Finite fields and progress on four loop form factors

Andreas v. Manteuffel

Johannes Gutenberg Universität Mainz

PRiSMA

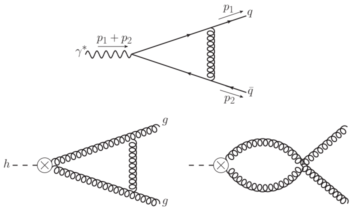
Michigan State University

LoopFest 2016

Buffalo, 15-17 August 2016

Form factors for quarks and gluons

- massless quark and gluon form factors



- purely virtual corrections to
  - Drell-Yan production
  - Higgs production in gluon-fusion
- form factors allow to study IR properties of QCD
  - cusp anomalous dimensions $1/\epsilon^2$
  - collinear anomalous dimensions $1/\epsilon$

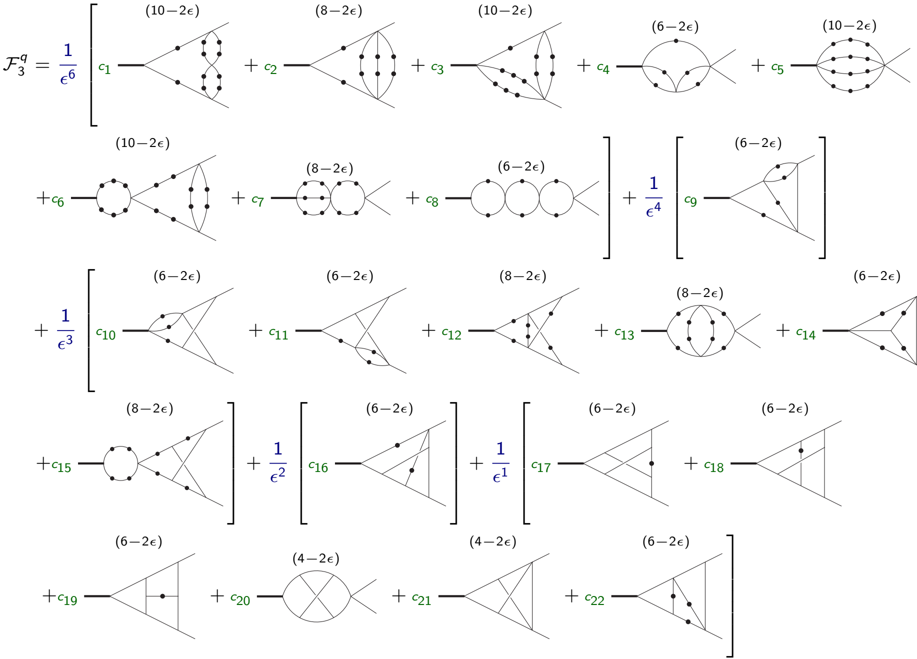
Form factors @ 3-loops

- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] $\Leftarrow$ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factors @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]

- recalculation of 3-loop results via finite integrals:
  - [AvM, Panzer, Schabinger '15]
  - automated setup, fully analytical
  - `Qgraf` [Nogueira]:
    - Feynman diagrams
  - `Reduze 2` [AvM, Studerus]:
    - interferences
    - IBP reductions
    - finite integral finder
    - basis change with dimensional recurrences
  - `HyperInt` [Panzer]:
    - integration of ϵ expanded master integrals

Quark form factor @ 3-loops [AvM, Panzer, Schabinger '15]



Towards the cusp anomalous dimension @ 4-loops

Cusp anomalous dimension @ 4-loops:

- required for $\mathrm{N}^3\mathrm{LL}$ resummation, see talk by [H.X. Zhu]
- Casimir scaling for quark and gluon cusp anomalous dimension:

$$\boxed{\Gamma_4^q \stackrel{?}{=} \frac{C_F}{C_A}\Gamma_4^g}$$

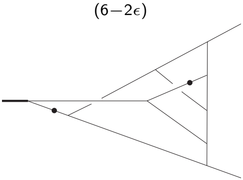
towards 4-loop form factors:

- reduced integrand for $\mathcal{N}=4$: [Boels, Kniehl, Tarasov, Yang '12, '15]
- leading N_c fermionic F_4^q: [Henn, Smirnov, Smirnov, Steinhauser '16]
- particularly challenging: gluon form factor in QCD

Analytical integration @ 4-loops

[AvM, Panzer, Schabinger '15]

a non-planar 12-line topology @ 4-loops:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \frac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3$$

$$- 47\zeta_2\zeta_5 - 21\zeta_7 + \frac{6883}{2100}\zeta_2^4 + \frac{49}{2}\zeta_2\zeta_3^2 + \frac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

- only shallow ϵ expansion needed
- numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation
- starts at weight 7, not expected to contribute to cusp anomalous dimension
- more: see talk by [R. Schabinger]

Feynman integrals form a linear vector space

$$I = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \qquad a_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

family of loop integrals:

- fulfill linear relations: integration-by-parts (IBP) identities
- systematic reduction to master integrals possible
- think of it as linear vector space with some finite basis
- specific basis choices:
  - canonical basis for method of differential equations [Henn]
  - basis of finite integrals for direct integration (analyt., numeric.) [Panzer; Panzer, AvM, Schabinger]

reductions are technical challenge:

1. often a bottleneck of the computation
2. this talk: improve reductions via finite field sampling

Integration-by-parts (IBP) identities

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right)$$

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right)$$

where p_j are external momenta, $a_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

integral reduction:

- express arbitrary integral for given problem via few basis integrals
- integration-by-parts (IBP) reductions [Chetyrkin, Tkachov '81]
- public codes: `Air` [Anastasiou], `Fire` [Smirnov], `Reduze 1` [Studerus], `Reduze 2` [AvM, Studerus], `LiteRed` [Lee]
- possible: exploit structure at algebra level
- here: Laporta's approach

Laporta's algorithm:

1. index integrals by propagator exponents: $I(a_1, \ldots, a_N)$
2. define ordering (e.g. fewer denominators means simpler)
3. generate IBPs for explicit values $a_1, \ldots, a_N$
4. results in linear system of equations
5. solve linear system of equations

major shortcomings of traditional Gauss solvers:

- suffers from intermediate expression swell
- requires large number of auxiliary integrals and equations
- limited possbilities for parallelisation

IBP reductions from finite field samples

A novel approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

IBP reductions from finite field samples

A novel approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

finite field techniques:

- no intermediate expression swell by construction
- early discard of redundant and auxiliary quantities
- big potential for parallelisation

established in math literature, becomes popular in physics:

- dense solver: [Kauers]
- filtering: `Ice` [Kant '13]
- supersymmetric integrand construction: [Bern et al]
- tensor reduction: [Heller]
- QCD integrand construction: [Peraro '16]

core algorithm:

Extended Euclidean Algorithm (EEA)

1. begin with $(g_0, s_0, t_0) = (a, 1, 0)$ and $(g_1, s_1, t_1) = (b, 0, 1)$,
2. then repeat

$$\begin{aligned} q_i &= g_{i-1} \, \text{quotient} \, g_i \\ g_{i+1} &= g_{i-1} - q_i g_i \\ s_{i+1} &= s_{i-1} - q_i s_i \\ t_{i+1} &= t_{i-1} - q_i t_i \end{aligned}$$

3. until $g_{k+1} = 0$ for some k. at that point:

$$s_k a + t_k b = g_k = \text{GCD}(a, b)$$

restrict first to linear systems with rational numbers coefficients

- use EEA to define inverse of integer b modulo m with $\text{GCD}(m, b) = 1$:

$$\begin{aligned} 1 &= s\,m + t\,b \\ \Rightarrow 1/b &:= t \mod m \end{aligned}$$

  this gives us a canonical homomorphism ϕ_m of $\mathbb{Q}$ onto $\mathbb{Z}_m$ with

$$\phi_m(a/b) = \phi_m(a)\phi_m(1/b)$$

- for large enough m, the map ϕ_m can be inverted !

given a finite field image of a/b modulo m for $m > 2\max(a^2, b^2)$,
a unique rational reconstruction is possible:

Rational reconstruction [Wang '81; Wang, Guy, Davenport '82]

to reconstruct a/b from its finite field image $u = a/b \mod m$:

- run EEA for u and m
- stop at first g_j with $|g_j| \leq \lfloor \sqrt{m/2} \rfloor$
- the unique solution is $a/b = g_j/t_j$

important details:

- since we don't know bound on m:
  veto $|t_j| > \lfloor \sqrt{m/2} \rfloor$ and $\mathrm{GCD}(t_j, g_j) \neq 1$ reconstructions, see *e.g.* [Monagan '04]
- construct large m with Chinese Remaindering:
  construct solution modulo $m = p_1 \cdots p_N$ from solutions modulo machine-sized primes p_i

A fast rational solver

INPUT: $I_{\mathbb{Q}}$ unreduced rational matrix

OUTPUT: $O_{\mathbb{Q}}$ row reduced rational matrix

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Function reconstruction

univariate rational function $\mathbb{Q}[d]$ reconstruction:

- works similar to the case $\mathbb{Q}$
- Chinese remaindering becomes Lagrange polynomial interpolation:

$$p_1 \cdots p_N \to (d - p_1) \cdots (d - p_N)$$

- rational reconstruction becomes Pade approximation:

$$\text{interpolating polynomial} \to \text{rational function}$$

multivariate rational function $\mathbb{Q}[d, s, t, \ldots]$ reconstruction:

- by iteration

A fast univariate solver

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p}[x] & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p}[x] & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p}[x] \\
 & \longrightarrow & I_{\mathbb{Z}_p,x_2} & \longrightarrow & O_{\mathbb{Z}_p,x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p,x3} & \longrightarrow & O_{\mathbb{Z}_p,x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

note: massively parallisable

```
 ____  __  __ _  ____  ____  ____ 
(  __)(  )(  ( \(  _ \(  __)(    \
 ) _)  )( /    / )   / ) _)  ) D (
(__)  (__)\_)__)(__\_)(____)(____/

Package: finred
Author:  Andreas v. Manteuffel
```

features:

- C++11 implementation for univariate sparse matrices
- employs flint library
- parallelisation: SIMD, threads, MPI, batch
- equation filtering: eliminate redundant rows
- plus lots of IBP specific features
- much faster than Reduze 2

Results for massless QCD @ 4 loops

[AvM, Schabinger]

completed:

- N_f^3 for quarks and gluons (three massless quark loops)
- complexity: 12 denominators, 6 numerators, non-planar, $O(10^8)$ eqs. per sector

checks:

- reductions verified against at least 5 independent samples
- calculation performed in different gauges
  - general R_ξ gauge, general external polarisation vectors
  - background field gauge

  result independent of these choices
- two independent diagram evaluations:
  - Qgraf + Mathematica
  - Qgraf + Form
- poles through to $1/\epsilon^3$ [Moch, Vermaseren, Vogt '05] reproduced

remarks:

- general R_ξ gauge introduces many dots
- more details: see talk by [R. Schabinger]

QCD result @ 4-loops for quarks

[AvM, Schabinger]

bare quark form factor

$$\mathcal{F}_4^q|_{N_f^3} = C_F \left[\frac{1}{\epsilon^5}\left(\frac{1}{27}\right) + \frac{1}{\epsilon^4}\left(\frac{11}{27}\right) + \frac{1}{\epsilon^3}\left(\frac{4}{9}\zeta_2 + \frac{254}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{26}{27}\zeta_3 + \frac{44}{9}\zeta_2 + \frac{29023}{1458}\right) + \frac{1}{\epsilon}\left(\frac{23}{3}\zeta_4 - \frac{286}{27}\zeta_3 + \frac{1016}{27}\zeta_2 + \frac{331889}{2916}\right) - \frac{146}{9}\zeta_5 - \frac{104}{9}\zeta_2\zeta_3 + \frac{253}{3}\zeta_4 - \frac{6604}{81}\zeta_3 + \frac{58046}{243}\zeta_2 + \frac{10739263}{17496} + \mathcal{O}(\epsilon)\right]$$

cusp anomalous dimension:

$$\Gamma_4^q|_{N_f^3} = C_F \left[\frac{64}{27}\zeta_3 - \frac{32}{81}\right]$$

agrees with [Grozin, Henn, Korchemsky, Marquard '15], [Henn, Smirnov, Smirnov, Steinhauser '16]

First QCD result @ 4-loops for gluons

[AvM, Schabinger]

Bare gluon form factor

$$\mathcal{F}_4^g|_{N_f^3} = C_F\Bigg[-\frac{2}{3\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{32}{3}\zeta_3-\frac{145}{9}\right)+\frac{1}{\epsilon}\left(\frac{352}{45}\zeta_2^2+\frac{1040}{9}\zeta_3+\frac{68}{9}\zeta_2-\frac{10003}{54}\right)$$
$$+\frac{4288}{27}\zeta_5-64\zeta_3\zeta_2+\frac{2288}{27}\zeta_2^2+\frac{24812}{27}\zeta_3+\frac{3074}{27}\zeta_2-\frac{508069}{324}+\mathcal{O}(\epsilon)\Bigg]$$
$$+C_A\Bigg[\frac{1}{27\epsilon^5}+\frac{5}{27\epsilon^4}+\frac{1}{\epsilon^3}\left(-\frac{14}{27}\zeta_2-\frac{55}{81}\right)+\frac{1}{\epsilon^2}\left(-\frac{586}{81}\zeta_3-\frac{70}{27}\zeta_2-\frac{24167}{1458}\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{802}{135}\zeta_2^2-\frac{5450}{81}\zeta_3-\frac{262}{81}\zeta_2-\frac{465631}{2916}\right)-\frac{14474}{135}\zeta_5+\frac{4556}{81}\zeta_3\zeta_2$$
$$-\frac{1418}{27}\zeta_2^2-\frac{99890}{243}\zeta_3+\frac{38489}{729}\zeta_2-\frac{20832641}{17496}+\mathcal{O}(\epsilon)\Bigg]$$

gluon cusp anomalous dimension:

$$\Gamma_4^g|_{N_f^3} = C_A\left[\frac{64}{27}\zeta_3-\frac{32}{81}\right]$$

- respects Casimir scaling
- non-planar C_F pieces do not contribute to $\Gamma_4^g|_{N_f^3}$

Conclusions

reductions via finite field sampling:

- fast & well established techniques
- avoids intermediate expression swell
- implementation for sparse matrices: `finred`
- speeds up integration-by-parts reductions of Feynman integrals
- useful also for other problems

four loop form factors in massless QCD:

- first result for gluons: N_f^3 contributions
- more to come