New numerical techniques for two- and threeloop integrals and applications

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I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, J. Usovitsch, arXiv:1607.08375
A. Freitas, arXiv:16mm.nnnnn

1. $\mathcal{O}(\alpha^2)$ **bosonic corrections to** $\sin^2\theta^{\rm b}_{\rm eff}$

2. **Numerical Mellin-Barnes integrals**

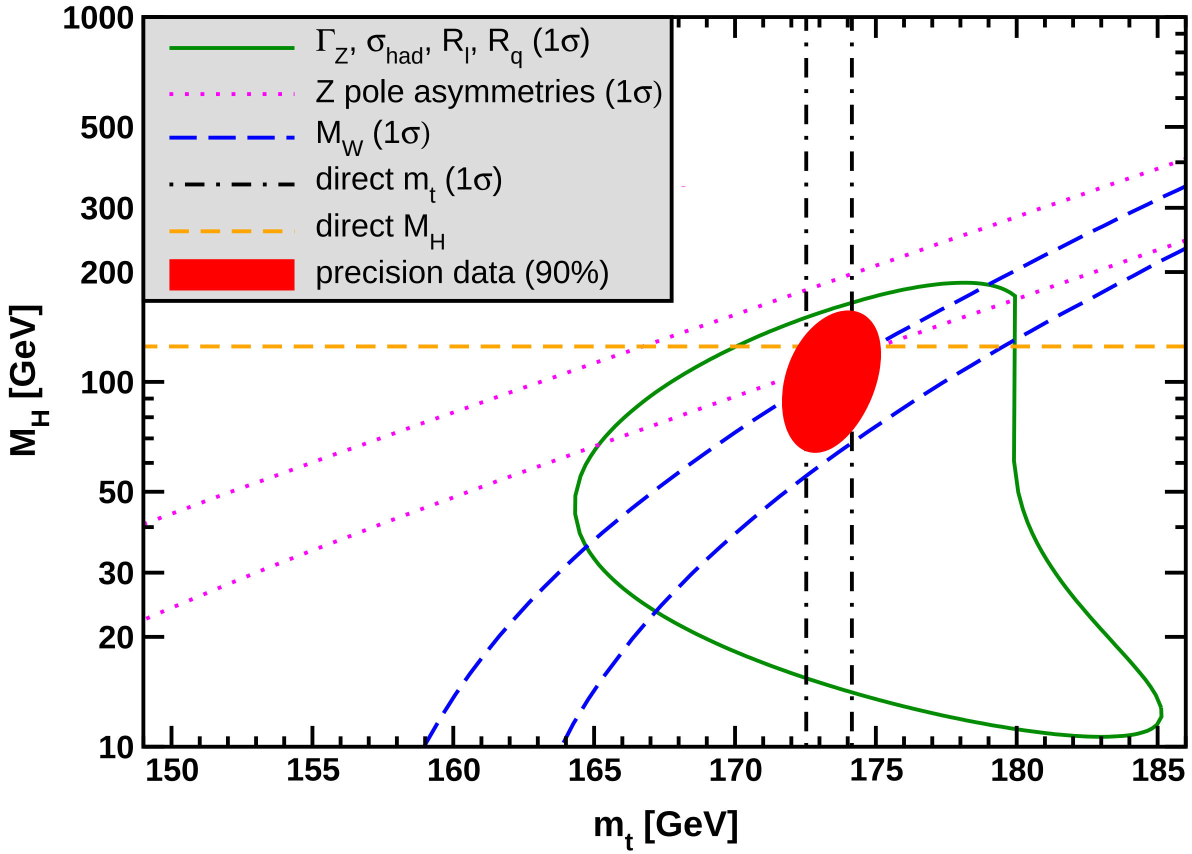
3. **Techniques for general 3-loop vacuum integrals**

$\mathcal{O}(\alpha^2)$ bosonic corrections to EWPOs

Standard Model after Higgs discovery:

- Very good agreement over large number of observables
- Sensitivity to TeV-scale new physics

Erler '16



Direct measurements:

$M_H = 125.09 \pm 0.24$ GeV

$m_t = 173.34 \pm 0.81$ GeV

Indirect prediction:

$M_H = 126.1 \pm 1.9$ GeV (with LHC BRs)

$M_H = 96^{+22}_{-19}$ GeV (w/o LHC data)

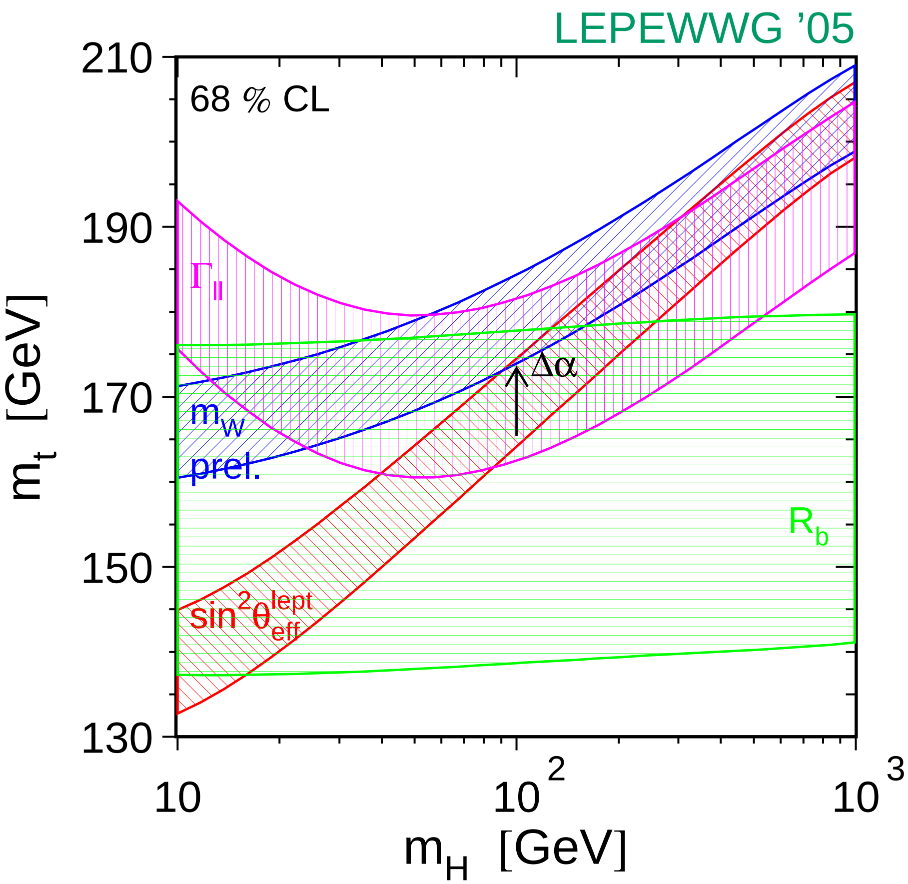
$m_t = 176.7 \pm 2.1$ GeV

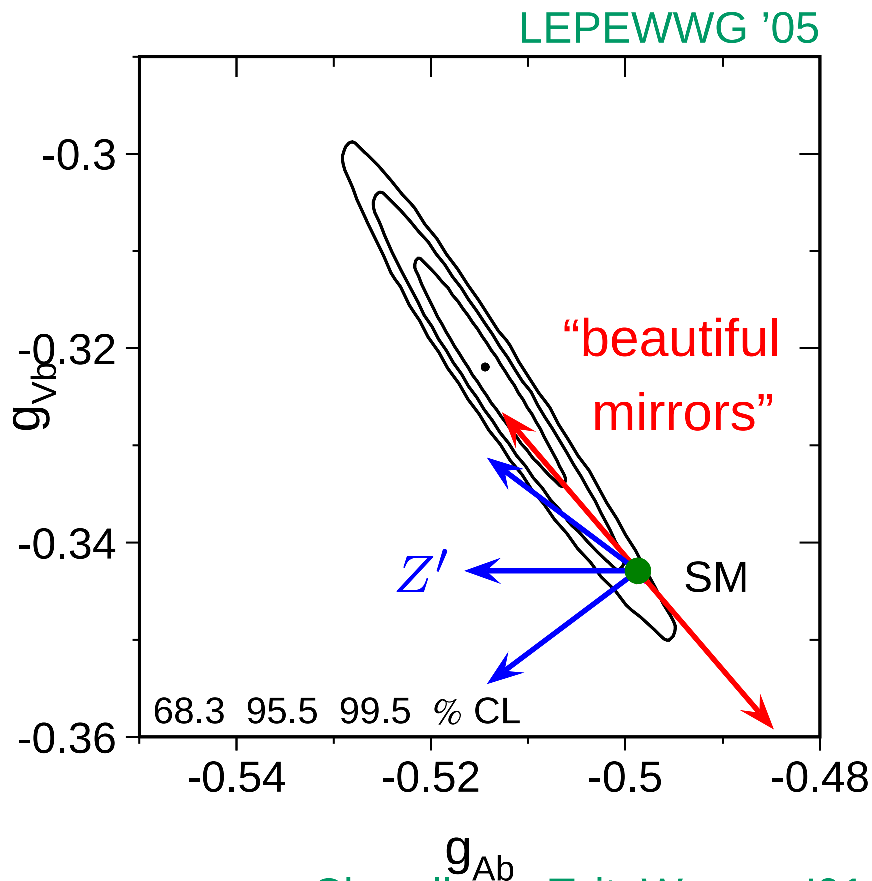
Electroweak precision observables

Impact of different observables:

M_W (from G_μ) $\qquad R_b = \Gamma[Z \to b\bar{b}]/\Gamma[Z \to \text{had.}]$

$\Gamma_{ll} = \Gamma_Z \, \text{BR}[Z \to ll]$ $\qquad \sin^2\theta^\ell_{\text{eff}}$ (from A_{LR} and A_{FB})





Choudhury, Tait, Wagner '01

Electroweak precision observables

Some important quantities:

	Exp. error	Th. error
M_W	15 MeV	4 MeV
Γ_Z	2.3 MeV	0.5 MeV
$R_b = \Gamma[Z \to b\bar{b}]/\Gamma[Z \to \text{had.}]$	6.6×10^{-4}	1.5×10^{-4}
$\sin^2\theta^\ell_\text{eff}$ (from A_LR and A_FB)	16×10^{-5}	5×10^{-5}

- Currently theory errors subdominant, but estimates are only educated guesses
- Future e^+e^- colliders (ILC / FCC-ee / CEPC) will improve precision by $\mathcal{O}(10)$

Bottom-quark asymmetry

Forward-backward asymmetry in $e^+e^- \to b\bar{b}$ after removal of QED effects:

$$A_{\rm FB}^{b\bar{b},0} = \frac{3}{4} A_{\rm e} A_{\rm b},$$

$$A_{\rm b} = \frac{2\,\Re e\, g_{\rm V}^b/g_{\rm A}^b}{1+\left(\Re e\, g_{\rm V}^b/g_{\rm A}^b\right)^2} = \frac{1-4|Q_b|\sin^2\theta_{\rm eff}^{\rm b}}{1-4|Q_b|\sin^2\theta_{\rm eff}^{\rm b} + 8Q_b^2\left(\sin^2\theta_{\rm eff}^{\rm b}\right)^2}$$

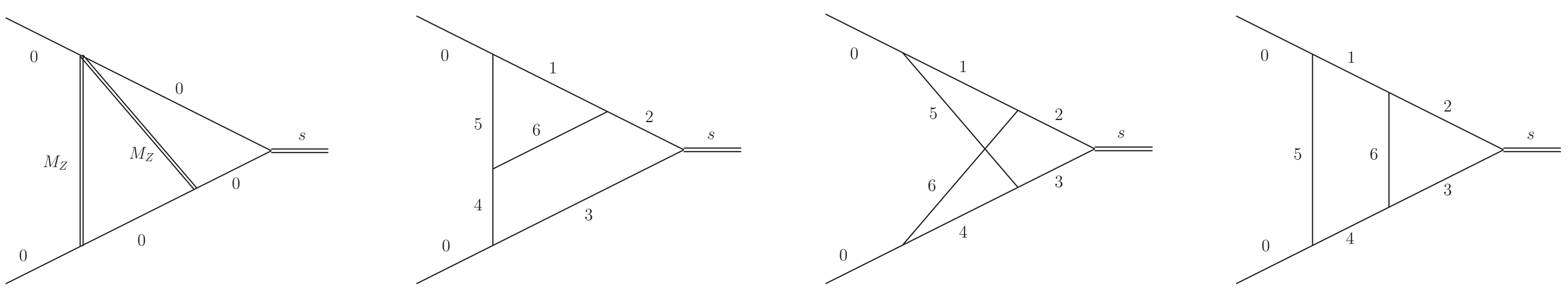
Known corrections to $\sin^2\theta_{\rm eff}^{\rm b}$:

- One-loop — Akhundov, Bardin, Riemann '86
- $\mathcal{O}(\alpha\alpha_{\rm s})$ QCD — Djouadi, Verzegnassi '87; Kniehl '90; Djouadi, Gambino '93; Fleischer, Tarasov, Jegerlehner, Raczka '92; Buchalla '93; Degrassi '93; Czarnecki, Kühn '96; Harlander, Seidensticker, Steinhauser '97
- "Fermionic" NNLO corrections — Awramik, Czakon, Freitas, Kniehl '08
- Partial 3/4-loop corrections to ρ/T-parameter
  $\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^2)$, $\mathcal{O}(\alpha_{\rm t}^2\alpha_{\rm s})$, $\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^3)$ — Chetyrkin, Kühn, Steinhauser '95; Faisst, Kühn, Seidensticker, Veretin '03; Boughezal, Tausk, v. d. Bij '05; Schröder, Steinhauser '05; Chetyrkin et al. '06; Boughezal, Czakon '06

  $(\alpha_{\rm t} \equiv \frac{y_{\rm t}^2}{4\pi})$

Calculational approach for bosonic $\mathcal{O}(\alpha^2)$ corrections

- Two-loop diagrams without closed fermion loops
- On-shell renormalization
- Self-energies (incl. from renormlization) and vertices with sub-loop bubbles using dispersion relation technique
  S. Bauberger et al. '95
  Awramik, Czakon, Freitas '06
- Non-trivial vertex diagrams:
  - Sector decomposition (FIESTA 3 / SecDec 3) Smirnov '14; Borowka et al. '15
  - Mellin-Barnes representations (MB / AMBRE 3 / MBnumerics) Czakon '06
    Dubovyk, Gluza, Riemann '15; Usovitsch '16
  - No tensor reduction (besides trivial cancellations)
    $\longrightarrow$ About 700 different two-loop vertex integrals



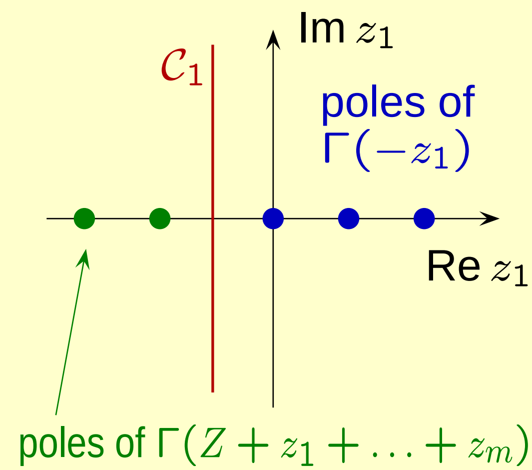
Mellin-Barnes representations

Transform Feynman integral with Mellin-Barnes representation

$$\frac{1}{(A_0 + \ldots + A_m)^Z} = \frac{1}{(2\pi i)^m} \int_{\mathcal{C}_1} dz_1 \cdots \int_{\mathcal{C}_m} dz_m$$

$$\times\, A_1^{z_1} \cdots A_m^{z_m} A_0^{-Z-z_1-\ldots-z_m}$$

$$\times\, \frac{\Gamma(-z_1) \cdots \Gamma(-z_m) \Gamma(Z + z_1 + \ldots + z_m)}{\Gamma(Z)},$$



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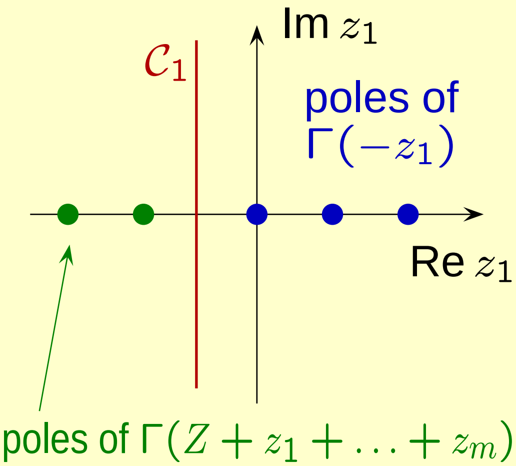
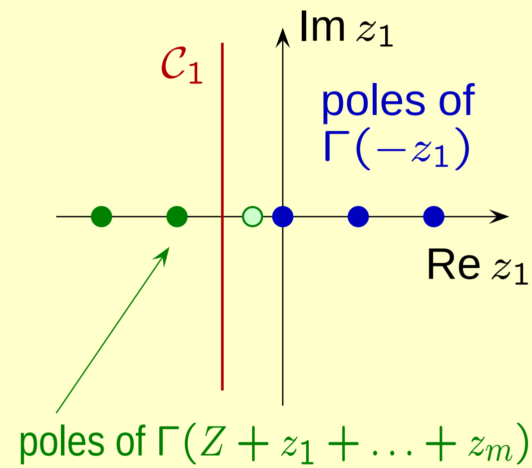


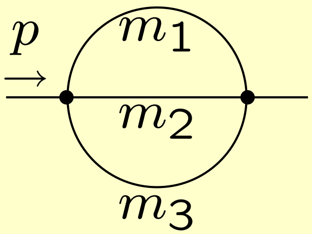
- Consistent choice of all $\mathcal{C}_i$ often requires $\varepsilon \neq 0$ $(Z = n + \epsilon)$
- For $\varepsilon \to 0$: residues from pole crossings $\to 1/\varepsilon^k$ terms

  Czakon '06
  Anastasiou, Daleo '06
- Do remaining $\mathcal{C}_i$ integrations numerically



$\varepsilon \to 0$

Mellin-Barnes representations

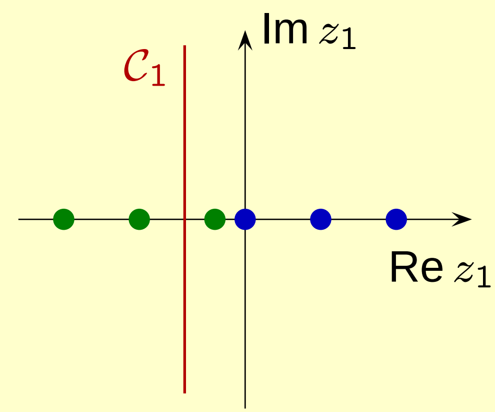


$$= \frac{-1}{(2\pi i)^3} \int dz_1 dz_2 dz_3 \, (m_1^2)^{-\varepsilon - z_1 - z_2} (m_2^2)^{z_2} (m_3^2)^{1-\varepsilon+z_1-z_3} (-p^2)^{z_3}$$
$$\times \Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_1+z_2)\Gamma(z_3-z_1)$$
$$\times \frac{\Gamma(1-\varepsilon-z_2)\Gamma(\varepsilon+z_1+z_2)\Gamma(\varepsilon-1-z_1+z_3)}{\Gamma(2-\varepsilon+z_3)}$$

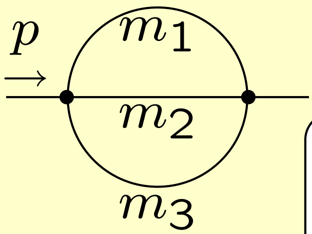
$$z_3 = c_3 + i y_3, \qquad y_i \in (-\infty, \infty)$$

$$(-p^2)^{z_3} = \underbrace{(p^2)^{c_3+iy_3} e^{-i\pi c_3}}_{\text{oscillating}} \underbrace{e^{\pi y_3}}$$

div. for $y_3 \to \infty$, eventually overcome by Γ funct.



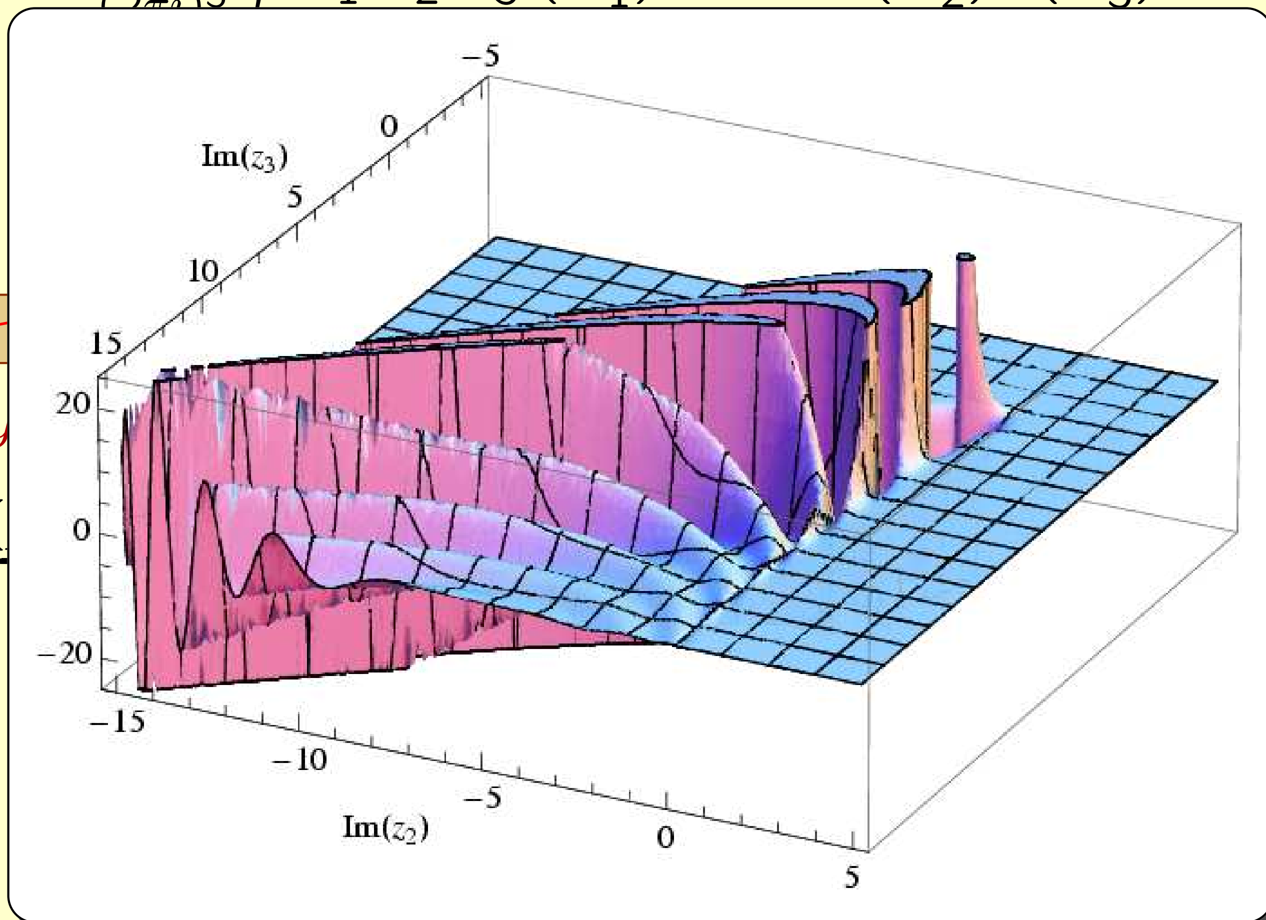
Mellin-Barnes representations



$$= \frac{-1}{(2\pi i)^3} \int dz_1 dz_2 dz_3\, (m_1^2)^{-\varepsilon - z_1 - z_2} (m_2^2)^{z_2} (m_3^2)^{1-\varepsilon+z_1-z_3} (-p^2)^{z_3}$$

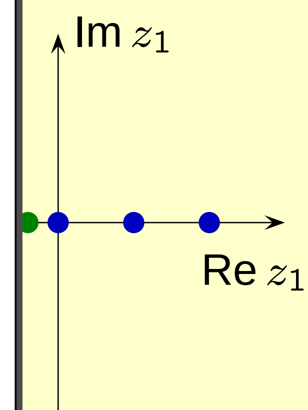
$)$

$z_1 + z_3)$

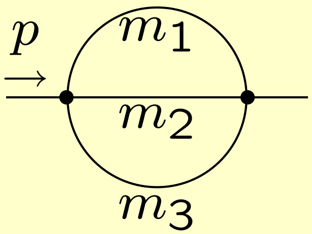


$z_3 = c_3 + iy$

$(-p^2)^{z_3} = ($



Mellin-Barnes representations



$$= \frac{-1}{(2\pi i)^3} \int dz_1 dz_2 dz_3 \, (m_1^2)^{-\varepsilon - z_1 - z_2} (m_2^2)^{z_2} (m_3^2)^{1-\varepsilon+z_1-z_3} (-p^2)^{z_3}$$
$$\times \Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_1+z_2)\Gamma(z_3-z_1)$$
$$\times \frac{\Gamma(1-\varepsilon-z_2)\Gamma(\varepsilon+z_1+z_2)\Gamma(\varepsilon-1-z_1+z_3)}{\Gamma(2-\varepsilon+z_3)}$$

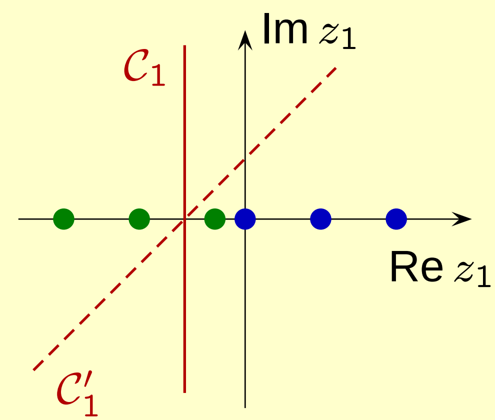
$z_3 = c_3 + i y_3, \qquad y_i \in (-\infty, \infty)$

$$(-p^2)^{z_3} = \underbrace{(p^2)^{c_3+iy_3} e^{-i\pi c_3}}_{\text{oscillating}} \underbrace{e^{\pi y_3}}_{\text{div. for } y_3 \to \infty}$$

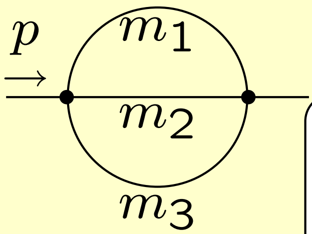
$y_i \to y_i - i\theta$

$$(-p^2)^{z_3} = (p^2)^{c_3+iy_3} e^{-i\pi(c_3+\theta y_i)} e^{(\pi+\theta \log p^2) y_3}$$

Huang, Freitas '10



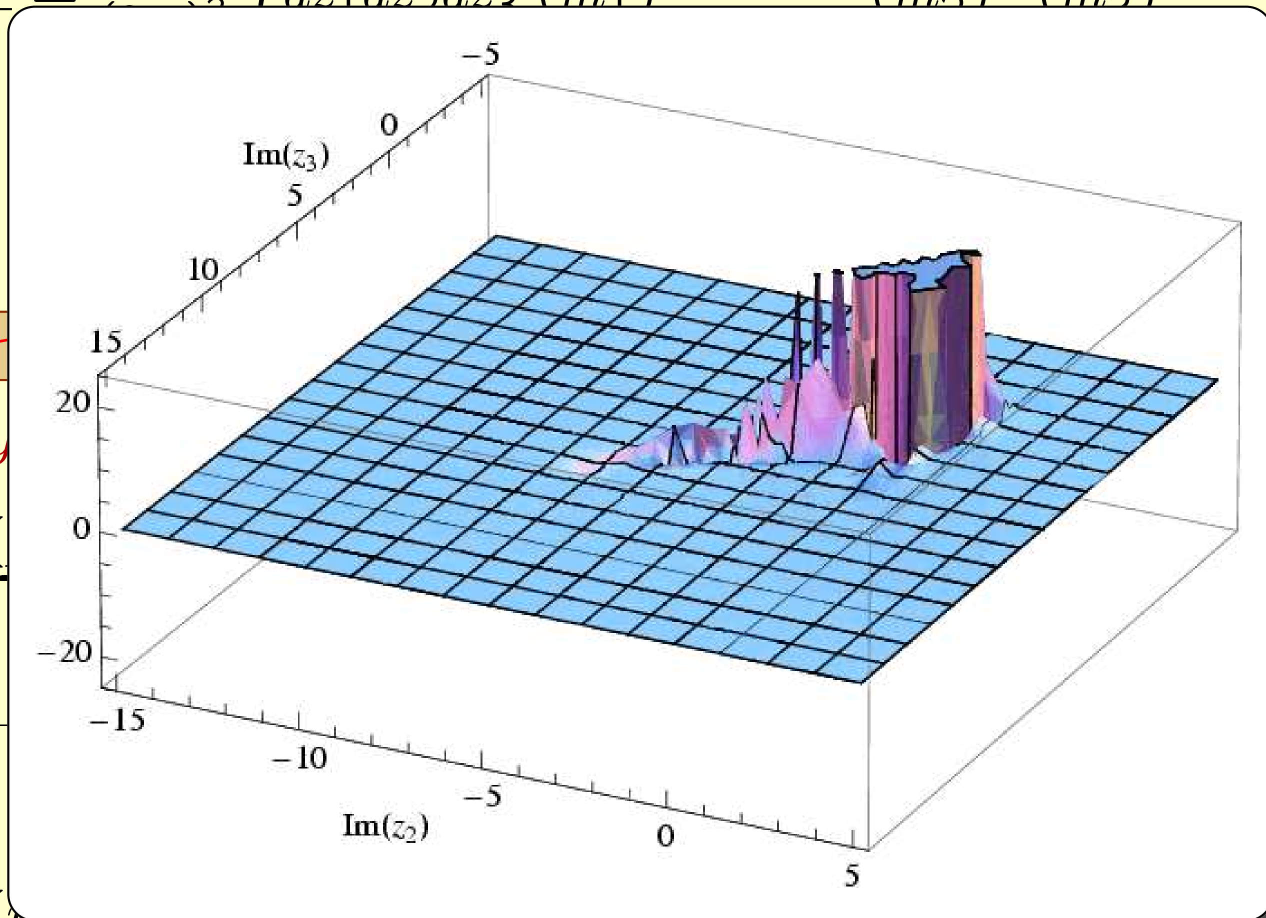
Mellin-Barnes representations



$$= \frac{-1}{(\ldots)^2} \int dz_1 dz_2 dz_3 \, (m_1^2)^{-\varepsilon - z_1 - z_2} (m_2^2)^{z_2} (m_3^2)^{1-\varepsilon+z_1-z_3} (-p^2)^{z_3}$$

$\ldots z_1)$

$\ldots z_1 + z_3)$



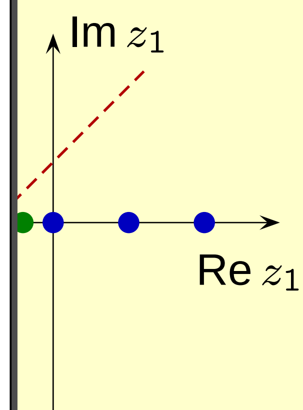
$z_3 = c_3 + iy \ldots$

$(-p^2)^{z_3} = (\ldots$

$y_i \to y_i - i\theta$

$(-p^2)^{z_3} = (p\ldots$

Freitas, Huang '10



Mellin-Barnes representations

Counter rotations not always successful:

$$\frac{1}{(2\pi i)^2}\int dz_1 dz_2\, 2(m^2)^{-2}\Big(-\frac{p^2}{m^2}\Big)^{-z_1-z_2}$$
$$\times\frac{\Gamma(-z_2)\Gamma^3(1+z_2)\Gamma(-z_1-z_2)\Gamma(1+z_1+z_2)\Gamma(-1-z_1-2z_2)}{\Gamma(1-z_1)}$$

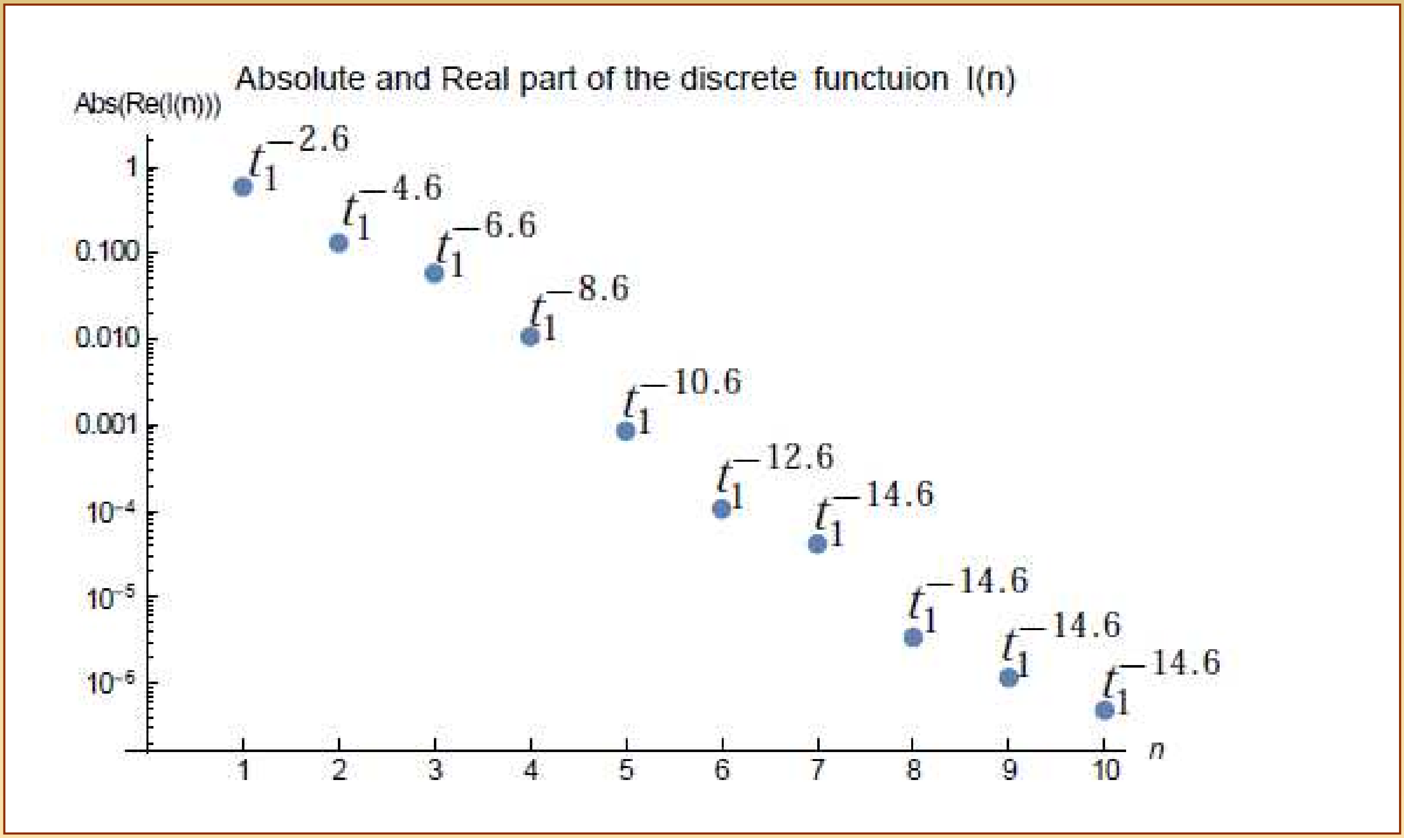
For $p^2 = m^2$ contour rotation has no effect

Shift countour: $z_1 = c_1 + iy_1,\; z_2 = c_2 + n + iy_2$

- Worst asymtotic behaviour of integrand for $y_1 \to -\infty$, $y_2 = 0$:
  $\sim y_1^{-2-2(c_2+n)}$ (for $n = 0$ and $c_2 = -0.7$: $\sim y_1^{-0.6}$)
- Pick up (finite number of) pole residues from contour shift

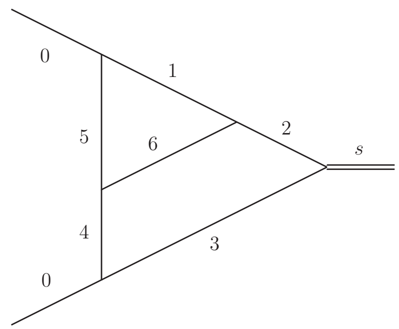
Mellin-Barnes representations

- Shifts improve asymptotic behaviour and size of numerical integral
- Automatic algorithms for finding suitable shifts in development (`MBnumerics`) Usovitsch '16



Examples

$m_1 = m_t,\ m_5 = m_6 = M_W,\ m_2 = m_3 = m_4 = 0$



SecDec: (24 hours)

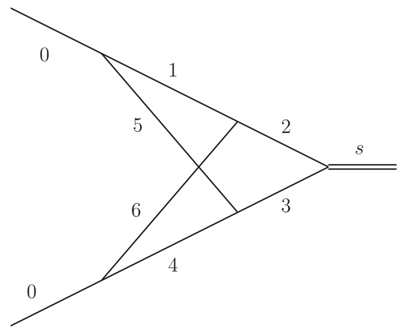
$$I_{SD} = 1.541 + 0.2487\,i + \frac{1}{\epsilon}(0.123615 - 1.06103\,i) + \frac{1}{\epsilon^2}(-0.3377373796 - 5 \times 10^{-10}\,i)$$

MBnumerics: (43 min.)

$$I_{MB} = 1.541402128186602 + 0.248804198197504\,i + \frac{1}{\epsilon}(0.1236145994284665\,9 - 1.0610332704387688\,i) + \frac{1}{\epsilon^2}(-0.3377373795505797\,0 + 3.6 \times 10^{-17}\,i)$$

Examples

$m_1 = M_Z$, rest zero



SecDec: error $\gg 1$

MBnumerics: (finite part)

$$-0.7785996083 - 4.12351260\, i$$

Analytical:

Fleischer, Kotikov, Veretin '98

$$-0.7785996090 - 4.12351259\, i$$

Result for $\sin^2\theta^{b}_{eff}$

$$\frac{\sin^2\theta^{b}_{eff}\big|_{bos}}{\sin^2\theta^{b}_{eff}} = -0.9855 \times 10^{-4}$$ (M_W fixed)

$\gtrsim$ 7 digits numerical precision

Comparison:

$$\frac{\sin^2\theta^{b}_{eff}\big|_{ferm}}{\sin^2\theta^{b}_{eff}} = 3.85 \times 10^{-4}$$

Awramik, Czakon, Freitas, Kniehl ’08

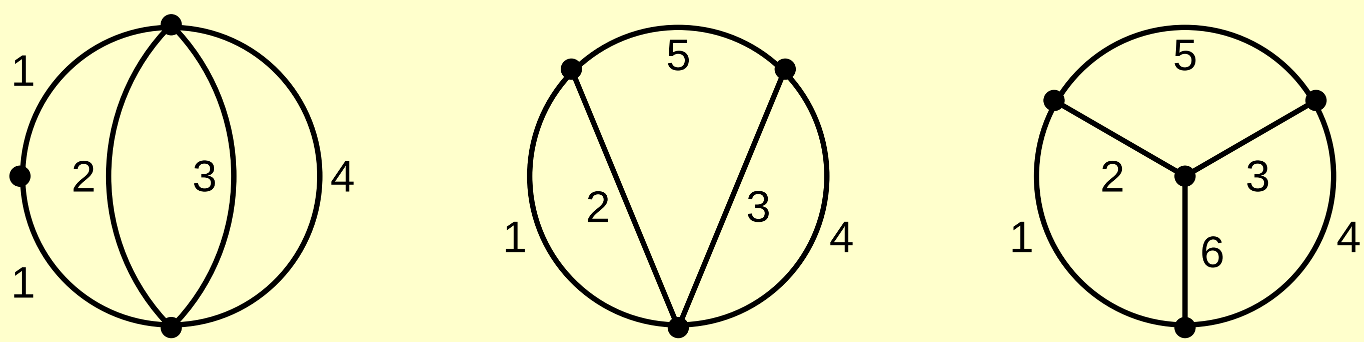
Experiment (LEP+SLD combination): $\sin^2\theta^{b}_{eff} = 0.281 \pm 0.016$

General 3-loop vacuum integrals

- Relevant for low-energy precision observables ($p^2 \ll M_Z$)
- Coefficients of low-momentum expansions
- Building block for more general 3-loop calculations

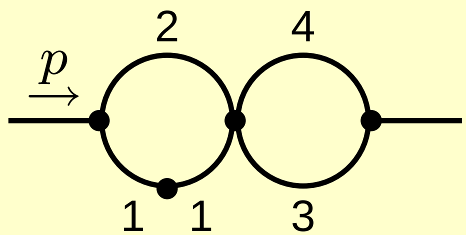
Master integrals:

$$M(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6;\, m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2)$$
$$= i\frac{e^{3\gamma_E \epsilon}}{\pi^{3D/2}} \int d^D q_1\, d^D q_2\, d^D q_3\, [q_1^2 - m_1^2]^{-\nu_1}[(q_1-q_2)^2 - m_2^2]^{-\nu_2}$$
$$\times [(q_2-q_3)^2 - m_3^2]^{-\nu_3}[q_3^2 - m_4^2]^{-\nu_4}[q_2^2 - m_5^2]^{-\nu_5}[(q_1-q_3)^2 - m_6^2]^{-\nu_6}$$



U_4 $= M(2,1,1,1,0,0)$ U_5 $= M(1,1,1,1,1,0)$ U_6 $= M(1,1,1,1,1,1)$

U_4

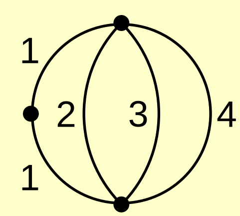


$$= B_{0,m_1}(p^2, m_1^2, m_2^2) B_0(p^2, m_3^2, m_4^2)$$
$$= \int_0^\infty ds \, \frac{\Delta I_{\mathrm{db}}(s)}{s - p^2 - i\varepsilon}$$

$$\Delta I_{\mathrm{db}}(s, m_1^2, m_2^2, m_3^2, m_4^2) = \Delta B_{0,m_1}(s, m_1^2, m_2^2)\, B_0(s, m_3^2, m_4^2) + B_{0,m_1}(s, m_1^2, m_2^2)\, \Delta B_0(s, m_3^2, m_4^2),$$

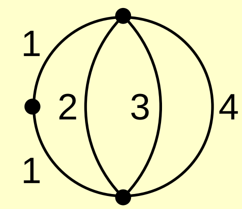
$$\Delta B_0(s, m_a^2, m_b^2) = \frac{1}{s} \lambda(s, m_a^2, m_b^2)\, \Theta\Big(s - (m_a + m_b)^2\Big)$$
$$\Delta B_{0,m_1}(s, m_a^2, m_b^2) = \frac{m_a^2 - m_b^2 - s}{s\, \lambda(s, m_a^2, m_b^2)}\, \Theta\Big(s - (m_a + m_b)^2\Big)$$

$$U_4(m_1^2, m_2^2, m_3^2, m_4^2) = -\frac{e^{\gamma_{\mathrm{E}} \epsilon}}{i\pi^{D/2}} \int d^D q_3 \int_0^\infty ds \, \frac{\Delta I_{\mathrm{db}}(s)}{q_3^2 - s + i\varepsilon}$$
$$= -\int_0^\infty ds\, A_0(s)\, \Delta I_{\mathrm{db}}(s)$$



U_4

Problem: U_4 is divergent



Solution:

$$U_4(m_1^2,m_2^2,m_3^2,m_4^2) = U_4(m_1^2,m_2^2,0,0) + U_4(m_1^2,0,m_3^2,0) + U_4(m_1^2,0,0,m_4^2) - 2\,U_4(m_1^2,0,0,0) + U_{4,\text{sub}}(m_1^2,m_2^2,m_3^2,m_4^2)$$

$\longrightarrow$ $U_4(m_X^2,m_Y^2,0,0)$ can be computed analytically

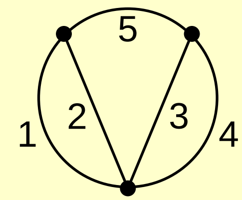
$\longrightarrow$ $U_{4,\text{sub}}$ is finite

$$U_{4,\text{sub}}(m_1^2,m_2^2,m_3^2,m_4^2) = -\int_0^\infty ds\, A_{0,\text{fin}}(s)\, \Delta I_{\text{db,sub}}(s)$$

$$\begin{aligned} I_{\text{db,sub}}(s,m_1^2,m_2^2,m_3^2,m_4^2) = \\ &\Delta B_{0,m_1}(s,m_1^2,m_2^2)\ \text{Re}\Big\{B_0(s,m_3^2,m_4^2) - B_0(s,0,0)\Big\} \\ &- \Delta B_{0,m_1}(s,m_1^2,0)\ \text{Re}\Big\{B_0(s,0,m_3^2) + B_0(s,0,m_4^2) - 2B_0(s,0,0)\Big\} \\ &+ \text{Re}\Big\{B_{0,m_1}(s,m_1^2,m_2^2)\Big\}\Big[\Delta B_0(s,m_3^2,m_4^2) - \Delta B_0(s,0,0)\Big] \\ &- \text{Re}\Big\{B_{0,m_1}(s,m_1^2,0)\Big\}\Big[\Delta B_0(s,0,m_3^2) + \Delta B_0(s,0,m_4^2) - 2\,\Delta B_0(s,0,0)\Big] \end{aligned}$$

U_5

Integration-by-parts relations:



$$U_5(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$$

$$= F\Big[A_0(m_i),\, T_3(m_i, m_j, m_k),\, U_4(m_i, m_j, m_k, m_l)\Big]$$

$$+ \frac{\lambda_{125}^2 \lambda_{345}^2}{(3-D)^2(m_2^2 - m_1^2 + m_5^2)(m_3^2 - m_4^2 + m_5^2)} M(2,1,1,2,1,0)$$

$F[...]$ = some linear combination (lengthy)

$M(2,1,1,2,1,0)$ is finite

$$U_5(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = -\frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int d^D q_3 \int_0^\infty ds\, \frac{\Delta I_{db2}(s)}{[q_3^2 - s][q_3^2 - m_5^2]}$$

$$= -\int_0^\infty ds\, B_0(0, s, m_5^2)\, \Delta I_{db2}(s)$$

$$\Delta I_{db2}(s, m_1^2, m_2^2, m_3^2, m_4^2) = \Delta B_{0,m_1}(s, m_1^2, m_2^2)\, B_{0,m_1}(s, m_4^2, m_3^2) + B_{0,m_1}(s, m_1^2, m_2^2)\, \Delta B_{0,m_1}(s, m_4^2, m_3^2)$$

Results

- U_4, U_5 given in terms of one-dimensional numerical integrals of elem. functions
- Special cases (e.g. $m_1 = 0$) can also be handled
- U_6 in progress...

Checks: (finite part shown)

$x = 0.8^2$	This work	Grigo, Hoff, Marquard, Steinhauser '12
$U_4(1,1,1,x)$	3.641562533670	3.641562533537
$U_4(1,x,x,x)$	4.209536621473	4.209536621428
$M(1,1,1,1,0,0;\ 1,1,1,x)$	37.77079673659	37.77079673639
$M(1,1,1,1,0,0;\ 1,x,x,x)$	33.73316262161	33.73316262154
	This work	Chetyrkin, Steinhauser '99
$U_5(1,1,0,0,1)$	55.65962246120629	55.65962246120630

Conclusions

- **Contour shifts** are very powerful for numerical evaluation of Mellin-Barnes integrals for Minkowskian external momenta
- **First application:** bosonic $\mathcal{O}(\alpha^2)$ corrections to $\sin^2\theta^{\rm b}_{\rm eff}$
- Package `MBnumerics` for contour shift technique under development
- **Numerical techniques** are promising but need to be improved substantially
- **3-loop vacuum integrals** with general masses can be evaluated through one-dimensional numerical integrals of elementary functions

Backup slides

Z-pole observables

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\text{ini}}(s,s') \otimes \sigma_{\text{hard}}(s')$$

- Subtraction of γ-exchange, γ–Z interference, box contributions:

$$\sigma_{\text{hard}} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{\text{box}}$$

- Z-pole contribution:

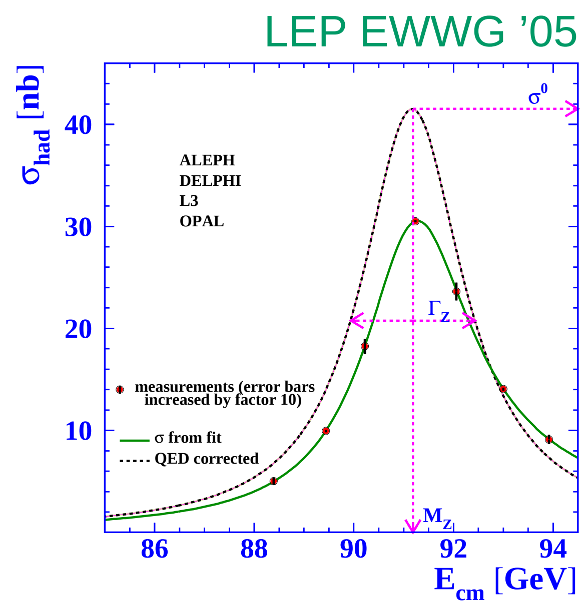
$$\sigma_Z = \frac{R}{(s-\overline{M}_Z^2)^2 + \overline{M}_Z^2 \overline{\Gamma}_Z^2} + \sigma_{\text{non-res}}$$

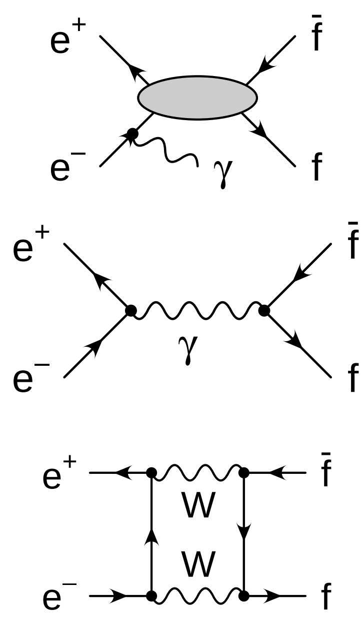
- In experimental analyses:

$$\sigma \sim \frac{1}{(s-M_Z^2)^2 + s^2\Gamma_Z^2/M_Z^2}$$

$$\overline{M}_Z = M_Z \Big/ \sqrt{1+\Gamma_Z^2/M_Z^2} \approx M_Z - 34\text{ MeV}$$

$$\overline{\Gamma}_Z = \Gamma_Z \Big/ \sqrt{1+\Gamma_Z^2/M_Z^2} \approx \Gamma_Z - 0.9\text{ MeV}$$





Variables mapping

Map MB integrals onto interval [0,1]:

$$z_i = x_i + i\,\frac{1}{\tan(-\pi t_i)}, \qquad t_i \in (0,1)$$

Jacobian: $\dfrac{\pi}{\sin^2(\pi t_i)}$

In addition, $\Gamma \to e^{\ln \Gamma}$ improves numerical stability

U_4 for $m_1 = 0$

U_4 with $m_1 = 0$ has IR singularity!

$$
\begin{aligned}
U_4(0, m_2^2, m_3^2, m_4^2) = &\, B_0(0,0,0)\, T_3(m_2^2, m_3^2, m_4^2) \\
&- B_0(0, \delta^2, \delta^2)\, T_3(m_2^2, m_3^2, m_4^2) \\
&+ U_4(\delta^2, m_2^2, m_3^2, m_4^2) + \mathcal{O}(\delta^2)
\end{aligned}
$$

($\delta \ll m_i$)

$\log \delta$ dependence of $U_4(\delta^2, m_2^2, m_3^2, m_4^2)$ can be extracted explicitly to avoid numerical instabilities