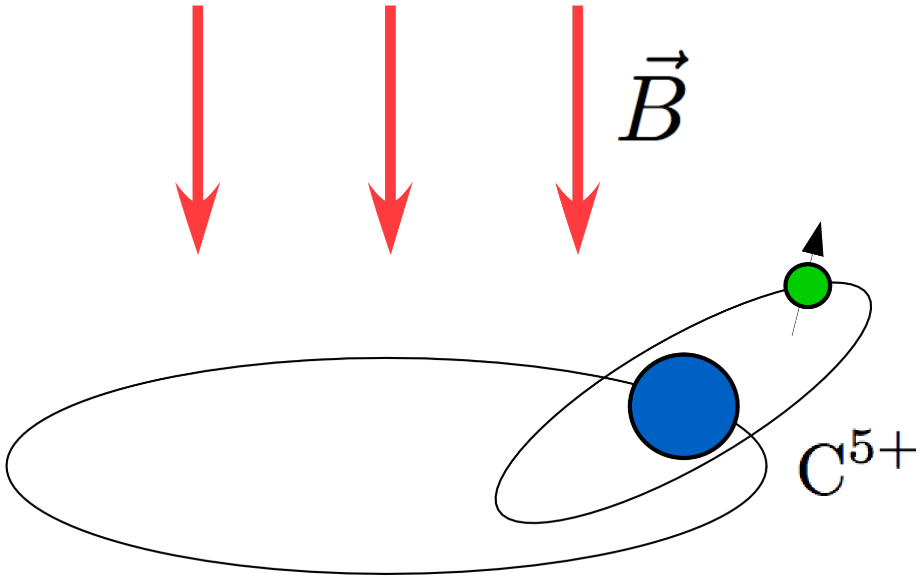
The g-factor of a bound electron



LoopFest Buffalo
August 17, 2016

Andrzej Czarnecki 🍁 University of Alberta

with M. Dowling, J. Piclum, R. Szafron

Outline

Three Loop-related themes in bound states:

Spectrum: Lamb shift, Rydberg, proton radius

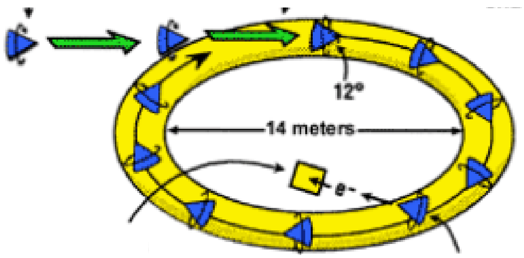
Interactions with external fields:
g-2 of a bound electron

Decay of a bound particle:
muon decay in orbit and
the muon --> electron conversion

Robert Szafron's talk

The puzzle of the muon magnetic moment

The 3.6 sigma discrepancy,



$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 287(80) \times 10^{-11}$$

PRD 86, 095009 (2012)

is large compared with other bounds on New Physics.

How to check g_μ-2?

Electron g-2 is likely sensitive to the same New Physics; but at present it is used to determine the fine-structure constant.

A new source of alpha is needed.

Note:

Centenary!
First introduced by
Sommerfeld 1916

How to check g_μ-2?

Nature 442, 516 (2006)
PRA 89, 052118 (2014)

The second best determination of alpha:
from atomic spectroscopy

$$R_\infty = \frac{m_e c \alpha^2}{2h}$$

Needed precision:

$14 \cdot 10^{-11}$

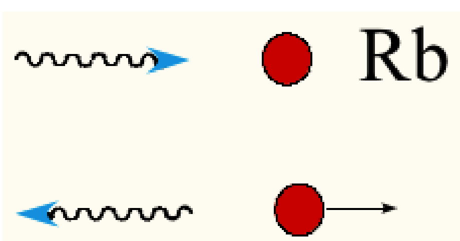
$$\alpha^2 = \frac{2R_\infty}{c} \cdot \frac{u}{m_e} \cdot \frac{M_X}{u} \cdot \frac{h}{M_X}$$

$7 \cdot 10^{-12}$
(but is it for sure?)

$8 \cdot 10^{-11}$
Nature 2014
Sturm et al

$12 \cdot 10^{-11}$
for Rb
(better for He)

$124 \cdot 10^{-11}$
improvement needed by factor ~10



gives h/m

$\alpha(\mathrm{Rb}) = 1/137.035\,999\,049(90) \quad [66 \cdot 10^{-11}]$

PRL 106, 080801 (2011)

Magnetic moment (bound electron)

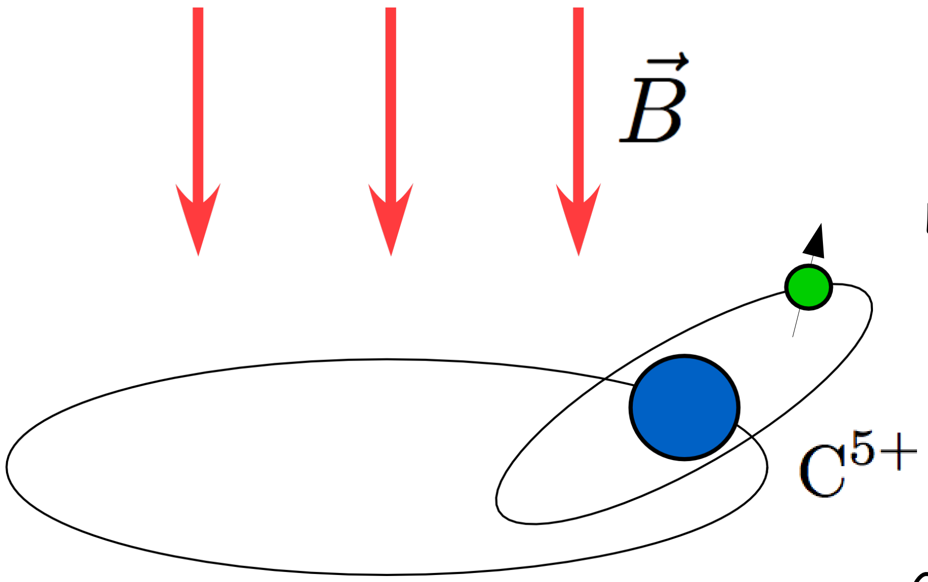
Why useful?

- determination of the electron mass
- future determination of alpha

Why interesting?

- quantum effects in external field
- simple system, model for more complex ones
- numerical estimates exist for large Z
- should be analytically feasible for small Z (many have tried)

Determination of electron's mass

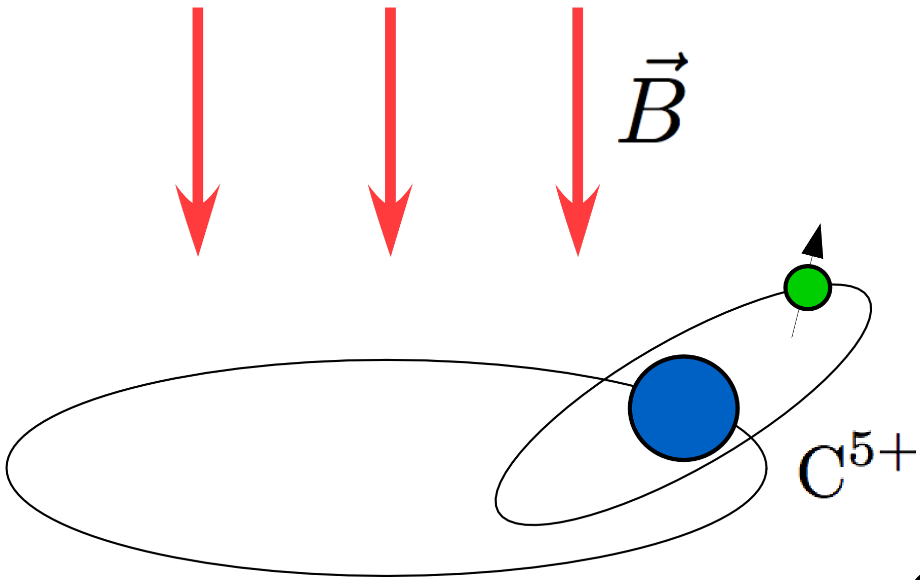


Larmor frequency $\omega_L = \frac{geB}{2m_e}$

Cyclotron frequency $\omega_{cycl} = \frac{ZeB}{M}$

$$m_e = \frac{g}{2Z}\frac{\omega_{cycl}}{\omega_L}M$$

Electron anchored in an ion



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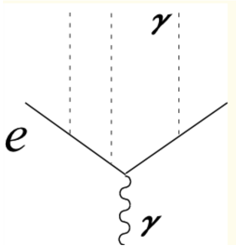
Cyclotron frequency $\omega_{\rm cycl} = \frac{ZeB}{M}$

$$m_e = \frac{g}{2Z}\frac{\omega_{\rm cycl}}{\omega_L}M$$

Interesting complication:
this *g*-factor is modified by the binding

Bound-electron g-2: the leading effect

Breit 1928: energy correction due to magnetic field
in the hydrogen ground state.



$$\delta E = e \int d^3x f^2 v^* \left[1 - i\gamma \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{r}} \gamma^5\right] \gamma^5 \boldsymbol{A} \cdot \boldsymbol{\Sigma} \left[1 + i\gamma \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{r}} \gamma^5\right] v$$

$$g = 2 \cdot \frac{1}{3}\left(1 + 2\sqrt{1 - (Z\alpha)^2}\right) \simeq 2\left(1 - \frac{(Z\alpha)^2}{3}\right)$$

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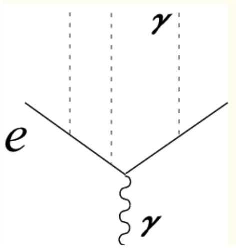

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Important: dependence on alpha; may be exploited to determine its value. (Use ions with various Z)

Bound-electron g-2: the leading effect

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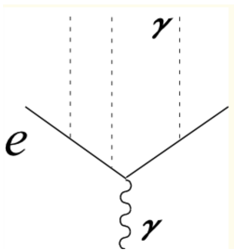

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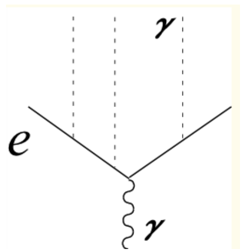
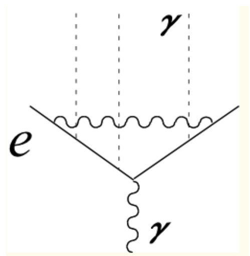
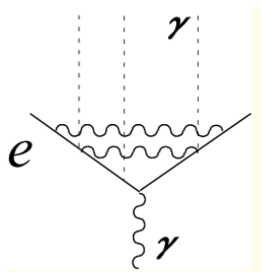
Valid to all orders in Zα

Harder to achieve when loops present.

Bound-electron g-2: binding and loops







$$g = 2 - \frac{2\,(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \ldots$$

$$+\frac{\alpha}{\pi}\left[1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4\,(a_{41}\ln Z\alpha + a_{40}) + \ldots\right]$$

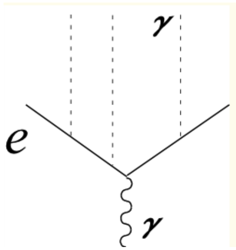
$$\underbrace{+\left(\frac{\alpha}{\pi}\right)^2\left[-0.65..\left(1 + \frac{(Z\alpha)^2}{6}\right) + (Z\alpha)^4\,(b_{41}\ln Z\alpha + b_{40}) + \ldots\right]}_{\text{two-loop corrections}}$$

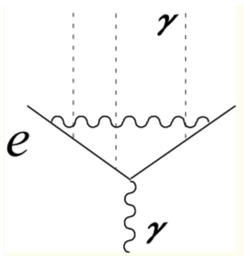
$$b_{41} = \frac{28}{9}$$

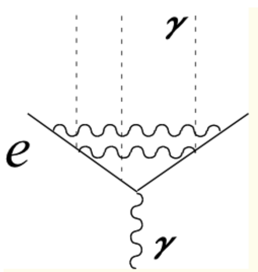
$$b_{40} = -16.4$$

Pachucki, AC Jentschura, Yerokhin (2005)

Bound-electron g-2: binding and loops







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$$b_{41} = \frac{28}{9}$$

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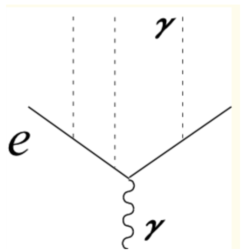
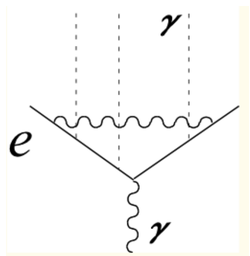
Together with experiments in Mainz, this improved the accuracy of m_e by about a factor 3,

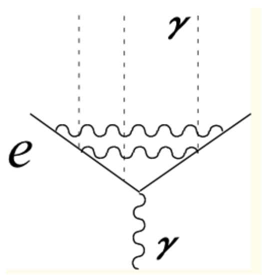
$$\frac{m_e}{u} = 0.000\ 548\ 579\ 909\ 32\,(29)\,(1)$$

theory error

Recent experimental improvement







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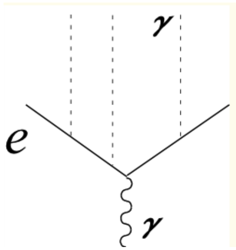
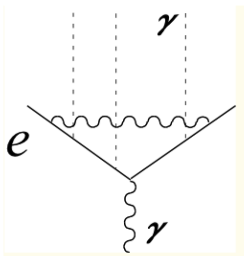
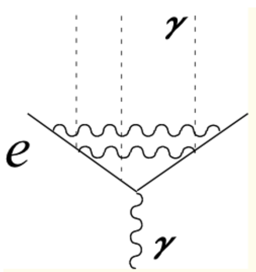
Nature 2014
Sturm et al

Recent experimental improvement







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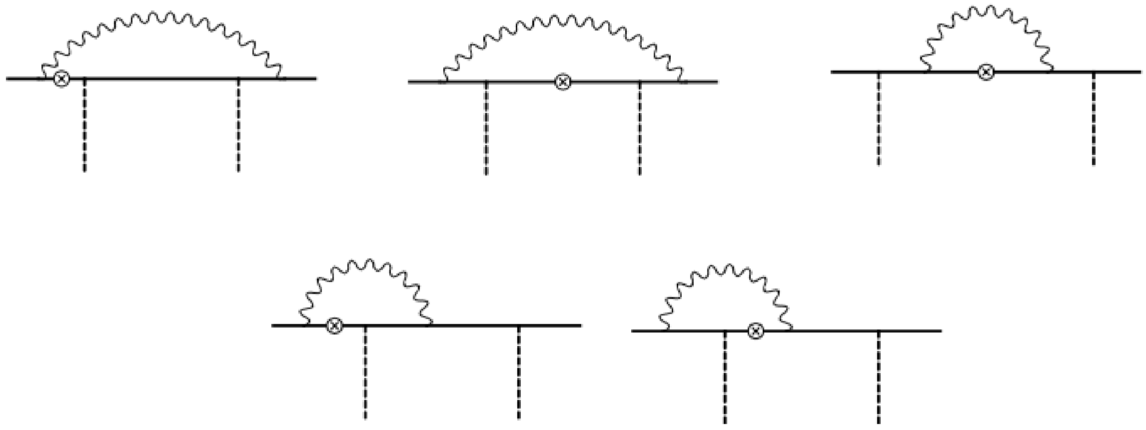
$$\frac{m_e}{u} = 0.000\ 548\ 579\ 909\ 067\,(17)$$

Nature 2014
Sturm et al

Next theory challenge:
$(Z\alpha)^5$ effects.

To find Δg, consider the energy in a magnetic field

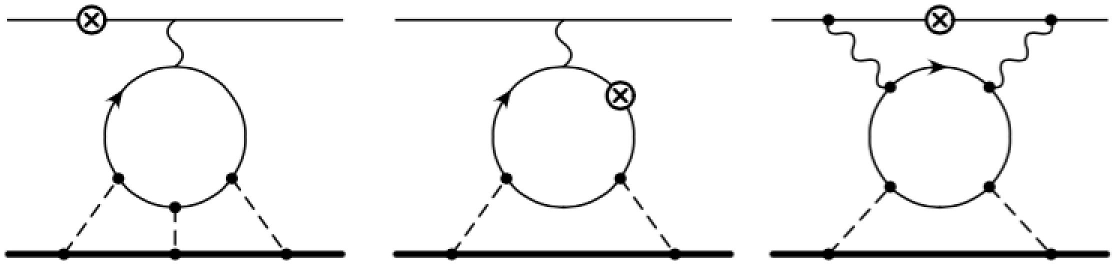


The result is *gauge*-invariant; but not yet complete.

What if the magnetic field couples to an external line?

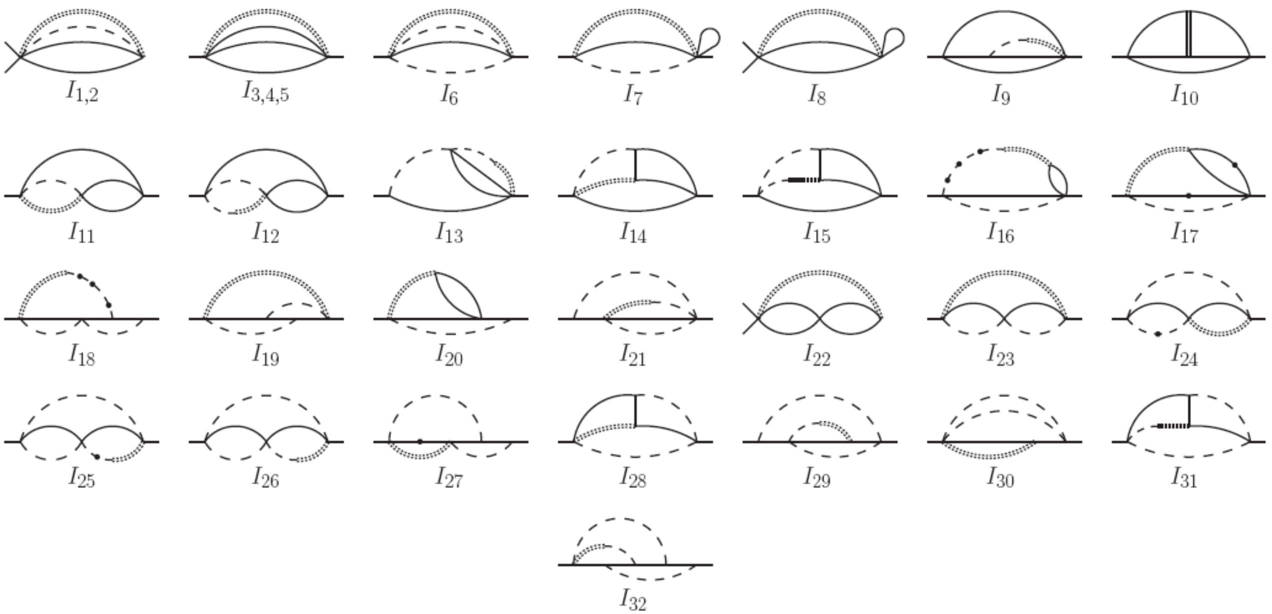
Next goal: $\alpha^2(Z\alpha)^5$ corrections to *g*

Examples:



More than 300 contributions.

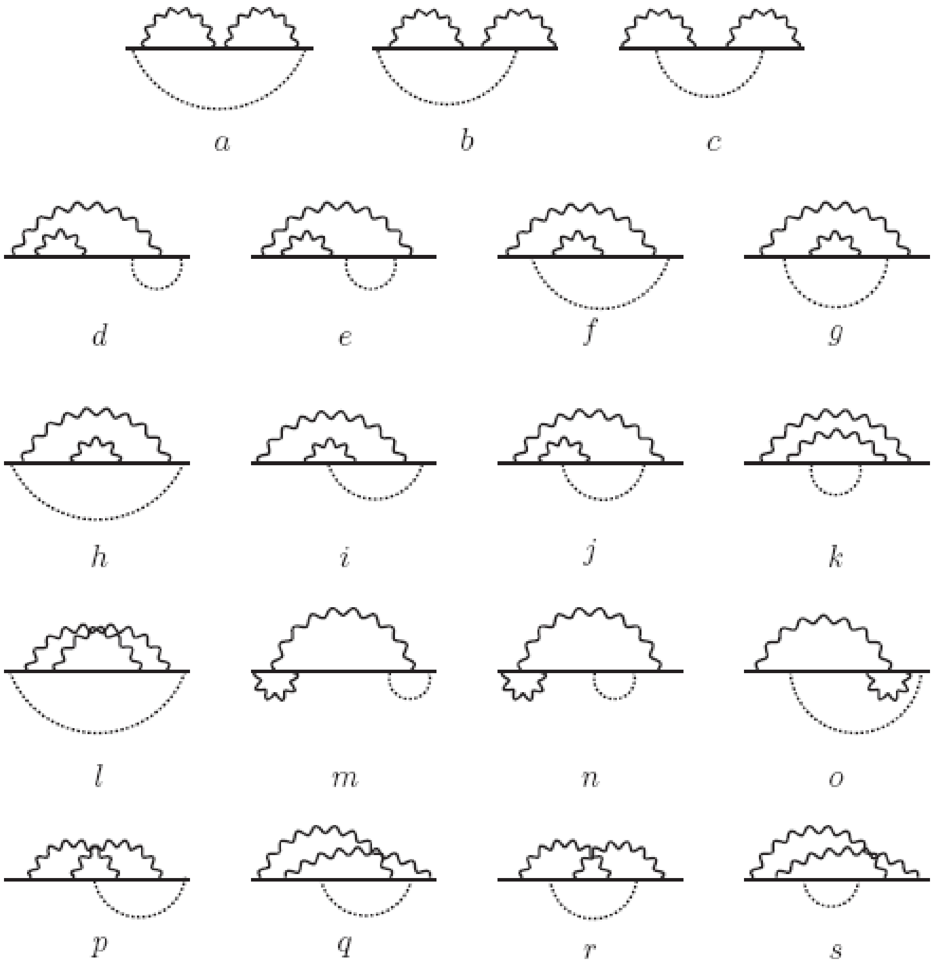
A set of 32 master integrals



Typical expression

$$
\begin{aligned}
I_{24} \;&=\; G(0,1,2,1,0,1,0) = \frac{2\pi^2}{\epsilon} - 162.745878930257(1) + 640.681562239(2)\epsilon \\
&\quad -9490.745115169417(3)\epsilon^2 + \mathcal{O}(\epsilon^3)\,,
\end{aligned}
$$

Reevaluation of the $\alpha^2(Z\alpha)^5$ Lamb shift



$$\delta E_{a-s} = \frac{\alpha^2(Z\alpha)^5}{\pi n^3}\left(\frac{\mu}{m}\right)^3 m\,[-7.72381(4)]$$

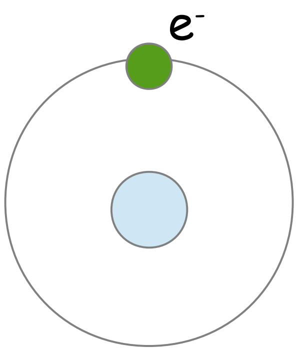
Dowling, Mondejar, Piclum, AC, PRA 81, 022509

Previous results

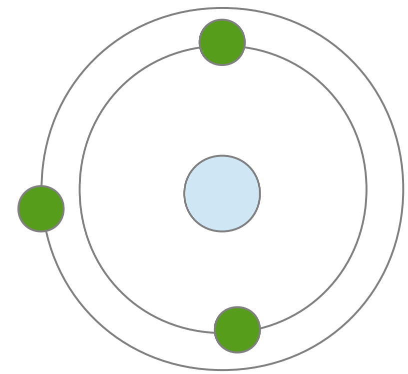
-7.61(16) Pachucki 1994
-7.724(1) Eides and Shelyuto, 1995

New source of alpha: medium-charged ions

$$g \simeq 2 - \frac{2\,(Z\alpha)^2}{3}$$



Hydrogen-like ion



Lithium-like ion

Combine H-like and Li-like ions to remove nuclear dependence;
then combine with a different nucleus, to remove free-g dependence!

Much interesting theoretical work remains to be done!

Yerokhin, Berseneva, Harman, Tupitsyn, Keitel: PRL (2016)

Summary

* Binding modifies the electron g-factor

* Theory of a bound electron is more fun than for free particles

* Synergy with beautiful experiments:
mass of the electron and, in future, the fine structure constant.

* $\alpha(Z\alpha)^5$ effects almost finished; $\alpha^2(Z\alpha)^5$ hopefully soon.

* Opportunities for more theoretical improvement...

Can we use the electron to check muon g-2?

$$a_e = \frac{g_e - 2}{2}$$

Measured with relative error $25 \cdot 10^{-11}$

Phys. Rev. Lett. 100, 120801 (2008)

Provides the fine structure constant with the same precision,

$$\alpha^{-1}(a_e) = 137.035\,999\,1736\,(331)\,(86)$$

Phys. Rev. Lett. 109, 111807 (2012)

Experimental error dominates (for now)

Numerical errors from 4- and 5-loop diagrams