Evaluation of the general 3-loop vacuum Feynman integral

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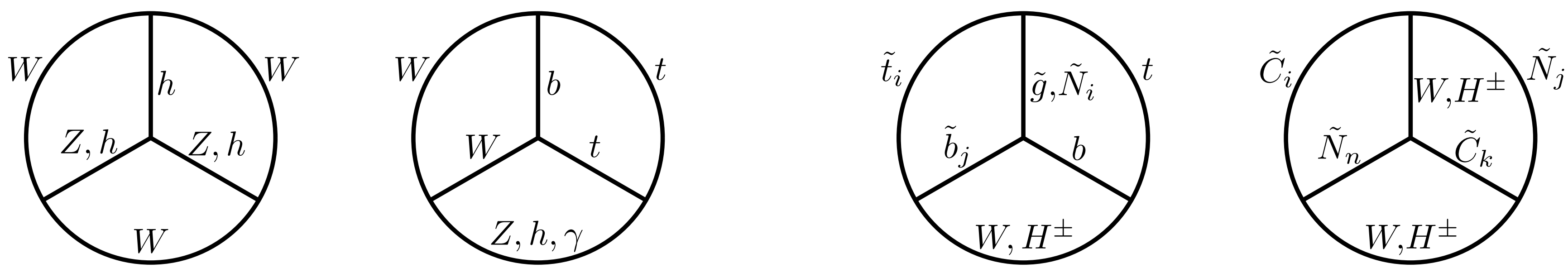
Based on work with David G. Robertson (Otterbein University),

to appear soon as preprint and public computer code.

One motivation: evaluation of Coleman-Weinberg effective potential

- In the Standard Model, $V_{\rm eff}$ relates the Higgs VEV to the Lagrangian $\overline{\rm MS}$ parameters. Known at:
  - 2-loop order (Ford, Jack, Jones, hep-ph/0111190)
  - 3-loop order only at leading order in α_S and y_t. (SPM 1310.7553)
- In SUSY, $V_{\rm eff}$ enables approximate calculation of lightest Higgs mass. Again, only known fully at 2-loop order. 3-loop contributions are numerically important, especially if SUSY is heavy.

Need to be able to systematically compute hundreds of integrals, for example:



In SUSY cases, mass hierarchies not known in advance.

All 1-scale vacuum integrals at 3-loop order are known analytically.

Broadhurst 1992, 1999; Avdeev+Fleischer+Mikhailov+Tarasov, 1994; Fleischer+Tarasov, 1994; Avdeev 1995; Fleischer+Kalmykov 1999; Schröder+Vuorinen 2005.

Available in a computer program: `MATAD` (Steinhauser hep-ph/0009092)
Can also get 3-loop vacuum integrals with multiple scales, by expansions in masses starting from the 1-scale integrals, for a given hierarchy.

A few examples of 2-scale integrals are also known analytically:

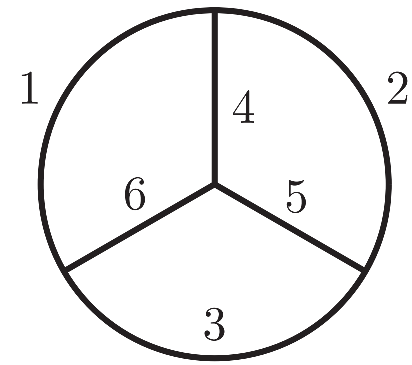
Davydychev+Kalmykov 2003, Kalmykov 2005, Bytev+Kalmykov+Kniehl 2009, a few more will appear in our own paper.

Our aim is for a fast, accurate, and flexible (valid for all masses, doesn't rely on predetermined hierarchical expansions) numerical computation method.

Outline

- Basis (“master”) integrals
- Renormalized basis integrals
- Analytic cases
- Evaluation of basis integrals using differential equations in squared mass arguments
- Public code: 3-loop Vacuum Integral Library = `3VIL`

Using partial fractions, any 3-loop vacuum integral can be reduced to this topology of scalar integral in $d = 4 - 2\epsilon$ Euclidean dimensions with $\int_p = \mu^{4-d} \int d^d p/(2\pi)^d$, where the $\overline{\text{MS}}$ renormalization scale is defined by $Q^2 = 4\pi e^{-\gamma_E} \mu^2$:



$$\mathbf{T}^{(n_1,n_2,n_3,n_4,n_5,n_6)}(x_1,x_2,x_3,x_4,x_5,x_6) \;=\; (16\pi^2)^3 \int_p \int_q \int_k$$

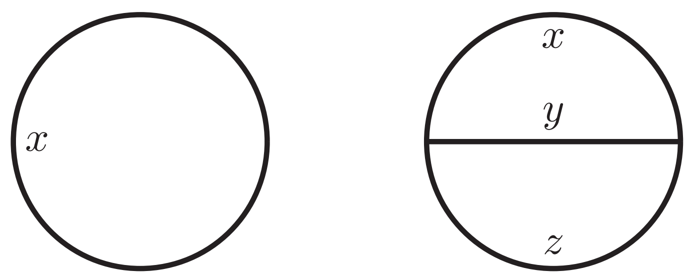
$$\frac{1}{[p^2+x_1]^{n_1}[q^2+x_2]^{n_2}[k^2+x_3]^{n_3}[(p-q)^2+x_4]^{n_4}[(q-k)^2+x_5]^{n_5}[(k-p)^2+x_6]^{n_6}}$$

The propagator powers n_i can be positive, negative, or zero. Using integration by parts, can always reduce all integrals of this type to a few basis integrals. . .

Basis integrals:

$$\begin{aligned}
\mathbf{H}(u,v,w,x,y,z) &= \mathbf{T}^{(1,1,1,1,1,1)}(u,v,w,x,y,z),\\
\mathbf{G}(w,u,z,v,y) &= \mathbf{T}^{(1,1,1,0,1,1)}(u,v,w,x,y,z),\\
\mathbf{F}(u,v,y,z) &= \mathbf{T}^{(2,1,0,0,1,1)}(u,v,w,x,y,z),\\
\mathbf{A}(u)\mathbf{I}(v,w,y) &= \mathbf{T}^{(1,1,1,0,1,0)}(u,v,w,x,y,z),\\
\mathbf{A}(u)\mathbf{A}(v)\mathbf{A}(w) &= \mathbf{T}^{(1,1,1,0,0,0)}(u,v,w,x,y,z),
\end{aligned}$$

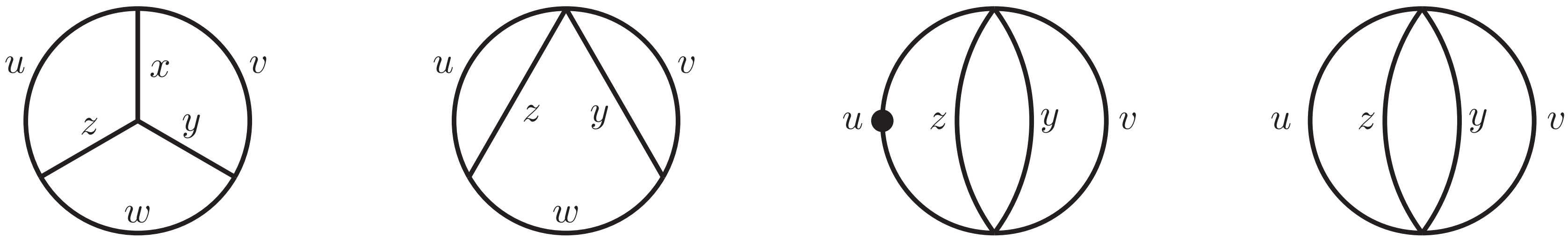
The last two are just products of 1-loop and 2-loop basis integrals:



$\mathbf{A}(x)$ $\mathbf{I}(x,y,z)$

These are known analytically, and present no problems.

The genuinely 3-loop integrals in the basis are $\mathbf{H}$, $\mathbf{G}$, and $\mathbf{F}$:



$\mathbf{H}(u, v, w, x, y, z)$ $\mathbf{G}(w, u, z, v, y)$ $\mathbf{F}(u, v, y, z)$ $\mathbf{E}(u, v, y, z)$

The dot on the $\mathbf{F}$ integral denotes a doubled propagator for the first squared mass argument; all other propagators are single.

The 4-propagator integral $\mathbf{E}$ is not part of the basis. By dimensional analysis:

$$\mathbf{E}(u, v, y, z) = \left[u\mathbf{F}(u, v, y, z) + v\mathbf{F}(v, u, y, z) + y\mathbf{F}(y, u, v, z) + z\mathbf{F}(z, u, v, y)\right]/(-2 + 3\epsilon),$$

so it is redundant. However, it is still useful. Note:

$$\mathbf{F}(u, v, y, z) = -\frac{\partial}{\partial u}\mathbf{E}(u, v, y, z).$$

Renormalized quantities are much more succinctly written in terms of modified basis integrals in which UV sub-divergences have been subtracted.

For example, at 2-loop order, define:

$$I(x,y,z) \quad = \quad \lim_{\epsilon\to 0}\left[\mathbf{I}(x,y,z) - I^{(1)}_{\rm div}(x,y,z) - I^{(2)}_{\rm div}(x,y,z)\right],$$

where

$$\begin{aligned} I^{(1)}_{\rm div}(x,y,z) &= \frac{1}{\epsilon}[\mathbf{A}(x)+\mathbf{A}(y)+\mathbf{A}(z)], \\ I^{(2)}_{\rm div}(x,y,z) &= \frac{1}{2}(x+y+z)\left(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\right). \end{aligned}$$

The modified basis integral $I(x,y,z)$ is finite, by construction. It is known in terms of dilogarithms. Note it is **not** just the same thing as the ϵ^0 term in the ϵ expansion!

For the 3-loop, 4-propagator integrals, define:

$$E(u,v,y,z) \quad = \quad \lim_{\epsilon\to 0}\left[\mathbf{E}(u,z,y,v) - E^{(1)}_{\rm div}(u,v,y,z) - E^{(2)}_{\rm div}(u,v,y,z) - E^{(3)}_{\rm div}(u,v,y,z)\right],$$

where the 1-loop, 2-loop, and 3-loop UV sub-divergences are, respectively,

$$\begin{aligned}
E^{(1)}_{\rm div}(u,v,y,z) \quad &= \quad \frac{1}{\epsilon}\mathbf{A}(u)\mathbf{A}(v) + \text{(5 permutations)}, \\
E^{(2)}_{\rm div}(u,v,y,z) \quad &= \quad \left[\frac{1}{2\epsilon^2}(v+y+z) + \frac{1}{2\epsilon}\left(\frac{u}{2} - v - y - z\right)\right]\mathbf{A}(u) + \text{(3 permutations)}, \\
E^{(3)}_{\rm div}(u,v,y,z) \quad &= \quad \left[\frac{1}{3\epsilon^3} - \frac{2}{3\epsilon^2} + \frac{1}{3\epsilon}\right](uv+uy+uz+vy+vz+yz) \\
&\quad + \left[\frac{1}{6\epsilon^2} - \frac{3}{8\epsilon}\right](u^2+v^2+y^2+z^2).
\end{aligned}$$

Renormalized quantities are written in terms of the ϵ-independent modified basis functions:

$$F(u,v,y,z) = -\frac{\partial}{\partial u}E(u,v,y,z).$$

Similarly, define the modified basis function:

$$G(w,u,z,v,y) = \lim_{\epsilon\to 0}\Big[\mathbf{G}(w,u,z,v,y) - G^{(1)}_{\rm div}(w,u,z,v,y) - G^{(2)}_{\rm div}(w,u,z,v,y) - G^{(3)}_{\rm div}(w,u,z,v,y)\Big],$$

where the 1-loop, 2-loop, and 3-loop UV sub-divergences are:

$$G^{(1)}_{\rm div}(w,u,z,v,y) = \frac{1}{\epsilon}\left[\mathbf{I}(w,u,z) + \mathbf{I}(w,v,y)\right],$$

$$G^{(2)}_{\rm div}(w,u,z,v,y) = \left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon}\right)\left[\mathbf{A}(u) + \mathbf{A}(v) + \mathbf{A}(y) + \mathbf{A}(z)\right] - \frac{1}{\epsilon^2}\mathbf{A}(w),$$

$$G^{(3)}_{\rm div}(w,u,z,v,y) = \left(-\frac{1}{6\epsilon^3} + \frac{1}{2\epsilon^2} - \frac{2}{3\epsilon}\right)(u+v+y+z) + \left(-\frac{1}{3\epsilon^3} + \frac{1}{3\epsilon^2} + \frac{1}{3\epsilon}\right)w.$$

$\mathbf{H}$ has no 1-loop and 2-loop sub-divergences, but does have a 3-loop UV divergence. So, define:

$$H(u,v,w,x,y,z) = \lim_{\epsilon\to 0}\left[\mathbf{H}(u,v,w,x,y,z) - H^{(3)}_{\rm div}(u,v,w,x,y,z)\right]$$

where

$$H^{(3)}_{\rm div}(u,v,w,x,y,z) = 2\zeta(3)/\epsilon.$$

The function $F(u, v, y, z)$ has an IR log divergence as $u \to 0$. Therefore, further define:

$$\overline{F}(u, v, y, z) \equiv F(u, v, y, z) + \overline{\ln}(u) I(v, y, z)$$

where

$$\overline{\ln}(u) = \ln(u/Q^2)$$

with $Q = \overline{\text{MS}}$ renormalization scale. The function $\overline{F}$ is well-defined for all values of its squared mass arguments, including $u = 0$.

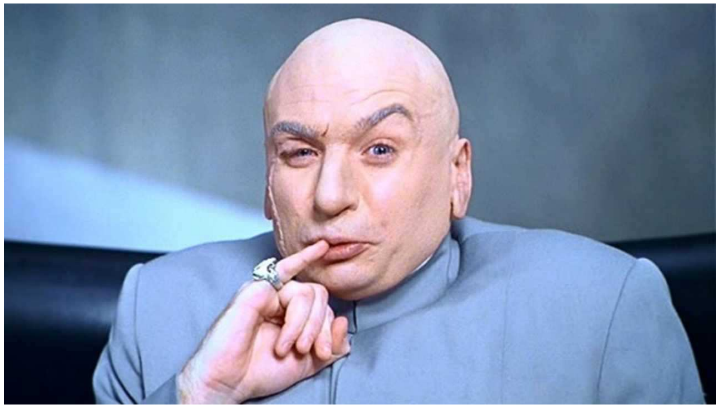
For convenience, our program `3VIL` outputs all E, F, and $\overline{F}$ functions, for given input arguments.

(Also can output the ϵ expansions of the original bold-faced integrals $\mathbf{I}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$.)

It remains to be able to evaluate the (modified) basis integrals.

Introducing: `3VIL` = 3-loop Vacuum Integral Library

- Written in C, can be called from C, C++, Fortran
- Uses analytic results where available, otherwise differential equations method
- Evaluation for generic mass inputs:
  - Time < 1 second on reasonably modern hardware
  - Relative accuracy $\lesssim 10^{-10}$
- For certain rare difficult cases, time ~ 5 seconds, accuracy $\sim 10^{-4}$
- Not quite ready for public release, but very soon...



The following are known analytically:

- All 1-scale integrals E, F, $\overline{F}$, G, H, with squared masses all equal to 0 or a single non-zero value x. Broadhurst 1992, 1999; Avdeev+Fleischer+Mikhailov+Tarasov, 1994; Fleischer+Tarasov, 1994; Avdeev 1995; Fleischer+Kalmykov 1999; Schröder+Vuorinen 2005.

- The following 2-scale integral cases, and integrals E, F related to them, and permutations implied by symmetries of the graphs: $\overline{F}(x,0,0,y)$, $\overline{F}(0,0,x,y)$, $\overline{F}(x,x,y,y)$, $\overline{F}(x,0,y,y)$, $\overline{F}(y,0,y,x)$, $G(0,0,0,x,y)$, $G(0,0,x,0,y)$, $G(x,0,0,0,y)$, $G(x,0,x,0,y)$, $G(0,x,x,y,y)$, $G(x,0,0,y,y)$, $G(y,x,x,x,x)$, $H(0,0,x,y,x,x)$. Davydychev+Kalmykov 2003, Kalmykov 2005, Bytev+Kalmykov+Kniehl 2009, our paper.

Our program `3VIL` knows about these cases and uses them whenever possible. Computation time $\approx$ 0.

The generic case: consider the master tetrahedral topology, and all corresponding basis integrals obtained by removing propagator lines:

$$H(u, v, w, x, y, z),$$

$$G(w, u, z, v, y),\ \ G(x, u, v, y, z),\ \ G(u, v, x, w, z),$$

$$G(y, v, w, x, z),\ \ G(v, u, x, w, y),\ \ G(z, u, w, x, y),$$

$$\overline{F}(w, u, x, y),\ \ \overline{F}(w, v, x, z),\ \ \overline{F}(x, u, w, y),\ \ \overline{F}(x, v, w, z),$$

$$\overline{F}(u, v, y, z),\ \ \overline{F}(u, w, x, y),\ \ \overline{F}(y, u, v, z),\ \ \overline{F}(y, u, w, x),$$

$$\overline{F}(v, u, y, z),\ \ \overline{F}(v, w, x, z),\ \ \overline{F}(z, u, v, y),\ \ \overline{F}(z, v, w, x),$$

products of I and A functions

The derivatives of all of these with respect to any squared mass argument u, v, w, x, y, z are also 3-loop integrals, and so are linear combinations of the basis.

Solve differential equations in the masses to compute these, starting from known analytical values at a fixed but arbitrary reference squared mass a as initial conditions:

$$H(a, a, a, a, a, a),\ \ G(a, a, a, a, a),\ \ \overline{F}(a, a, a, a),\ \ I(a, a, a),\ \ A(a).$$

Define an integration variable t, and:

$$\begin{aligned} U &= a + t(u - a), & V &= a + t(v - a), & W &= a + t(w - a), \\ X &= a + t(x - a), & Y &= a + t(y - a), & Z &= a + t(z - a). \end{aligned}$$

and consider basis integrals as functions of U, V, W, X, Y, Z.

- At $t = 0$, have $U = V = W = X = Y = Z = a$, so all integrals are known.
- At $t = 1$, have desired values of squared mass arguments: $(U, V, W, X, Y, Z) = (u, v, w, x, y, z)$.

Denoting the basis integrals generically by Φ_i, have first-order coupled linear differential equations in t:

$$\frac{d}{dt}\Phi_j = \sum_k c_{jk}\Phi_k + c_j$$

where the coefficients c_{jk} and c_j are ratios of polynomials in t and fixed values a, u, v, w, x, y, z.

Integrate differential equations numerically from $t = 0$ to $t = 1$.

Differential equations method for evaluation of loop integrals

Kotikov 1991, Remiddi 1997, Caffo+Czyz+Laporta+Remiddi 1998, Caffo+Czyz+Remiddi 2002, SPM 2003, SPM+Robertson 2005, ...

Allows analytic evaluation in favorable cases; otherwise Runge-Kutta numerical integration.

When computing tetrahedral integral $H(u, v, w, x, y, z)$, we simultaneously get all subordinate basis integrals G, F, $\overline{F}$, E.

However, there are complications...

$$\frac{d}{dt}\Phi_j = \sum_k c_{jk}\Phi_k + c_j$$

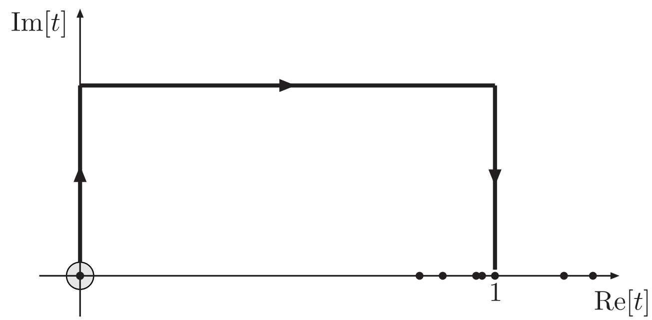
A complication: the coefficients c_{jk} and c_j have poles in t.

- All poles can be made simple by use of partial fractions on the coefficients.
- There are always poles at $t = 0$.

  Use a power series expansion around $t = 0$, up to order t^8.

  Start integration at $t = 0.01$
- All poles are on the real t axis. Sometimes poles exist for $0 < t < 1$.

  In that case, integrate on a contour in the complex plane to avoid them:



Otherwise, integrate straight along $\mathrm{Re}[t]$ axis.

Recall $U = a + t(u - a)$, etc.

The fixed reference squared mass a is arbitrary. In principle, results should not depend on it. Can be changed as a check. By default `3VIL` uses:

$$a = 2\mathrm{Max}(u, v, w, x, y, z).$$

Avoids numerical problems that can arise in certain special cases.

Other checks:

- analytical special cases compared to Runge-Kutta evaluation
- vanishing of imaginary parts of basis integrals when squared mass inputs are positive
- change shape of contour in complex plane, including height in the $\mathrm{Im}[t]$ direction

Initialization at $t = 0.01$:

$$H(U,V,W,X,Y,Z) \;=\; H(a,a,a,a,a,a) + \sum_{n\geq 1} t^n H^{(n)}(u,v,w,x,y,z;a),$$
$$G(W,U,Z,V,Y) \;=\; G(a,a,a,a,a) + \sum_{n\geq 1} t^n G^{(n)}(w,u,z,v,y;a),$$
$$\overline{F}(U,V,Y,Z) \;=\; \overline{F}(a,a,a,a) + \sum_{n\geq 1} t^n \overline{F}^{(n)}(u,v,y,z;a),$$

with:

$$\overline{F}(a,a,a,a) \;=\; a\left[53/12 + (3\sqrt{3}\mathrm{Ls}_2 - 3/2)\overline{\ln}(a) + \frac{3}{2}\overline{\ln}^2(a) - \frac{1}{2}\overline{\ln}^3(a)\right]$$
$$G(a,a,a,a,a) \;=\; a\left[-97/3 + 12\sqrt{3}\mathrm{Ls}_2 + 6\zeta_3 + (26 - 6\sqrt{3}\mathrm{Ls}_2)\overline{\ln}(a) - 8\overline{\ln}^2(a) + \overline{\ln}^3(a)\right]$$
$$H(a,a,a,a,a,a) \;=\; 16\mathrm{Li}_4(1/2) - \frac{17\pi^4}{90} + \frac{2}{3}\ln^2(2)[\ln^2(2) - \pi^2] - 9(\mathrm{Ls}_2{}^2) + 6\zeta_3[1 - \overline{\ln}(a)]$$

and

$$H^{(1)}(u,v,w,x,y,z;a) = \zeta_3(6a - u - v - w - x - y - z)/a,$$

etc. All expansion coefficients through $n = 7$ included, so that at $t = 0.01$ the relative error from truncation is same order as that of long double arithmetic, 10^{-16}.

For most of the integration, 3VIL uses a 6-stage, 5th order Runge-Kutta algorithm with automatic step-size adjustment.

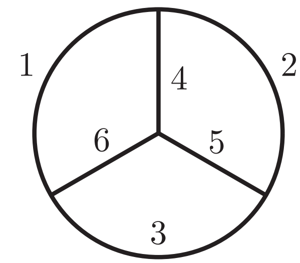
However, can have poles in the coefficients at the integration endpoint $t = 1$. Usual Runge-Kutta routines fail!

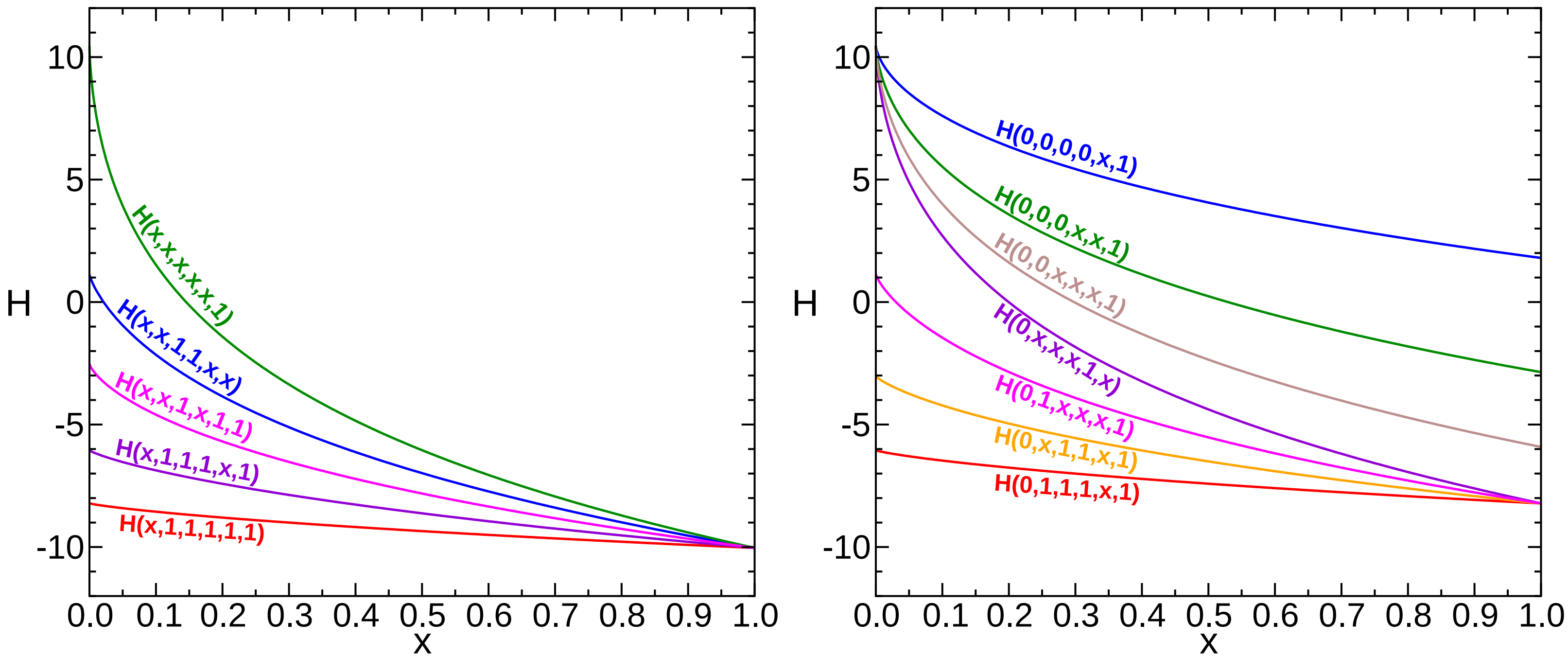
Key property needed: no evaluations of derivatives at the endpoint of the integration step.

No 4-stage Runge-Kutta algorithms with this property exist, but we found a 5-stage, 4th order algorithm. (Invented for a very similar situation for our program TSIL = Two-loop Self-energy Integration Library, hep-ph/0501132.)

Note: although the coefficients in the differential equations have poles, the basis functions themselves are completely finite and smooth! Only pseudo-thresholds, no thresholds.

Some examples of the basis integral H, as a function of a squared mass argument x, with other squared mass arguments fixed to 0 or 1.





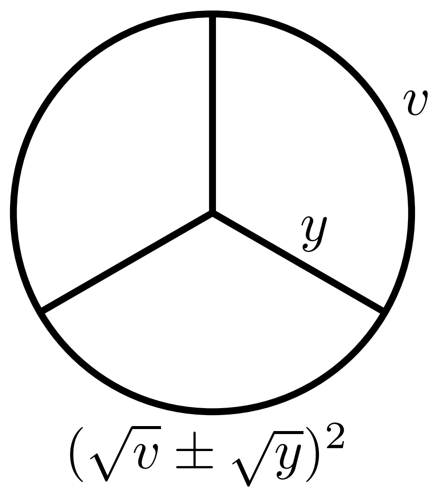
The endpoints at $x = 0$ and $x = 1$ are known analytically in terms of logs.

For all other $0 < x < 1$, computed analytically with $\texttt{3VIL}$.

Outlook

- Identified a basis for 3-loop vacuum integrals with arbitrary masses; convenient modified basis for renormalized quantities
- Evaluation using differential equations method
  - fast, accurate, flexible
  - get all subordinate integrals simultaneously
- Public code `3VIL` coming very soon
- Applications
  - 3-loop effective potential for Standard Model, SUSY, general theory
  - Higher point functions when external momenta are small, or are suitable for expansions

Pseudo-thresholds = numerically difficult cases:



with $v \neq 0$ and $y \neq 0$.

Note that these cases are “unnatural”; not consequences of any possible symmetry in a quantum field theory. Don’t arise in Standard Model, but may occur in parameter scans in Beyond Standard Model theories.