Two-Loop Integrand Decomposition Into Master Integrands And Surface Terms

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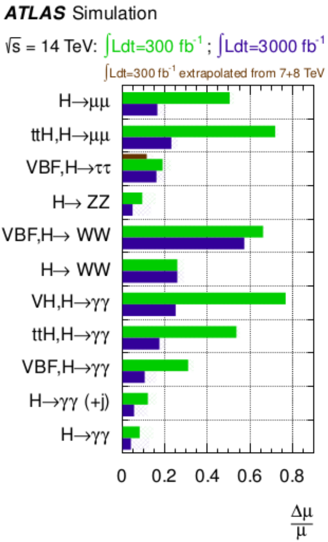
Based on work with S. Abreu, Z. Bern, F. Febres-Cordero, H. Ita, B. Page and M. Zeng.

LHC era

High-luminosity run of the LHC will narrow down experimental errors substantially.

Need to provide NNLO predictions for many processes.



Progress in NNLO phenomenology

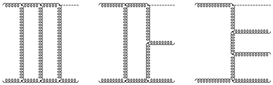
- Di-photon: [Catani, Cieri, de Florian, Ferrera, Grazzini 11; Campbell, Ellis, Li, Williams 16]
- Dijet: [Currie, Gehrmann-De Ridder, Gehrmann, Glover, Pires, Wells 14]
- W+J: [Boughezal, Focke, Liu, Petriello 15]
- Z+J: [Gehrmann-De Ridder, Gehrmann, Glover, Huss, Morgan 15; Boughezal, Campbell, Ellis, Focke, Giele, Liu, Petriello 15]
- H+J: [Chen, Cruz-Martinez, Gehrmann, Glover, MJ 16; Caola, Melnikov, Schulze 15; Boughezal, Focke, Giele, Liu, Petriello 15]
- tt: [Czakon, Fiedler, Heymes, Mitov 16]
- WW: [Gehrmann, Grazzini, Kallweit, Maierhöfer, v. Mannteuffel, Pozzorini, Rathlev, Tancredi 14; Caola, Melnikov, Röntsch, Tancredi 15]
- ZZ: [Cascioli, Gehrmann, Grazzini, Kallweit, Maierhöfer, v. Mannteuffel, Pozzorini, Rathlev, Tancredi, Weihs 14; Grazzini, Kallweit, Rathlev 15; Caola, Melnikov, Röntsch, Tancredi 16]
- ZH: [Ferrera, Grazzini, Tramontano 14; Campbell, Ellis, Williams 16]
- $Z\gamma$, $W\gamma$: [Grazzini, Kallweit, Rathlev, Torre 14]
- HH: [de Florian, Grazzini, Hanga, Kallweit, Lindert, Maierhöfer, Mazzitelli, Rathlev 16]

Can we go beyond $2 \to 2$?
Multiscale processes?

- 5-point amplitudes [Badger, Frellesvig, Zhang 15; Gehrmann, Henn, Lo Presti 15]
- 6-point amplitudes [Dunbar, Perkins 2016; Badger, Mogull, Peraro 16]

Current limitations

☞ Main bottleneck: two loop contribution



Feynman diagrams

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Feynman Integrals

- Many diagrams
- Large cancellations
- Process specific

- Tensor reduction [Tarasov 96; Anastasiou, Glover, Oleari 99]
- IBP identities [Tkachov, Chetyrkin 81]

⇒
  - Few master integrals
  - Useful for several processes
- Differential equations [Gehrmann, Remiddi 01]

Status at NLO

Significant improvements due to on-shell methods:

- Fully automatised computations.
- All necessary master integrals known.
- Amplitudes assembled from on-shell and gauge-invariant pieces. (Tree amplitudes $\leftarrow$ recursion [Berends, Giele 88; Britto, Cachazo, Feng, Witten 05]) $\rightarrow$ Feynman diagrams avoided!

Implementations:

- Blackhat [Bern, Dixon, Febres Cordero, Höche, Ita, Kosower, Maître, Ozeren 13]
- NJET [Badger, Biedermann, Uwer, Yundin 12]
- OpenLoops [Cascioli, Maierhöfer, Pozzorini 12]
- MadGraph [Alwall, Frederix, Frixione, Hirschi, Maltoni, Mattelaer, Shao, Stelzer, Torrielli, Zaro 14]
- GoSam [Cullen, v. Deurzen, Greiner, Heinrich, Luisoni, Mastrolia, Mirabella, Ossola, Peraro, Schlenk, v, Soden-Fraunhofen, Tramontano 14]

☞ Can the same be done at two loops? ☜

[Badger, Bobadilla, Caron-Huot, Frelleswig, Johansson, Kosower, Larsen, Mastrolia, Ossola, Primo, Zhang,. . .]

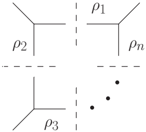
Goal

Find parametrisation

$$\mathcal{I}_{i_1 \dots i_n} = \int \mathrm{d}^D \ell \frac{\sum_k c_k t^k(\ell)}{\rho_1 \cdots \rho_n} = \int \mathrm{d}^D \ell \frac{\sum_i c_i t^i_{master}(\ell) + \sum_j c_j t^j_{surface}(\ell)}{\rho_1 \cdots \rho_n}.$$

[Ossola, Papadopoulos, Pittau 06; Bern, Dixon, Kosower]

- The coefficients c_k can be determined on the cut [Bern, Dixon, Kosower 06],

$$\sum_k c_k t^k(\ell) =$$


$$(\ell).$$

- Parametrisation in terms of integrands of master integrals and terms vanishing upon integration.

Adapted coordinates

Parametrise the loop momentum in terms of inverse propagators ρ_i and (depending on the number of propagators available) additional transverse variables α:

$$\ell^\mu = \sum_{i=1}^{D_p} r_i v_i^\mu + \sum_{a=1}^{D_t} \alpha_a n_a^\mu$$ [van Neerven, Vermaseren 84; see also *The analytic* S−*matrix*.]

$$r_i = -\frac{1}{2}\left((\rho_i + m_i^2 - q_i^2) - (\rho_{i-1} + m_{i-1}^2 - q_{i-1}^2)\right).$$

With $r_i = (\ell \cdot p_i)$ and $\alpha_i = (\ell \cdot n_i)$.

- Putting propagators on shell easily implemented as $\rho_i \to 0$.

Adapted coordinates

We have $D+1$ loop variables, $\rho_0, \ldots, \rho_{D_p}, \alpha_1, \ldots, \alpha_{D_t}$, and one constraint,

$$0 = \ell^2 - m_0^2 - \rho_0 = \left(\sum_{i=1}^{D_p} r_i v_i\right)^2 + \sum_{a=1}^{D_t} \alpha_a^2 - m_0^2 - \rho_0 = c(\rho, \alpha).$$

- $c(\rho, \alpha) = 0$ defines the physical momentum space.
- Tensor terms are given by algebraic functions:

$$t_{\mu_1 \ldots \mu_n} \ell^{\mu_1} \ldots \ell^{\mu_n} = \prod_{a,i} (\alpha_a)^{k_a} (\rho_i)^{k_i},$$

  with k_a and k_i bounded by QCD power counting.

$\Rightarrow$ Parametrisation of the integrand together with the scalar one.

Parametrisation of the Integrand

What are the master integrals?

$\hookrightarrow$ Solve integration-by-part (IBP) identities:

$$\int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \partial_\mu \left(\frac{u^\mu t(\ell)}{\rho_1 \ldots \rho_n} \right) = 0,$$

$\Rightarrow$ integrand parametrisation:

All integrands = master integrands + surface terms (IBP's) [Ita 15],

where the surface terms vanish upon integration.

- Automatically performs the reduction to master integrals
  $\rightarrow$ advantage for numerical computation.
- Keep surface terms only during intermediate steps of the computation.

IBP generating vectors

Generic vector fields u^μ yield IBP's with doubled propagators. This can be avoided by choosing vectors satisfying [Gluza, Kajda, Kosower 11]

$$(u^\mu \partial_\mu)\rho_i = f_i(\ell)\rho_i \quad \forall \rho_i \quad \Rightarrow \quad u^\mu = (f_i \rho_i, u_a)$$ [Ita 15; Larsen, Zhang 15]

Then,

$$\partial_\mu \left(\frac{u^\mu t(\ell)}{\rho_1 \dots \rho_n} \right) = \frac{(\partial_\mu u^\mu) t(\ell)}{\rho_1 \dots \rho_n} + \frac{u^\mu \partial_\mu t(\ell)}{\rho_1 \dots \rho_n} - \sum_j \frac{t(\ell) u^\mu \partial_\mu \rho_j}{\rho_1 \dots \rho_j^2 \dots \rho_n}$$
$$= \frac{(\partial_\mu u^\mu) t(\ell)}{\rho_1 \dots \rho_n} + \frac{u^\mu \partial_\mu t(\ell)}{\rho_1 \dots \rho_n} - \sum_j \frac{t(\ell) f_j}{\rho_1 \dots \rho_n}.$$

Impose $u^\mu \partial_\mu c(\rho, \alpha) = 0$ to stay inside the physical momentum space.

IBP generating vectors at one loop

3 Types of IBP generators: Horizontal ($\rho_i = 0$), vertical ($\alpha_a = 0$) and mixed.

Horizontal	$u^\mu_{[ab]}\partial_\mu = \alpha_a\partial_b - \alpha_b\partial_a$	Generic topologies
Vertical	$u^\mu\partial_\mu = \sum_i f_i\rho_i\partial_i$	Links different topologies
Mixed	$u^\mu\partial_\mu = \sum_i f_i\rho_i\partial_i + \sum_a g_a\alpha_a\partial_a$	Degenerate on-shell PS

How to find the integrand decomposition?

- Write down all monomials in α compatible with power counting.
- Act with the IBP generating vectors $\rightarrow$ surface terms.
- The master integrands are in the complement.

This reproduces the well-known one-loop results for all topologies.

Two loop topologies

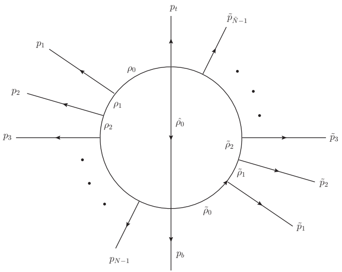
Planar: two sets of one-loop para's:

$\rho_0, \ldots, \rho_{D_p}, \alpha_1, \ldots, \alpha_{D_t}, \ c(\rho, \alpha)$

$\tilde{\rho}_0, \ldots, \tilde{\rho}_{\tilde{D}_p}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{\tilde{D}_t}, \ \tilde{c}(\rho, \alpha)$

Additional constraint from central propagator:

$$\hat{c}(\rho, \tilde{\rho}, \alpha, \tilde{\alpha}) = 0$$



Works for non-planar and higher loops as well.

Two loop IBP generating vectors

The generating vectors must satisfy $u^\mu \partial_\mu \{c, \tilde{c}\} = 0$. Thus they are made of combinations of one loop generating vectors. Imposing furthermore $u^\mu \partial_\mu \hat{c}(\rho, \tilde{\rho}, \alpha, \tilde{\alpha}) = 0$ singles out the following combinations:

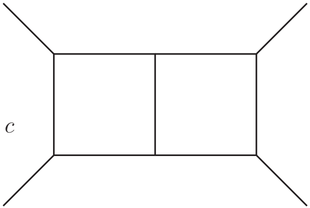
- Two loop rotations: $u^\mu_{[abc]} = \partial_{[a|} u^\mu_{|bc]}$,
- Diagonal rotations: $u^{\mathrm{diag},\mu}_{[ab]} = u^\mu_{[ab]} + \tilde{u}^\mu_{[ab]}$,
- Crossed rotations: $u^\mu_{[ab][cd]} = (\tilde{u}_{[cd]}\hat{c}) u^\mu_{[ab]} - (u_{[ab]}\hat{c})\tilde{u}^\mu_{[cd]}$.

Additional IBP generators for:

- Nonplanar
- Degenerate phase space
- D dimensions

Example

Two loop double box in 4D:



32 indep. monomials:

1

$\alpha_1, \alpha_2, \tilde{\alpha}_1, \tilde{\alpha}_2$

$\alpha_2^2, \alpha_1\alpha_2, \tilde{\alpha}_2^2, \tilde{\alpha}_1\tilde{\alpha}_2, \alpha_1\tilde{\alpha}_1, \alpha_1\tilde{\alpha}_2, \alpha_2\tilde{\alpha}_1$

$\alpha_2^3, \alpha_1\alpha_2^2, \tilde{\alpha}_2^3, \tilde{\alpha}_1\tilde{\alpha}_2^2, \alpha_2^2\tilde{\alpha}_1, \alpha_1\tilde{\alpha}_2^2,$

$\alpha_1\tilde{\alpha}_1\tilde{\alpha}_2, \alpha_1\alpha_2\tilde{\alpha}_1$

$\alpha_2^4, \alpha_1\alpha_2^3, \tilde{\alpha}_2^4, \tilde{\alpha}_1\tilde{\alpha}_2^3, \alpha_2^3\tilde{\alpha}_1, \alpha_1\tilde{\alpha}_2^3,$

$\alpha_1\tilde{\alpha}_1\tilde{\alpha}_2^2, \alpha_1\alpha_2^2\tilde{\alpha}_1$

$\alpha_2^4\tilde{\alpha}_1, \alpha_1\tilde{\alpha}_2^4, \alpha_1\tilde{\alpha}_1\tilde{\alpha}_2^3, \alpha_1\alpha_2^3\tilde{\alpha}_1$

One IBP generator $(\tilde{u}_{[12]}\hat{c})u^\mu_{[12]} - (u_{[12]}\hat{c})\tilde{u}^\mu_{[12]}$.
9 masters found in complement of surface terms.

Validation

- Reproduce the count of master integrands obtained by studying the cohomology of on-shell phase-space.
- Solve the equations

$$\left(\rho_i f_i(\ell)\frac{\partial}{\partial \rho_i} + u_a \frac{\partial}{\partial \alpha_a}\right) c(\rho, \alpha) = 0,$$

for instance with *Macaulay2* [Grayson, Stillman 92], *Singular* [Decker, Greuel, Pfister, Schönemann 15].

$\Rightarrow$ No further IBP generating vectors than the above ones.

Conclusions and outlook

- We present a decomposition of the two loop integrand.
- The decomposition implements the step of integral reduction in a natural way.
- The integrand is written in terms of master integrands and surface terms.
- We give a method which allow to generate all required surface terms.

Next steps:

- Write down the explicit decomposition for all necessary topologies.
- Proof of principle computation.

Thanks!

Special cases

- Nonplanar generating vector:

$$u^{\mu} = e_{[\tilde{a}\tilde{b}]} e'_{[cd]} u^{\mu}_{[fg]} - e_{[cd]} e'_{[\tilde{a}\tilde{b}]} u^{\mu}_{[fg]} + e_{[cd]} e'_{[fg]} u^{\mu}_{[\tilde{a}\tilde{b}]} - e_{[\tilde{a}\tilde{b}]} e'_{[fg]} u^{\mu}_{[cd]} + e_{[fg]} e'_{[\tilde{a}\tilde{b}]} u^{\mu}_{[cd]} - e_{[fg]} e'_{[cd]} u^{\mu}_{[\tilde{a}\tilde{b}]},$$

$$e_{[ab]} = u_{[ab]} \hat{c} \qquad e_{[ab]} = u_{[ab]} \hat{c}',$$

  with an additional propagator constraint $\hat{c}'$.

- Massless internal and at least one massless external particle: Further one loop rescalings

$$\sum_i f_i \rho_i \partial_i + \sum_a g_a \alpha_a \partial_a$$

  can appear in the crossed rotations.

D-dimensional surface terms

Split the loop momenta as

$$\ell^2 = \ell_4^2 + \ell_{D-4}^2 \quad , \quad \tilde{\ell}^2 = \tilde{\ell}_4^2 + \tilde{\ell}_{D-4}^2$$

Parametrise $(D-4)$-dimensional components with additional transverse variables μ^i, $\tilde{\mu}^j$, with $i, j = 5, \ldots, D$. Rotational invariance in the $(D-4)$-dimensional space allows only the combinations μ^2, $\tilde{\mu}^2$ and $(\mu \cdot \tilde{\mu})$.

Ensure rotational invariance of the IBP's by contracting the generating vectors with μ^i, $\tilde{\mu}^j$:

$$\mu^i u^{\nu}_{[\mu_i a]}$$
$$\mu^i u^{\nu}_{[\mu_i ab]}$$
$$\ldots$$