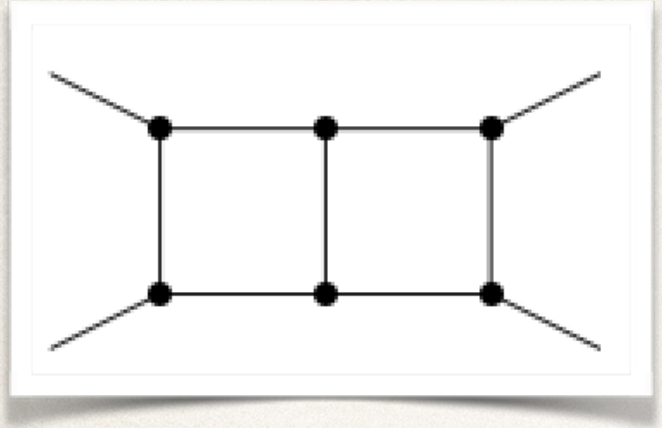


$$= -ig\,\left(t^b\right)_{ji}\,\gamma^5\,\Gamma^B.$$

Results

- 4-gluons amplitudes **[Bern and Kosower (1992)]**
- Annihilation of quark & antiquark in two gluons **[Kunszt, Signer and Trocsanyi (1993)]**
- Higgs + 3-gluon amplitudes **[Schmidt (1997)]**
- 5-gluon amplitudes **[Njet]**
- 6-gluon amplitudes **[Njet]**
- Higgs + 4-gluon amplitudes **[Badger, Glover, Mastrolia, Williams (2009)]**
- Higgs + 5-gluon amplitudes (preliminary results) **[GoSam]**

What about multi-loop level?



$$k_i \cdot p_j, k_i \cdot \varepsilon_j, k_i \cdot k_j, \mu_{ij} = -\tilde{k}_i \cdot \tilde{k}_j$$

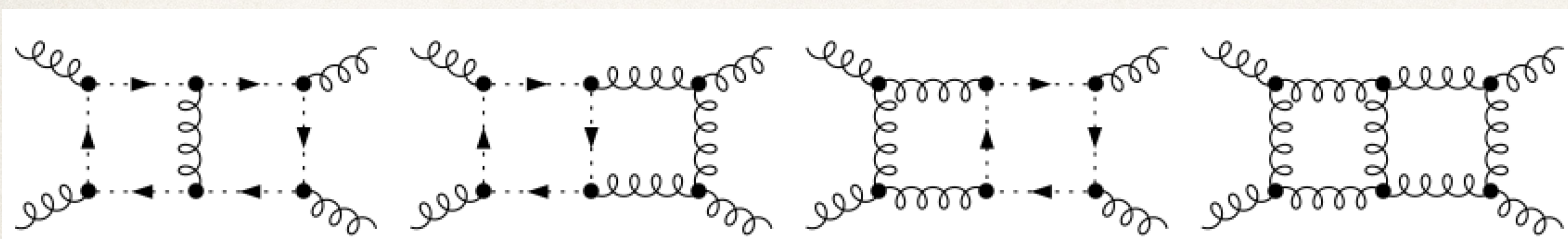
$$A_n^{(2),[D]}(\{p\}) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{N(\{k\},\{p\})}{\prod_{l=1}^{7} D_l(\{k_i\},\{p\})}$$

Write everything in terms of
Irreducible **S**calar **P**roducts

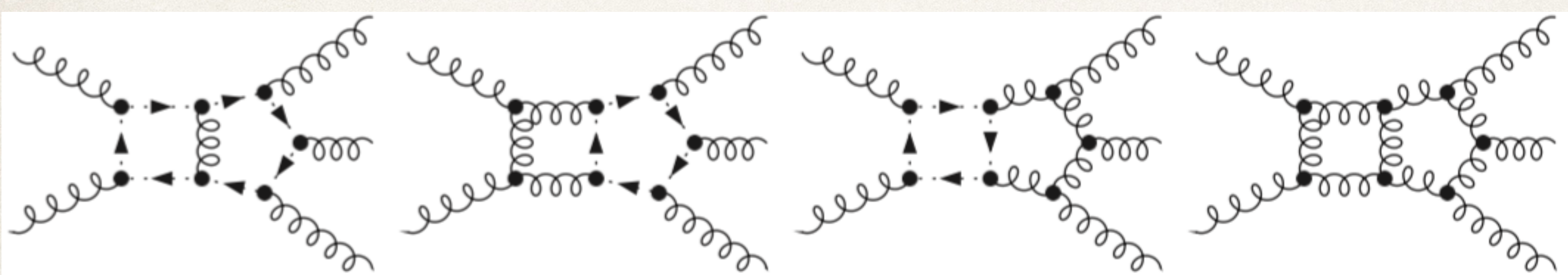
Diagrammatic approach

[Mastrolia, Peraro, Primo (2016)]
[Mastrolia, Peraro, Primo, W.J.T. (2016)]

More details in A. Primo's Talk



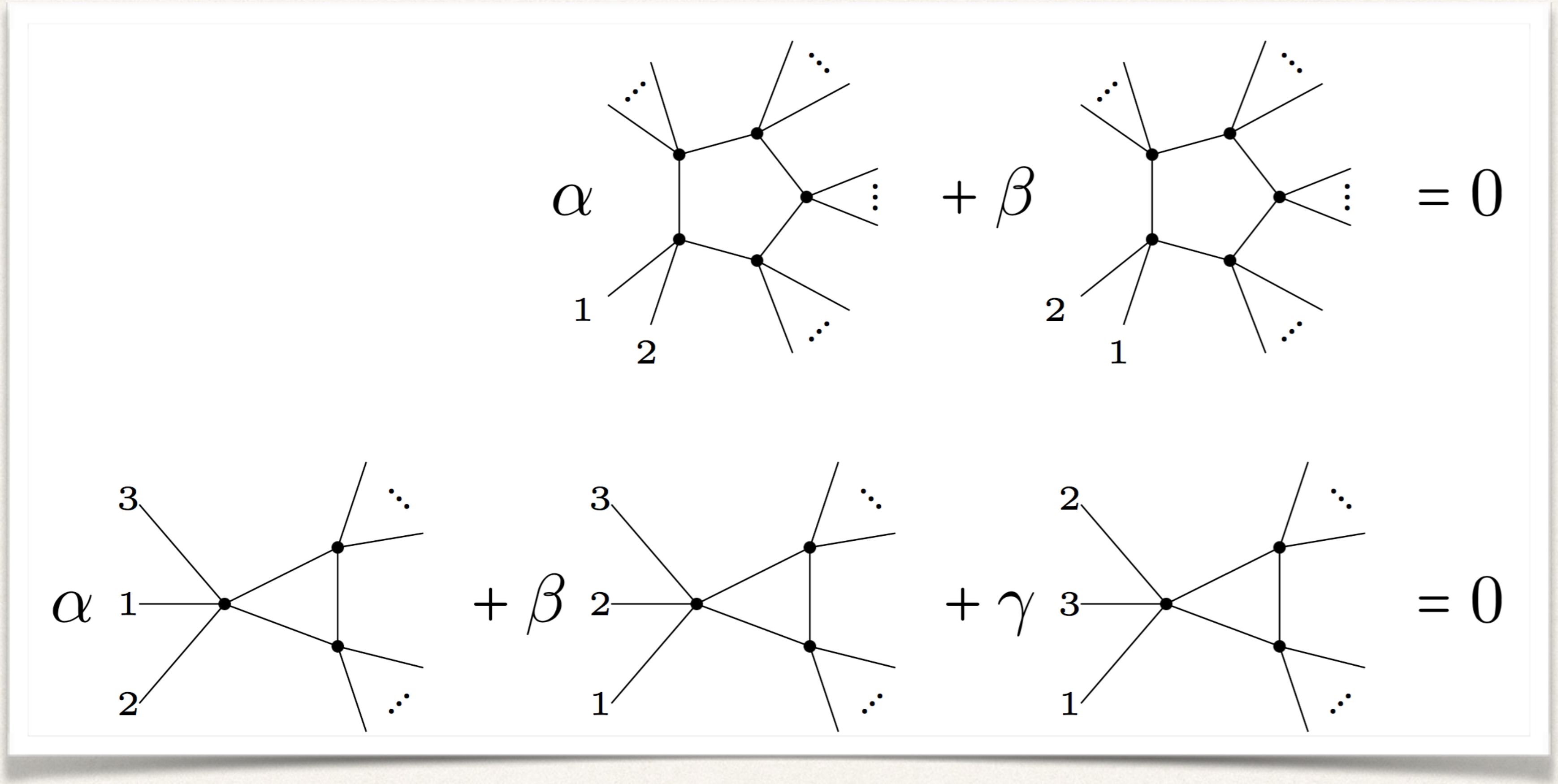
[Bern, Dixon, Kosower (2000)]



[Badger, Frellesvig, Zhang (2013)]

Further simplifications from colour/kinematics duality

At integrand level,



Generalised Unitarity and C/K duality **dance** together.

Which "gauge" theories obey C-K duality

- Pure $\mathcal{N}$=0,1,2,4 super-Yang-Mills (any dimension) — Bern, Carrasco, HJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Stieberger; Feng et al. Mafra, Schlotterer, etc ('08-'11)
- Self-dual Yang-Mills theory O'Connell, Monteiro ('11)
- Heterotic string theory Stieberger, Taylor ('14)
- Yang-Mills + F^3 theory Broedel, Dixon ('12)
- QCD, super-QCD, higher-dim QCD HJ, Ochirov ('15)
- Generic matter coupled to $\mathcal{N}$= 0,1,2,4 super-Yang-Mills — Chiodaroli, Gunaydin, Roiban; HJ, Ochirov ('14)
- Spontaneously broken $\mathcal{N}$= 0,2,4 SYM Chiodaroli, Gunaydin, HJ, Roiban ('15)
- Yang-Mills + scalar ϕ^3 theory Chiodaroli, Gunaydin, HJ, Roiban ('14)
- Bi-adjoint scalar ϕ^3 theory — Bern, de Freitas, Wong ('99), Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell
- NLSM/Chiral Lagrangian Chen, Du ('13)
- D=3 Bagger-Lambert-Gustavsson theory (Chern-Simons-matter) Bargheer, He, McLoughlin; Huang, HJ, Lee ('12 -'13)

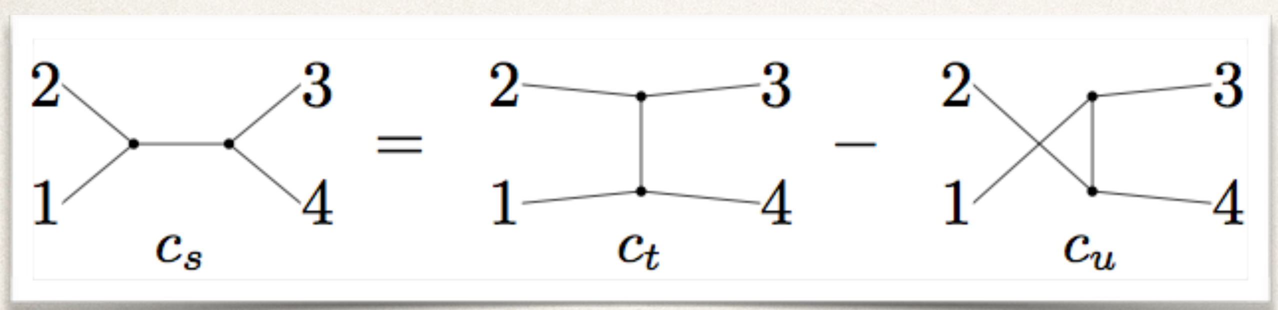
Colour-kinematics duality

- Colour-kinematics duality strong relation gravity amplitudes and Yang-Mills amplitudes **[Bern, Carrasco, Johansson (2008),(2010)]**
- Write QCD amplitudes in terms of cubic graphs

$$\mathcal{A}_n = g^{n-2} \sum \frac{n_i c_i}{D_i}$$

  - Colour factors $c_i \sim f^{abc} f^{ced}$
  - Kinematic factors $n_i \sim (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_3)(\varepsilon_3 \cdot \varepsilon_4) + \ldots$

Jacobi Relation (colour)

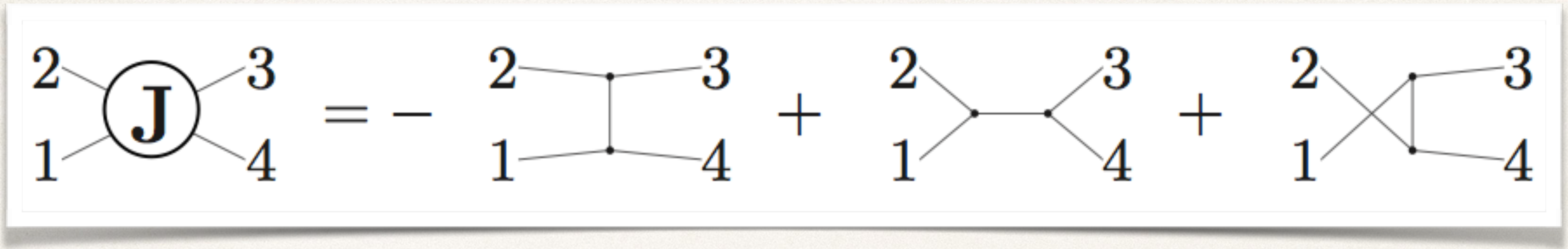


$$c_s = c_t - c_u$$
$$f^{a_1 a_2 b} f^{a_3 a_4 b} = f^{a_4 a_1 b} f^{a_2 a_3 b} - f^{a_1 a_3 b} f^{a_2 a_4 b}$$
$$f^{a_1 a_2 b} T^b = T^{a_1} T^{a_2} - T^{a_2} T^{a_1}$$

— Satisfied automatically for 4-point tree amplitudes $n_s = n_t - n_u$

Off-shell Colour-kinematics duality

Consider a tensor as the Jacobi identity of numerators



Four-gluon identity $N_g^{\text{tree}} = J^{\mu_1\cdots\mu_4}\varepsilon_{\mu_1}(p_1)\varepsilon_{\mu_2}(p_2)\varepsilon_{\mu_3}(p_3)\varepsilon_{\mu_4}(p_4),$

$$
\begin{aligned}
N_g^{\text{tree}} = \varepsilon\left(p_1\right)\cdot p_1 [& \left(\varepsilon\left(p_2\right)\cdot p_1 + 2\varepsilon\left(p_2\right)\cdot p_4\right)\varepsilon\left(p_3\right)\cdot\varepsilon\left(p_4\right) \\
& - \varepsilon\left(p_2\right)\cdot\varepsilon\left(p_4\right)\left(\varepsilon\left(p_3\right)\cdot p_1 + 2\varepsilon\left(p_3\right)\cdot p_4\right) \\
& + \varepsilon\left(p_2\right)\cdot\varepsilon\left(p_3\right)\left(\varepsilon\left(p_4\right)\cdot p_1 + 2\varepsilon\left(p_4\right)\cdot p_3\right)] \\
& \qquad + \text{cyclic permutations.}
\end{aligned}
$$

[Zhu (1980)]

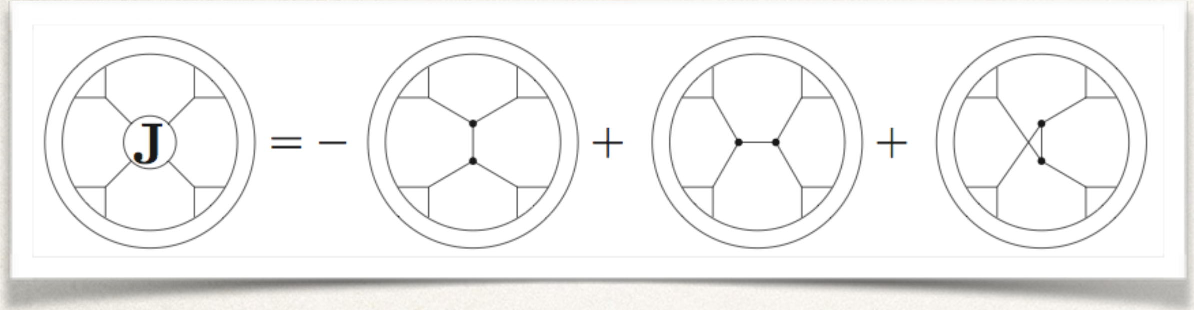
$N_g^{\text{tree}} = 0$

by imposing Momentum Conservation and Transversality condition.

Off-shell Colour-kinematics duality

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

- At multi-loop level or higher-points



External particles become internal

$$u\left(p_i\right), v\left(p_i\right) \rightarrow \not{p}_i$$

$$\varepsilon^{\mu_i}\left(p_i ; q_i\right) \rightarrow \Pi^{\mu_i \nu_i}\left(p_i ; q_i\right)$$

Propagator in axial gauge

- Numerator built from the J-block is decomposed in terms of squared momenta

$$\left(N_{\mathrm{g}}^{\mathrm{loop}}\right)_{\alpha_1 \ldots \alpha_4} = J^{\mu_1 \ldots \mu_4} \Pi_{\mu_1 \alpha_1}(p_1, q_1)\, \Pi_{\mu_2 \alpha_2}(p_2, q_2)\, \Pi_{\mu_3 \alpha_3}(p_3, q_3)\, \Pi_{\mu_4 \alpha_4}(p_4, q_4)\,,$$

$$\left(N_{\mathrm{g}}^{\mathrm{loop}}\right)_{\alpha_1 \ldots \alpha_4} = \sum_{i=1}^{4} p_i^2 (A_g^i)_{\alpha_1 \ldots \alpha_4} + \sum_{\substack{i,j=1 \\ i \neq j}}^{4} p_i^2 p_j^2 (C_g^{ij})_{\alpha_1 \ldots \alpha_4}\,.$$

$$A_g = A_g(\{p_i\})$$
$$C_g = C_g(\{p_i\})$$

Off-shell Colour-kinematics duality

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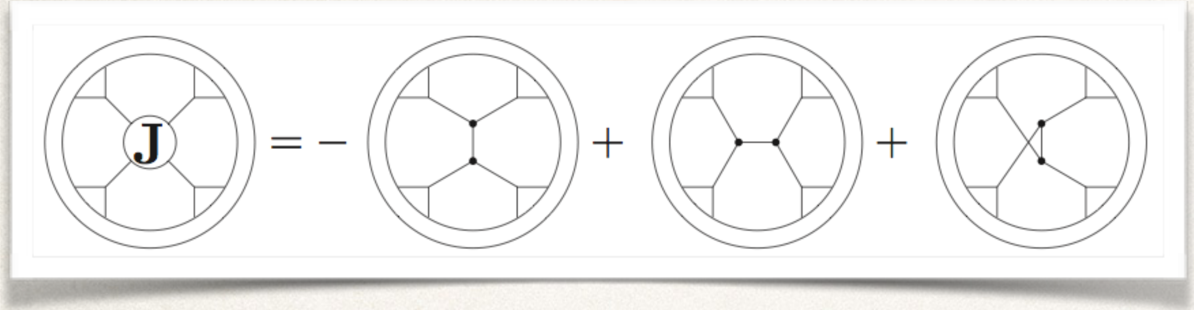
- At multi-loop level or higher-points



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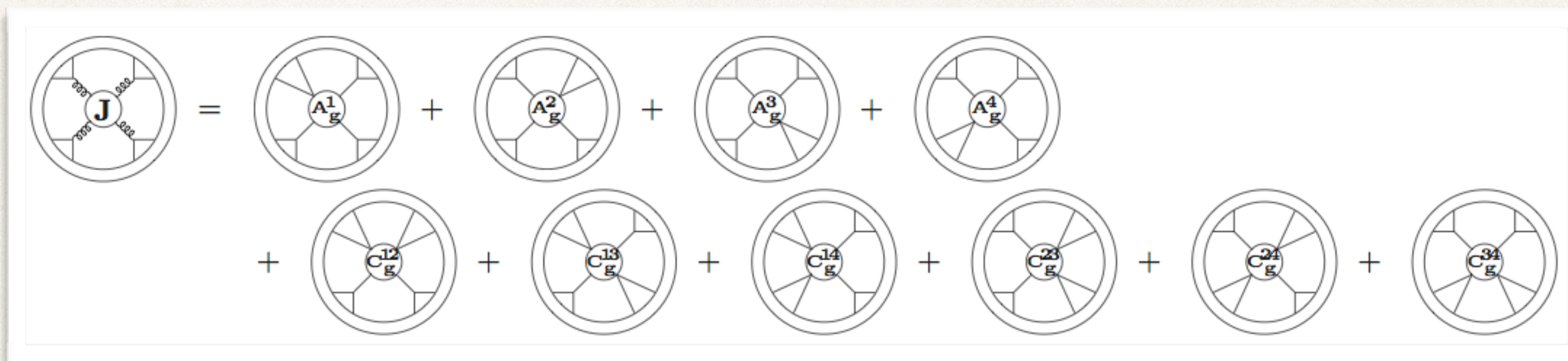
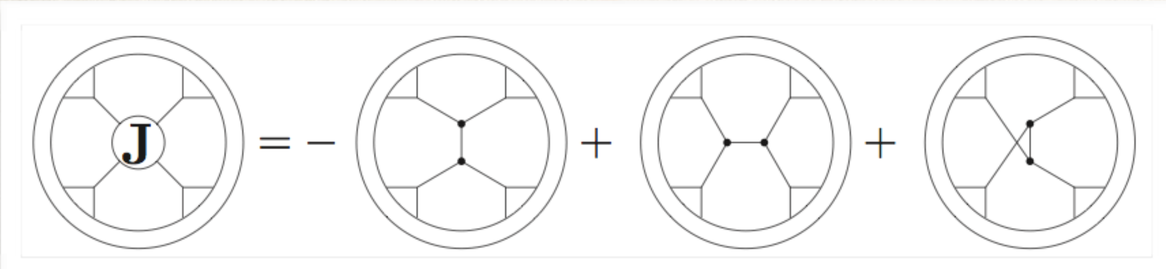


- Any loop diagram built from the J-block can be written as the sum of diagrams with one or two propagators less.

Off-shell Colour-kinematics duality

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

- At multi-loop level or higher-points



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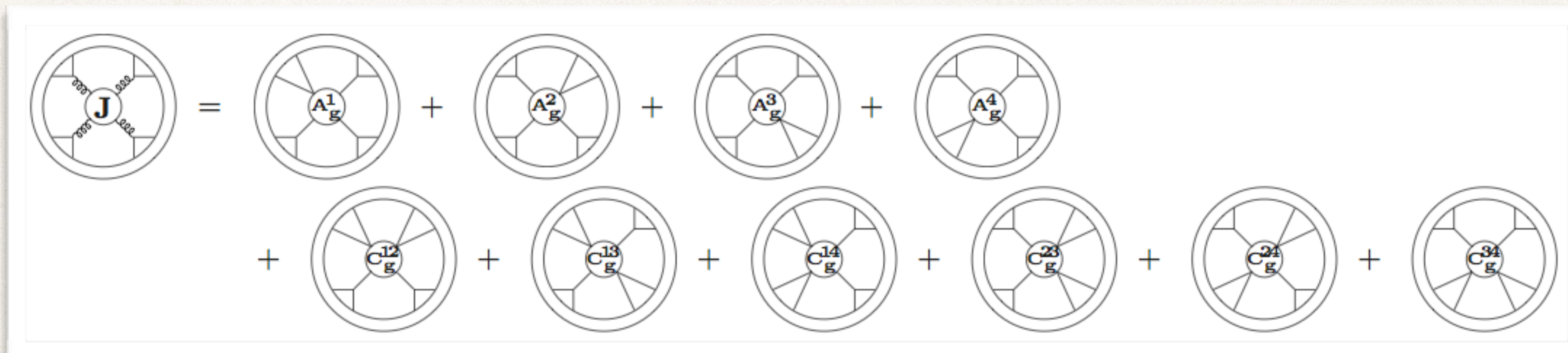
Propagator in axial gauge

- Numerator built from the J-block is decomposed in terms of squared momenta



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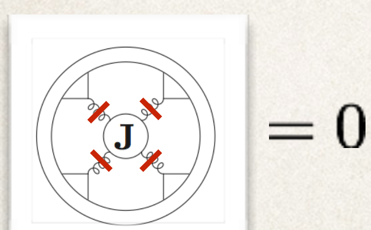
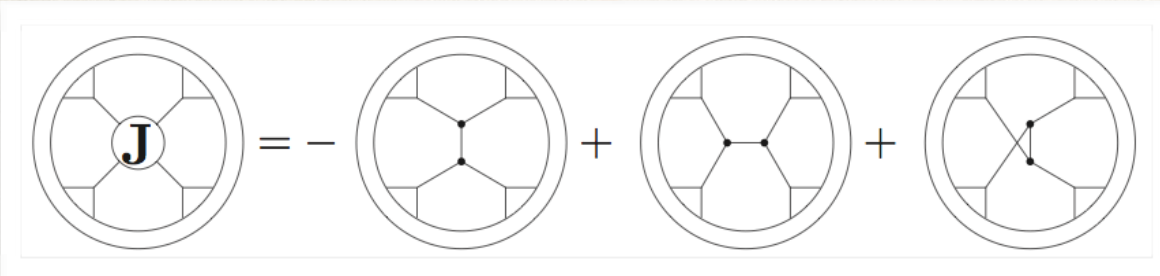


- By imposing on-shellness of the four particles

Off-shell Colour-kinematics duality

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

- At multi-loop level or higher-points



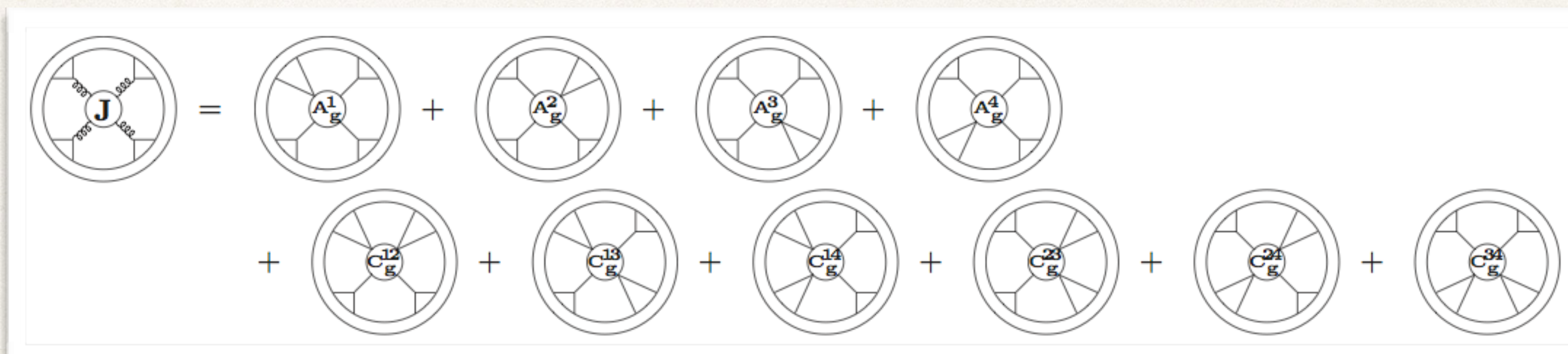
External particles become internal

$$u\left(p_i\right), v\left(p_i\right) \to \not{p}_i$$

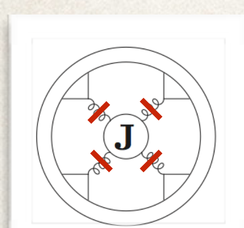
$$\varepsilon^{\mu_i}\left(p_i\,;q_i\right) \to \Pi^{\mu_i\nu_i}\left(p_i\,;q_i\right)$$

Propagator in axial gauge

- Numerator built from the J-block is decomposed in terms of squared momenta



- Any loop diagram built from the J-block can be written as the sum of diagrams with one or two propagators less.



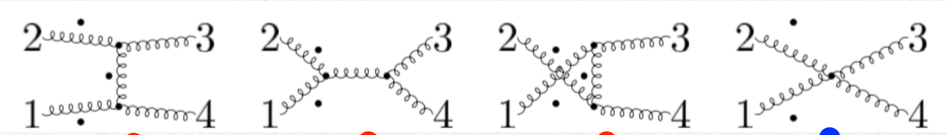
$= 0$

- By imposing on-shellness of the four particles

- Colour-kinematics duality is also manifest for d-dimensional regulated amplitudes —> **Novel approach w/in FDF**

C/K relations @ tree-level in DimReg w/in FDF

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

Consider the 4-point amplitude 

and the Jacobi identity

$$-c_1 + c_2 + c_3 = 0.$$

$$\mathcal{A}_4(p_1,p_2,p_3,p_4) = c_1 \frac{n_1}{P_{23}^2 - \mu^2} + c_2 \frac{n_2}{P_{12}^2} + c_3 \frac{n_3}{P_{24}^2 - \mu^2}$$

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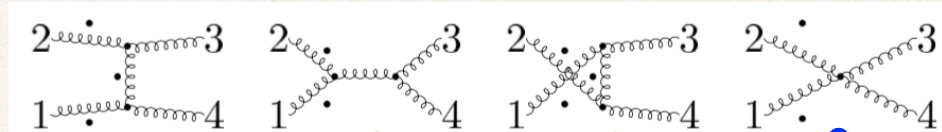

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Solving for c_2

$$\mathcal{A}_4(p_1,p_2,p_3,p_4) = c_1 K_1 + c_3 K_3$$

being

$$K_1 = \frac{n_1}{P_{23}^2 - \mu^2} + \frac{n_2}{P_{12}^2}, \qquad K_3 = \frac{n_3}{P_{24}^2 - \mu^2} - \frac{n_2}{P_{12}^2}.$$

Kinematic numerators obey Jacobi identity

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Colour-ordered amplitudes

$$K_1 = A(1,2,3,4) \qquad K_3 = A(2,1,3,4)$$

William J. Torres Bobadilla

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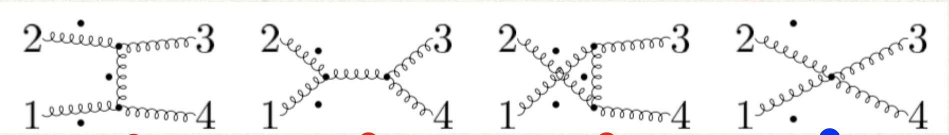

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$$K_1 = A(1,2,3,4) \qquad K_3 = A(2,1,3,4)$$

$$\boxed{\begin{pmatrix} \frac{1}{P_{23}^2-\mu^2} & \frac{1}{P_{12}^2} & 0 \\ 0 & -\frac{1}{P_{12}^2} & \frac{1}{P_{24}^2-\mu^2} \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_3 \\ 0 \end{pmatrix}}$$

C/K relations @ tree-level in DimReg w/in FDF

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

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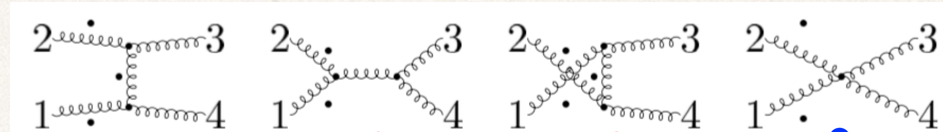

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4-pt C/K-relations

$$A(2, 1, 3, 4) = \frac{P_{23}^2 - \mu^2}{P_{24}^2 - \mu^2} A(1, 2, 3, 4).$$

C/K relations @ tree-level in DimReg w/in FDF

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

As well, for the 5-point

$$A_5(1,3,4,2,5) = \frac{-P_{12}^2 P_{45}^2 A_5(1,2,3,4,5) + (P_{14}^2 - \mu^2)(P_{24}^2 + P_{25}^2 - 2\mu^2) A_5(1,4,3,2,5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)},$$

$$A_5(1,2,4,3,5) = \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1,4,3,2,5) + P_{45}^2 (P_{12}^2 + P_{24}^2 - \mu^2) A_5(1,2,3,4,5)}{P_{35}^2 (P_{24}^2 - \mu^2)},$$

$$A_5(1,4,2,3,5) = \frac{-P_{12}^2 P_{45}^2 A_5(1,2,3,4,5) + (P_{25}^2 - \mu^2)(P_{14}^2 + P_{25}^2 - 2\mu^2) A_5(1,4,3,2,5)}{P_{35}^2 (P_{24}^2 - \mu^2)},$$

$$A_5(1,3,2,4,5) = \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1,4,3,2,5) + P_{12}^2 (P_{24}^2 + P_{45}^2 - \mu^2) A_5(1,2,3,4,5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)}.$$

Making use of the photon decoupling identity

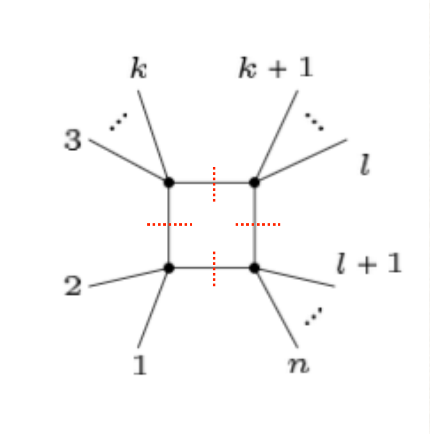
$$A_5(1,2,4,3,5) = \frac{(P_{14}^2 + P_{45}^2 - \mu^2) A_5(1,2,3,4,5) + (P_{14}^2 - \mu^2) A_5(1,2,3,5,4)}{(P_{24}^2 - \mu^2)}$$

C/K relations @ 1-loop

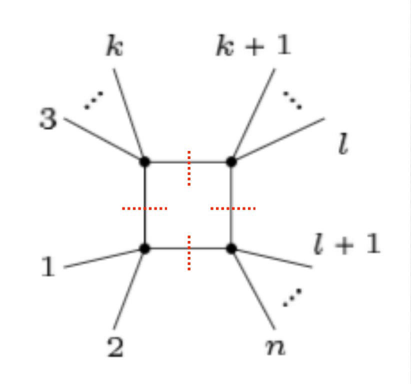
[Primo, W.J.T. (2016)]

Inspired by the generalised unitarity

$$C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = A_4^{\text{tree}}\left(-l_1^{\pm}, 1, 2, l_3^{\pm}\right) A_k^{\text{tree}}\left(-l_3^{\pm}, P_{3\cdots k}, l_{k+1}^{\pm}\right) \times A^{\text{tree}}_{l-k+2}\left(-l^{\pm}_{k+1}, P_{k+1\ldots,l}, l^{\pm}_{l+1}\right) A^{\text{tree}}_{n-l+2}\left(-l^{\pm}_{l+1}, P_{l+1\ldots,n}, l_1^{\pm}\right)$$



$$C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n}.$$



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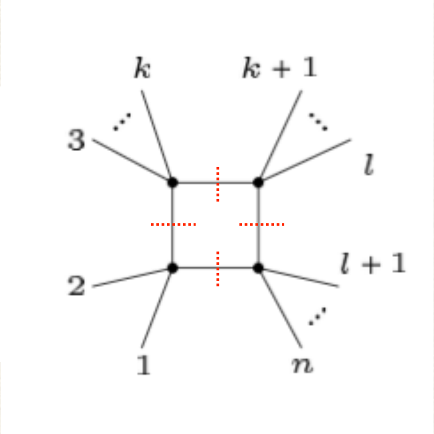
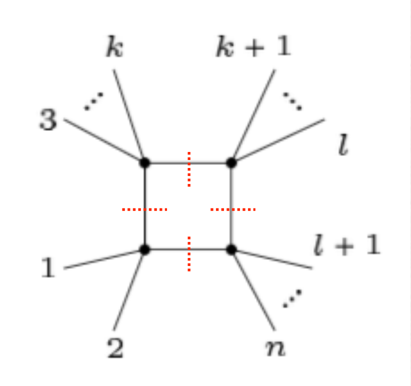


$$C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n}.$$



C/K relation

$$A(2,1,3,4) = \frac{P_{23}^2 - \mu^2}{P_{24}^2 - \mu^2} A(1,2,3,4).$$

— One-loop amplitudes in N=4 sYM

[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)]

— Cut constructible part of One-loop QCD amplitudes [Chester (2016)]

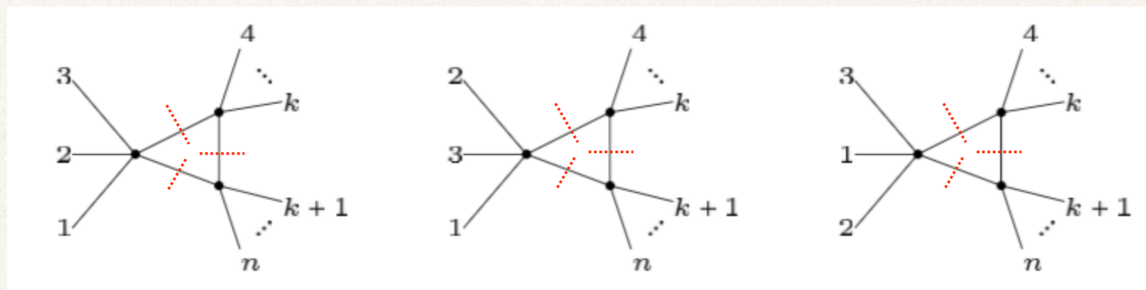
— One-loop QCD amplitudes [Primo, W.J.T. (2016)]

C/K relations @ 1-loop

[Primo, W.J.T. (2016)]

Same behaviour for lower topologies

$$C^{\pm}_{123|4\ldots k|(k+1)\ldots n} = A_5^{\text{tree}}\left(-l_1^{\pm},1,2,3,l_4^{\pm}\right) A_{k-1}^{\text{tree}}\left(-l_4^{\pm},P_{4\cdots k},l_{k+1}^{\pm}\right) A_{n-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm},P_{k+1\ldots,n},l_1^{\pm}\right)$$



$$C^{\pm}_{213|4\ldots k|(k+1)\ldots n} = \frac{\left(P^2_{l_4^{\pm}2}+P^2_{23}-\mu^2\right)C^{\pm}_{123|4\ldots k|(k+1)\ldots n}+\left(P^2_{l_4^{\pm}2}-\mu^2\right)C^{\pm}_{132|4\ldots k|(k+1)\ldots n}}{\left(P^2_{-l_1^{\pm}2}-\mu^2\right)}$$

due to

$$A_5(1,2,4,3,5) = \frac{(P^2_{14}+P^2_{45}-\mu^2)A_5(1,2,3,4,5)+(P^2_{14}-\mu^2)A_5(1,2,3,5,4)}{(P^2_{24}-\mu^2)}$$

Unitarity + C/K-relations @ work

[Mastrolia, Primo, W.J.T. (in progress)]

- Target :: Reduce the number of independent residues needed to compute any colour-dressed one-loop amplitude

$$A_n^{\text{1-loop}} = \int d^d\bar{l}\,\frac{\mathcal{N}(l,\mu^2)}{D_0 D_1 \ldots D_{n-1}},$$

$$D_i = (\bar{l}+p_i)^2 - m_i^2 = (l+p_i)^2 - m_i^2 - \mu^2.$$

$$\frac{N(l,\mu^2)}{D_0 D_1 \ldots D_{n-1}} = \sum_{i\ll m}^{n-1} \frac{\Delta_{ijklm}(l,\mu^2)}{D_i D_j D_k D_l D_m} + \sum_{i\ll l}^{n-1} \frac{\Delta_{ijkl}(l,\mu^2)}{D_i D_j D_k D_l} + \sum_{i\ll k}^{n-1} \frac{\Delta_{ijk}(l,\mu^2)}{D_i D_j D_k} + \sum_{i<j}^{n-1} \frac{\Delta_{ij}(l,\mu^2)}{D_i D_j} + \sum_{i}^{n-1} \frac{\Delta_{i}(l,\mu^2)}{D_i},$$

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$$D_i = (\bar{l}+p_i)^2 - m_i^2 = (l+p_i)^2 - m_i^2 - \mu^2.$$

- Ingredients :: Residues @cut —> Keep under control their polynomial structure

$$\begin{aligned}
\Delta_{ijklm} &= c\mu^2, \\
\Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 \mu^2 + c_3 x_4 \mu^2 + c_4 \mu^4, \\
\Delta_{ijk} &= c_{0,0} + c^+_{1,0} x_4 + c^+_{2,0} x_4^2 + c^+_{3,0} x_4^3 + c^-_{1,0} x_3 + c^-_{2,0} x_3^2 + c^-_{3,0} x_3^3 + c_{0,2}\mu^2 + c^+_{1,2} x_4 \mu^2 + c^-_{1,2} x_3 \mu^2, \\
\Delta_{ij} &= c_{0,0,0} + c_{0,1,0} x_1 + c_{0,2,0} x_1^2 + c^+_{1,0,0} x_4 + c^+_{2,0,0} x_4^2 + c^-_{1,0,0} x_3 + c^-_{2,0,0} x_3^2 + c^+_{1,1,0} x_1 x_4 \\
&\quad + c^-_{1,1,0} x_1 x_3 + c_{0,0,2}\mu^2, \\
\Delta_i &= c_{0,0,0,0} + c_{0,1,0,0} x_1 + c_{0,0,1,0} x_2 + c^-_{1,0,0,0} x_3 + c^+_{1,0,0,0} x_4,
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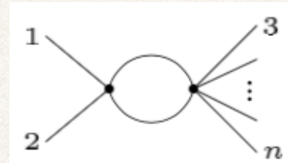
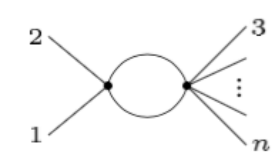
\Delta_{ij} &= c_{0,0,0} + c_{0,1,0} x_1 + c_{0,2,0} x_1^2 + c^+_{1,0,0} x_4 + c^+_{2,0,0} x_4^2 + c^-_{1,0,0} x_3 + c^-_{2,0,0} x_3^2 + c^+_{1,1,0} x_1 x_4 \\
&\quad + c^-_{1,1,0} x_1 x_3 + c_{0,0,2}\mu^2, \\
\Delta_i &= c_{0,0,0,0} + c_{0,1,0,0} x_1 + c_{0,0,1,0} x_2 + c^-_{1,0,0,0} x_3 + c^+_{1,0,0,0} x_4,
\end{aligned}$$

- Procedure :: C/K-relations @work —> Generate a system of equations that relates residues of different ordering through C/K-relations



$$= \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2}$$



Unitarity + C/K-relations @ work

[Mastrolia, Primo, W.J.T. (in progress)]

- Target :: Reduce the number of independent residues needed to compute any colour-dressed one-loop amplitude

$$A_n^{\text{1-loop}} = \int d^d\bar{l}\,\frac{\mathcal{N}(l,\mu^2)}{D_0 D_1 \dots D_{n-1}},$$

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- Ingredients :: Residues @cut —> Keep under control their polynomial structure

$$\Delta_{ijklm} = c\mu^2,$$
$$\Delta_{ijkl} = c_0 + c_1 x_4 + c_2\mu^2 + c_3 x_4 \mu^2 + c_4 \mu^4,$$
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$$\Delta_{ij} = c_{0,0,0} + c_{0,1,0}x_1 + c_{0,2,0}x_1^2 + c^+_{1,0,0}x_4 + c^+_{2,0,0}x_4^2 + c^-_{1,0,0}x_3 + c^-_{2,0,0}x_3^2 + c^+_{1,1,0}x_1x_4 + c^-_{1,1,0}x_1x_3 + c_{0,0,2}\mu^2,$$
$$\Delta_i = c_{0,0,0,0} + c_{0,1,0,0}x_1 + c_{0,0,1,0}x_2 + c^-_{1,0,0,0}x_3 + c^+_{1,0,0,0}x_4,$$

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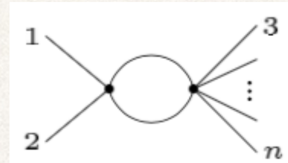
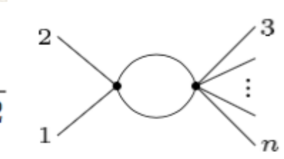


$$= \frac{P^2_{l_3^\pm 2} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2}$$



— The solution of the system gives us a reduce set of independent residues

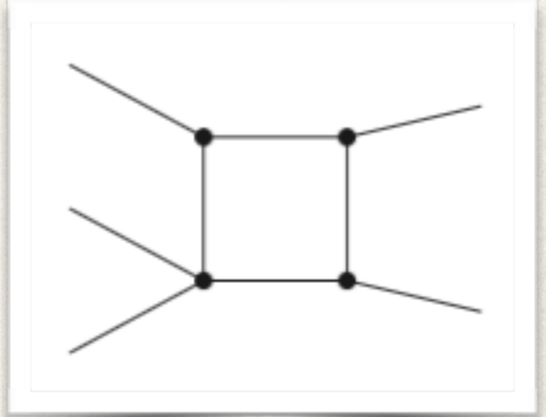
— Unitarity @work —> Compute the independent residues through Unitarity Based Methods

$$\Delta^{(13\ldots)} \equiv \sum_{l_i\in\mathcal{S}} A_4(-l_1,1,2,l_2)\times A(\ldots)\times\cdots\times A(\ldots),$$

Unitarity + C/K-relations @ work

[Mastrolia, Primo, W.J.T. (in progress)]

5pt one-mass-boxes



Start with 120 residues

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William J. Torres Bobadilla

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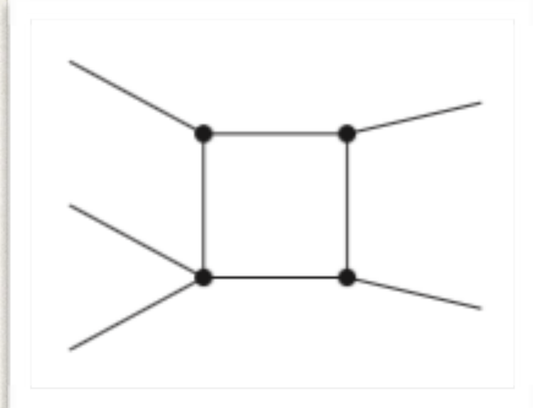
5pt one-mass-boxes



Start with 120 residues

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- **-60 residues** because of reflection.
  At the intengrand level this is equivalent to performing the shift $l_i \to -l_{i+1}$.

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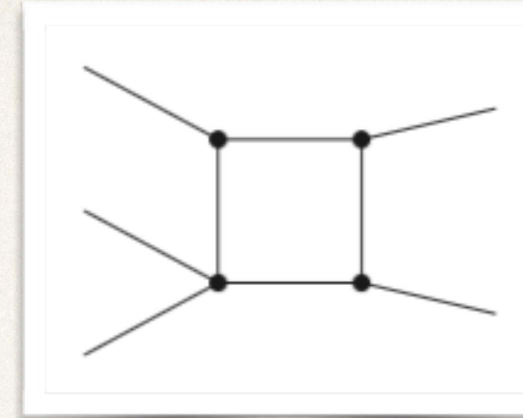

5pt one-mass-boxes

Start with 120 residues

{Δ[{{1, 2}, 3, 4, 5}], Δ[{{1, 2}, 3, 5, 4}], ~~Δ[{{1, 2}, 4, 3, 5}]~~, Δ[{{1, 2}, 4, 5, 3}], ~~Δ[{{1, 2}, 5, 3, 4}]~~, ~~Δ[{{1, 2}, 5, 4, 3}]~~, Δ[{{1, 3}, 2, 4, 5}], Δ[{{1, 3}, 2, 5, 4}],
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- **-60 residues** because of reflection. At the intengrand level this is equivalent to performing the shift $l_i \to -l_{i+1}$.
- **-30 residues** because of C/K-relations

$$\Delta(\{2,1\},3,4,5) = \frac{(l_1-1)^2-\mu^2}{(l_1-2)^2-\mu^2}\Delta(\{1,2\},3,4,5),$$

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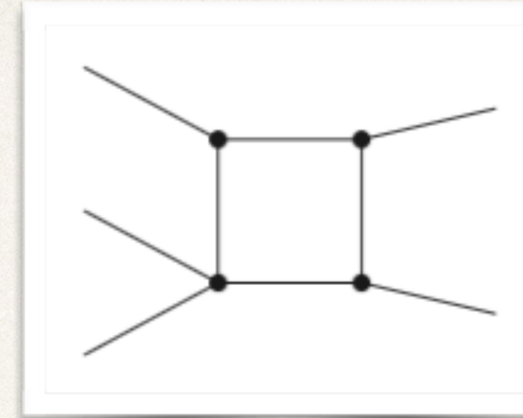

Start with 120 residues

{Δ[{{1, 2}, 3, 4, 5}], Δ[{{1, 2}, 3, 5, 4}], ~~Δ[{{1, 2}, 4, 3, 5}]~~, Δ[{{1, 2}, 4, 5, 3}], ~~Δ[{{1, 2}, 5, 3, 4}]~~, ~~Δ[{{1, 2}, 5, 4, 3}]~~, Δ[{{1, 3}, 2, 4, 5}], Δ[{{1, 3}, 2, 5, 4}],
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~~Δ[{{5, 1}, 2, 3, 4}]~~, ~~Δ[{{5, 1}, 2, 4, 3}]~~, ~~Δ[{{5, 1}, 3, 2, 4}]~~, ~~Δ[{{5, 1}, 3, 4, 2}]~~, ~~Δ[{{5, 1}, 4, 2, 3}]~~, ~~Δ[{{5, 1}, 4, 3, 2}]~~, ~~Δ[{{5, 2}, 1, 3, 4}]~~, ~~Δ[{{5, 2}, 1, 4, 3}]~~,
~~Δ[{{5, 2}, 3, 1, 4}]~~, ~~Δ[{{5, 2}, 3, 4, 1}]~~, ~~Δ[{{5, 2}, 4, 1, 3}]~~, ~~Δ[{{5, 2}, 4, 3, 1}]~~, ~~Δ[{{5, 3}, 1, 2, 4}]~~, ~~Δ[{{5, 3}, 1, 4, 2}]~~, ~~Δ[{{5, 3}, 2, 1, 4}]~~, ~~Δ[{{5, 3}, 2, 4, 1}]~~,
~~Δ[{{5, 3}, 4, 1, 2}]~~, ~~Δ[{{5, 3}, 4, 2, 1}]~~, ~~Δ[{{5, 4}, 1, 2, 3}]~~, ~~Δ[{{5, 4}, 1, 3, 2}]~~, ~~Δ[{{5, 4}, 2, 1, 3}]~~, ~~Δ[{{5, 4}, 2, 3, 1}]~~, ~~Δ[{{5, 4}, 3, 1, 2}]~~, ~~Δ[{{5, 4}, 3, 2, 1}]~~}

- **-60 residues** because of reflection. At the intengrand level this is equivalent to performing the shift $l_i \to -l_{i+1}$.
- **-30 residues** because of C/K-relations

$$\Delta(\{2,1\},3,4,5) = \frac{(l_1-1)^2-\mu^2}{(l_1-2)^2-\mu^2}\Delta(\{1,2\},3,4,5),$$

End up with **30 independent residues**

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Results

Inspired by Momentum Twistors

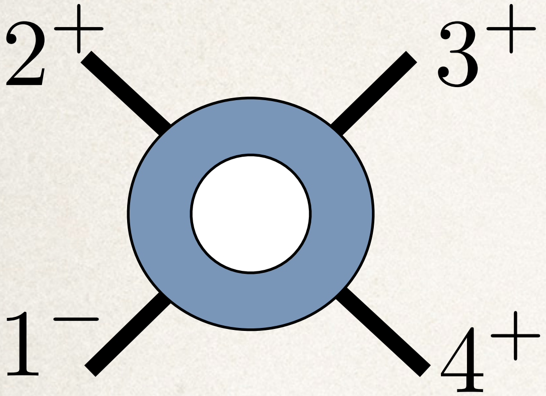
[Hodges (2009)]
[Badger, Frellesvig, Zhang (2013)]
[Peraro (2016)]

$$x_i = \begin{cases} s_{12} & i = 1 \\ -\frac{\langle i\, i+1\rangle\langle i+2\, 1\rangle}{\langle 1\, i\rangle\langle i+1\, i+2\rangle} & i = 2,\ldots,n-2 \\ \delta_{n,4} + (1-\delta_{n,4})\frac{s_{23}}{s_{12}} & i = n-1 \\ -\frac{[2|P_{2,i-n+4}|i-n+5\rangle}{[21]\langle 1|i-n+5\rangle} & i = n,\ldots,2n-6 \\ \frac{\langle 1|P_{23}P_{2,i-2n+9}|i-2n+10\rangle}{s_{23}\langle 1|i-2n+10\rangle} & i = 2n-5,\ldots,3n-11 \\ \frac{s_{123}}{s_{12}} & i = 3n-10 \end{cases}$$

[Badger (Amplitudes 2015)]

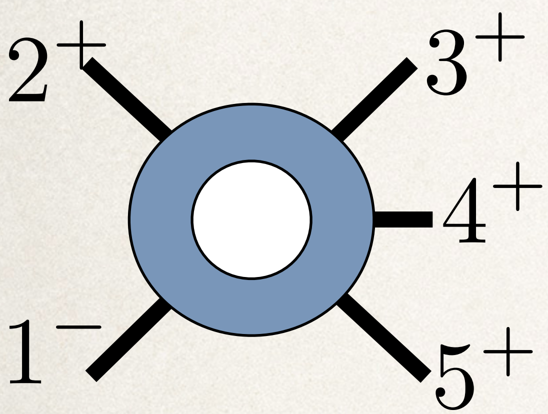
Write Lorentz invariant quantities in terms of 3n-10 variables

4pt single minus



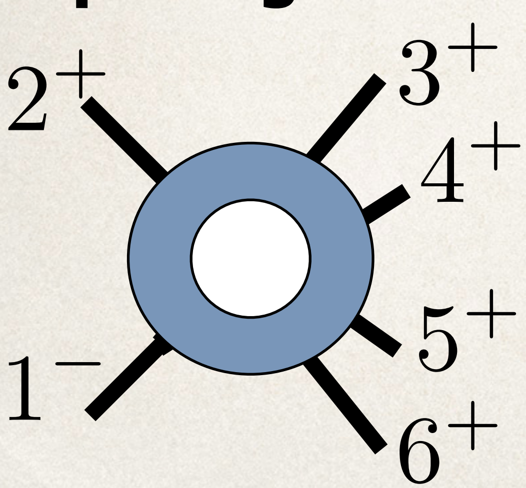
$$= -\frac{i}{48\pi^2}x_1^5 x_2^2\,(x_2+1)$$

5pt single minus



$$= -\frac{i}{48\pi^2}x_1^6 x_2^2 x_3\left[(x_2+1)\,x_4 + (x_3+1)\,x_2^2 x_3 x_5 - \frac{(x_2-x_4)^3 x_3}{x_2-x_4+x_5}\right]$$

6pt single minus



$$\begin{aligned}
&-\frac{1}{48\,\pi^2} i\, x_1^7\, x_2^3\, x_3^2\, x_4 \left(\frac{x_2\, x_3^2\, x_4^3\, x_5\, (x_2\, x_3\, x_6 + x_5\, (x_3\, (x_2\, (x_7-1)-1)-1))^2\, (x_6-1)^4}{x_6^2\, (((x_2+1)\, x_3+1)\, x_4\, (x_6-1)+x_6)\, (x_5\, (x_6+x_4\, (x_6+x_3\, (x_6+x_2\, (x_7-1)-1)-1))+x_2\, x_3\, x_4\, x_6\, x_8)} - \right.\\
&\quad \left(x_3\, x_4\, x_5\, \left(\left(x_2\, x_3\, ((x_3+1)\, x_4\, (x_6-1)+x_6)\, x_7^2 - 2\, x_2\, x_3\, x_6\, x_7 + ((x_2+1)\, x_3+1)\, x_6\right) x_5^2 - 2\, x_2\, x_3\, x_6\, (x_6 + x_2\, x_3\, x_4\, (x_6-1)\, x_7)\, x_5 + x_2^2\, x_3\, x_6^2\, ((x_2\, x_3+1)\, x_4\, (x_6-1)+x_6)\right)\right.\\
&\qquad \left.(x_6-1)^3\right) \Big/ \left(x_6^2\, (((x_2+1)\, x_3+1)\, x_4\, (x_6-1)+x_6)\, (x_5\, (x_6-x_7)-x_6\, x_8)\right) +\\
&\quad 3\, x_2\, x_3 \left(\frac{x_2\, x_5\, x_6}{x_2\, x_6 - x_5\, x_7} + \frac{x_3\, x_4\, \left(x_2^2\, x_3\, (x_4+1)\, x_6 - x_5^2\right) x_7}{x_6}\right) - \frac{1}{x_6^2\, (x_2\, x_6 - x_5\, x_7)} \left(x_3^3\, x_4\, (x_4+1)\, x_6^3\, (x_6+2)\, x_2^4 - \right.\\
&\qquad x_3\, x_6^2 \left(x_4\, (x_4+1) \left(x_6^2 + (-x_7\, x_5 + x_5 - 1)\, x_6 + x_5\, x_7\, (2\, x_7+1)\right) x_3^2 + x_6\, (-x_7\, x_5 + x_5 + x_6 + x_4\, (x_4\, (x_6-1) + 2\, x_6 - 1))\, x_3 - x_6\, (x_6 + x_5\, (x_7-1))\right) x_2^3 -\\
&\qquad x_5\, x_6 \left(x_4\, (x_4+1)\, x_5\, x_6\, (x_7-1)\, x_7\, x_3^3 + x_5\, (x_6\, (x_7-1)\, x_7 + x_4\, (3\, x_6 + x_7 - 1))\, x_3^2 + x_5\, x_6\, (x_7-1)\, x_7\, x_3 + x_6^2\right) x_2^2 +\\
&\qquad \left. x_5 \left(x_3^2\, x_4\, x_7\, (3\, x_6 + x_7 - 1)\, x_5^2 + 3\, x_3\, x_6^2\, x_5 + x_6^2\, (x_5\, x_7 - x_6)\right) x_2 + x_5^2\, x_6\, (x_6\, x_7 - x_3\, x_5)\right) +\\
&\quad \left.\frac{x_2\, x_3^2\, x_4 \left(-x_5^2 - x_2\, x_3\, (x_4+1)\, x_6\, x_5 + x_2^2\, x_3\, (x_4+1)\, x_6^2\right) x_8}{x_5\, x_6} + \frac{x_2^3\, x_3^2\, (x_4+1)\, x_6\, (x_4\, (-x_6 + x_3\, (-x_6 + x_2\, (x_6 - 2\, x_7 + 1) + 1) + 1) - x_6)}{x_2\, x_6 - x_5\, x_7} - \frac{x_2\, x_3^2\, x_4\, x_5^3\, (x_7-1)^2}{x_8}\right)
\end{aligned}$$

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Summary and Outlook

Unitarity, On-shellness & Integrand Decomposition

— Dramatic developments for **One-Loop** Amplitudes
— NLO: automating **analytic** one-loop calculations
— NN...LO
— many legs
— massive particles in the loops

Formal Properties of Scattering Amplitudes

— Hidden properties can emerge only from direct calculations.
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— Scattering Amplitudes in Gauge theories still reserve a lot of surprises.

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loops

Thanks