Relating the top quark $\overline{\mathrm{MS}}$ and on-shell masses

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DESY

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Introduction

When we talk about the mass of a particle we have to define the renormalization scheme used.
Two schemes are particularly important

- $\overline{\mathrm{MS}}$ scheme (running mass)
- on-shell scheme

The value of the mass depends on the choice of the renormalization scheme!
We must be able to translate between the schemes.
Current (and future) experimental precision requires NNNLO precision.

Scheme definitions

One considers the renormalized quark propagator

$$S_F(q) = \frac{-i\, Z_2}{\not{q} - Z_m m + \Sigma(q, m)}$$

with $\Sigma(q, m)$ the quark two-point function.
For the $\overline{\mathrm{MS}}$ scheme we require

$$S_F(q) \quad \text{finite}$$

and for the on-shell scheme we require a pole at the position of the mass

$$S_F(q) \stackrel{q^2 \to M^2}{\longrightarrow} \frac{-i}{\not{q} - M}$$

Setup of the calculation

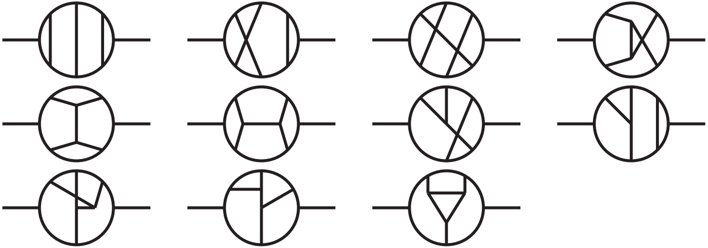
- Need to calculate mass renormalization constant Z_m^{OS} by calculating four-loop on-shell integrals
- Together with the renormalization constant in the $\overline{\text{MS}}$-scheme $Z_m^{\overline{\text{MS}}}$ [Chetyrkin '97; Larin, van Rittbergen, Vermaseren '97]
  we get

$$\left.\begin{array}{l} m_{\text{bare}} = Z_m^{\text{OS}} M \\ m_{\text{bare}} = Z_m^{\overline{\text{MS}}} m \end{array}\right\} \Rightarrow m = M \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}}$$

  - 1-loop [Tarrach'81]
  - 2-loop [Gray,Broadhurst,Grafe,Schilcher'90]
  - 3-loop [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]

Setup of the calculation

Need to calculate 4-loop on-shell diagrams of the form



where we have to consider all possible ways to route the external quark through the diagrams $\Rightarrow$ 100 integral families.

Setup of the calculation

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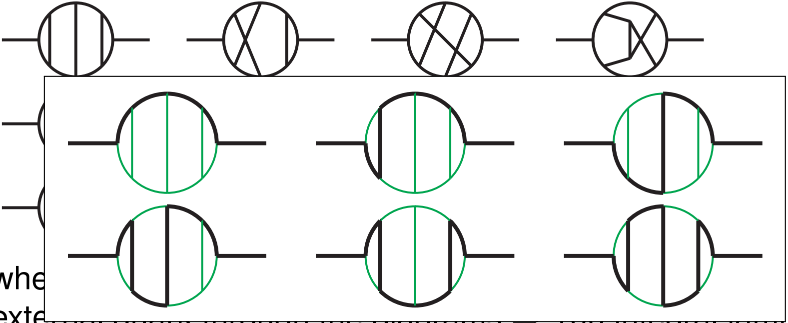


whe

external quark through the diagrams ⇒ 100 integral families.

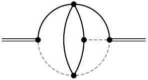
Setup of the calculation

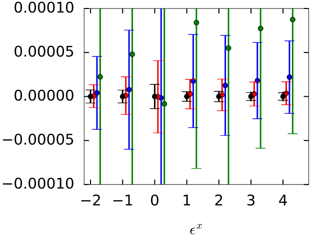
Follow the *standard* procedure for multi-loop calculations

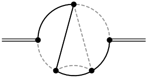
- exploit that the appearing integrals are not linear independent but related through Integration-By-Parts identities [Chetyrkin, Tkachov '81]
- reduce all appearing integrals to a small set of basis integrals using `FIRE` [Smirnov] or `Crusher` [PM,Seidel]
- evaluate the remaining basis integrals ($\mathcal{O}(350)$) using analytic or numerical techniques (Mellin-Barnes or Sector Decomposition)

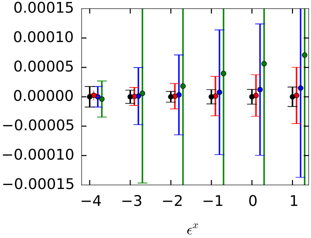
Fiesta results

How to make sure Fiesta results/error are correct? Look at convergence with number of MC points!









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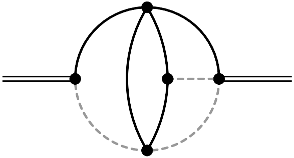
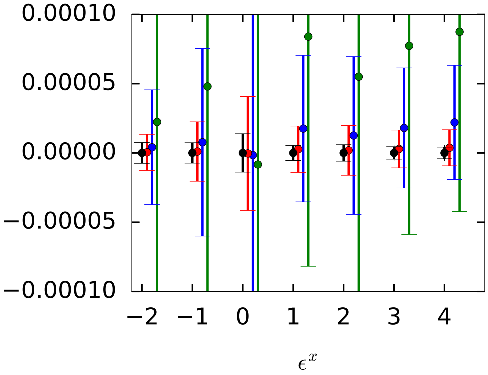
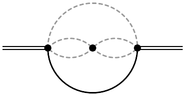




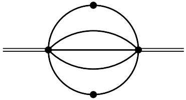
Error often underestimated
$\rightarrow$ multiply the error by a safety factor 10

Mellin-Barnes results

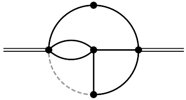
main improvement of our previous result due to the extended use of Mellin-Barnes techniques to calculate the master integrals (80 integrals)



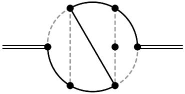
0-dim



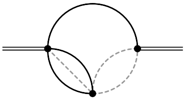
2-dim



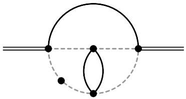
4-dim



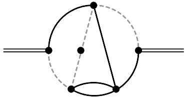
6-dim



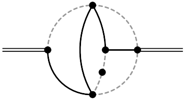
1-dim



3-dim



5-dim



7-dim

$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \to$ on-shell

$$\begin{aligned} m_t(m_t) &= M_t \Big(1 - 0.4244\,\alpha_s - 0.9246\,\alpha_s^2 - 2.593\,\alpha_s^3 \\ &\quad -(8.949 \pm 0.018)\,\alpha_s^4\Big) \\ &= 173.34 - 7.924 - 1.859 - 0.562 \\ &\quad - (0.209 \pm 0.0004)\ \text{GeV} \end{aligned}$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

small remaining error due to numerical integration of the master integrals using FIESTA [A. Smirnov] for the sector decomposition.

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$$
\begin{aligned}
M_b &= m_b(m_b) \Big(1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 \\
&\quad +(12.685 \pm 0.025)\,\alpha_s^4\Big) \\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

Threshold mass schemes

- Potential-subtracted mass

$$m^{\text{PS}} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi}(1 + \alpha_s \ldots)$$

need static potential @ three loops

[Smirnov,Smirnov,Steinhauser '09; Anzai,Kiyo,Sumino '09]

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- 1S mass

$$m^{1\text{S}} = M + \frac{1}{2} E_1^{\text{pt}}$$

$$E_1^{\text{pt}} = -\frac{C_F^2 M \alpha_s^2}{8}(1 + \alpha_s \ldots)$$

need binding energy @ three loops [Beneke et al]

PS mass $\leftrightarrow$ $\overline{\mathrm{MS}}$ mass

$$
\begin{aligned}
m_t^{\mathrm{PS}}(\mu_f = 80\ \mathrm{GeV}) &= 163.508 + (7.531 - 3.685) \\
&\quad +(1.607 - 0.989) + (0.495 - 0.403) \\
&\quad +(0.195 - 0.211 \pm 0.0004)\ \mathrm{GeV} \\
&= 163.508 + 3.847 + 0.618 + 0.092 \\
&\quad -(0.016 \pm 0.0004)\ \mathrm{GeV}
\end{aligned}
$$

$$
\begin{aligned}
m_b^{\mathrm{PS}}(\mu_f = 2\ \mathrm{GeV}) &= 4.163 + 0.207 + 0.080 \\
&\quad +0.032 - (0.0004 \pm 0.0003)\ \mathrm{GeV}
\end{aligned}
$$

- large cancellations between contributions from OS-MS and PS-OS
- good convergence

1S mass $\leftrightarrow$ $\overline{\text{MS}}$ mass

$$
\begin{aligned}
m_t^{1\mathrm{S}} &= 163.508 + (7.531 - 0.428) + (1.588 - 0.368) \\
&\quad + (0.479 - 0.262) + (0.185 - 0.174 \pm 0.0004)\ \mathrm{GeV} \\
&= 163.508 + 7.103 + 1.220 \\
&\quad + 0.217 + (0.011 \pm 0.0004)\ \mathrm{GeV}
\end{aligned}
$$

$$
\begin{aligned}
m_b^{1\mathrm{S}} &= 4.163 + 0.352 + 0.123 \\
&\quad + 0.039 - (0.008 \pm 0.0003)\ \mathrm{GeV}
\end{aligned}
$$

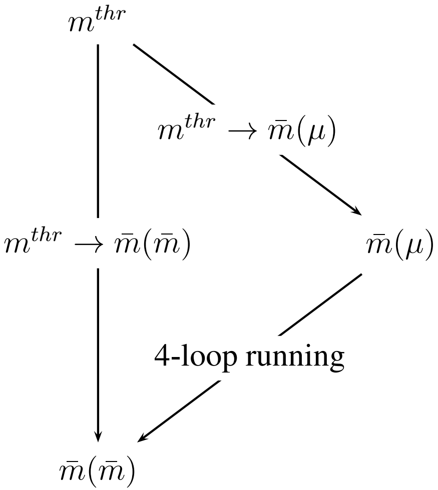
- large cancellations between contributions from OS-MS and 1S-OS
- good convergence

$m^{\text{thr}} \to \bar{m}(\bar{m})$

alternative error estimate by

- first calculating $\bar{m}(\mu)$
- and in a second step $\bar{m}(\bar{m})$

has to give the same result up to higher-order corrections



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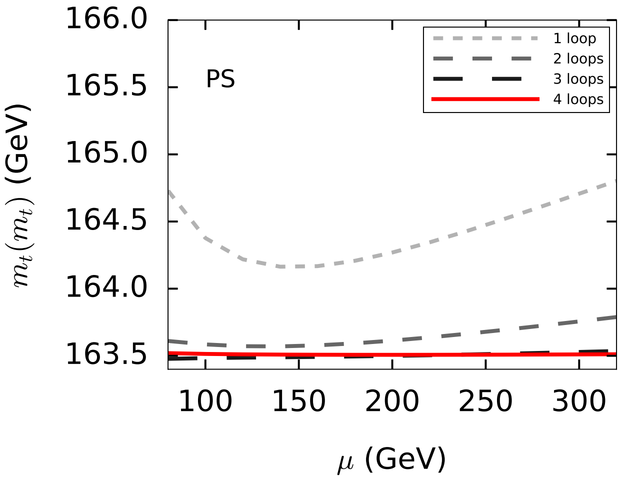

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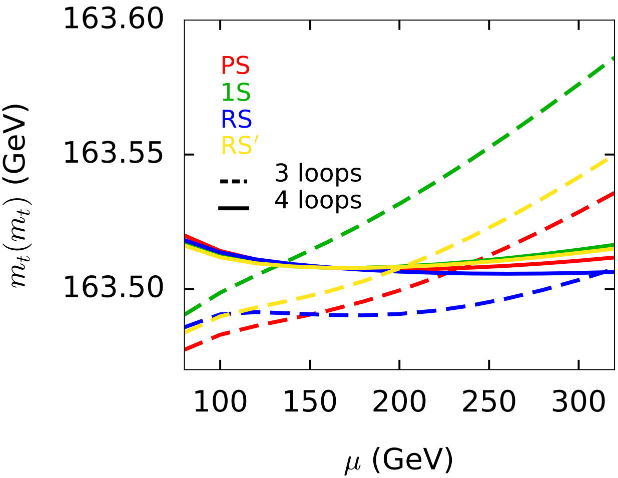
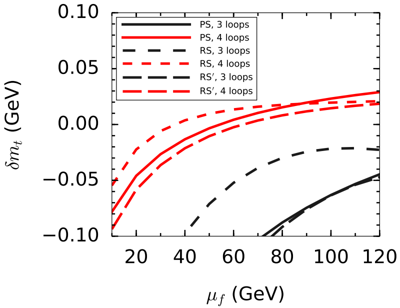
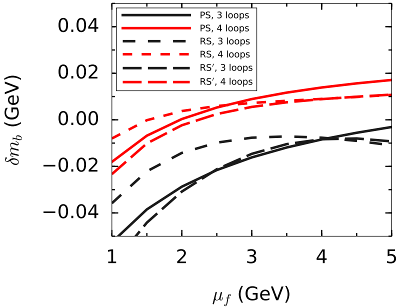


μ_f dependence

$$m^{\text{PS}} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi}(1 + \alpha_s \ldots)$$





General color structure

$$
\begin{aligned}
z_m^{(4)} \;=\;& C_F^4 z_m^{FFFF} + C_F^3 C_A z_m^{FFFA} + C_F^2 C_A^2 z_m^{FFAA} + C_F C_A^3 z_m^{FAAA} \\
&+ \frac{d_F^{abcd} d_A^{abcd}}{N_c} z_m^{d_{FA}} + n_l \frac{d_F^{abcd} d_F^{abcd}}{N_c} z_m^{d_{FF}L} + n_h \frac{d_F^{abcd} d_F^{abcd}}{N_c} z_m^{d_{FF}H} \\
&+ C_F^3 T n_l z_m^{FFFL} + C_F^2 C_A T n_l z_m^{FFAL} + C_F C_A^2 T n_l z_m^{FAAL} \\
&+ C_F^2 T^2 n_l^2 z_m^{FFLL} + C_F C_A T^2 n_l^2 z_m^{FALL} + C_F T^3 n_l^3 z_m^{FLLL} \\
&+ C_F^3 T n_h z_m^{FFFH} + C_F^2 C_A T n_h z_m^{FFAH} + C_F C_A^2 T n_h z_m^{FAAH} \\
&+ C_F^2 T^2 n_h^2 z_m^{FFHH} + C_F C_A T^2 n_h^2 z_m^{FAHH} + C_F T^3 n_h^3 z_m^{FHHH} \\
&+ C_F^2 T^2 n_l n_h z_m^{FFLH} + C_F C_A T^2 n_l n_h z_m^{FALH} + C_F T^3 n_l^2 n_h z_m^{FLLH} \\
&+ C_F T^3 n_l n_h^2 z_m^{FLHH}
\end{aligned}
$$

General color structure cont'd

$$
\begin{aligned}
z_m^{FFFF} &= -6.983 \pm 0.805\,, & z_m^{FALL} &= -42.46326\,,\\
z_m^{FFFA} &= 13.40 \pm 2.07\,, & z_m^{FLLL} &= 4.06885\,,\\
z_m^{FFAA} &= -11.17 \pm 1.74\,, & z_m^{FFFH} &= -1.3625 \pm 0.0132\,,\\
z_m^{FAAA} &= -99.272 \pm 0.493\,, & z_m^{FFAH} &= 14.9800 \pm 0.0334\,,\\
z_m^{d_{FA}} &= 0.39 \pm 1.07\,, & z_m^{FAAH} &= -2.3597 \pm 0.0342\,,\\
z_m^{d_{FF}L} &= -0.937 \pm 0.178\,, & z_m^{FFHH} &= 1.65752 \pm 0.00031\,,\\
z_m^{d_{FF}H} &= -3.924 \pm 0.642\,, & z_m^{FAHH} &= -0.20934 \pm 0.00273\,,\\
z_m^{FFFL} &= -0.05094 \pm 0.00298\,, & z_m^{FHHH} &= -0.14902 \pm 0.00000\,,\\
z_m^{FFAL} &= 9.26642 \pm 0.00454\,, & z_m^{FFLH} &= -2.89209 \pm 0.00010\,,\\
z_m^{FAAL} &= 122.1872 \pm 0.0100\,, & z_m^{FALH} &= 0.62076 \pm 0.00042\,,\\
z_m^{FFLL} &= -2.25441\,, & z_m^{FLLH} &= -0.10321\,,\\
& & z_m^{FLHH} &= -0.21703 \pm 0.00000
\end{aligned}
$$

n_c dependence

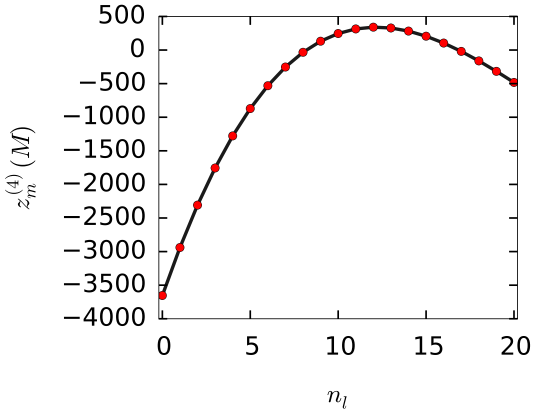
show here only n_l^1, n_l^0

$$
\begin{aligned}
z_m^{L1/N_c^3} &= 0.1788 \pm 0.0333\,, \\
z_m^{L1/N_c^2} &= -0.18076 \pm 0.00000\,, \\
z_m^{L1/N_c^1} &= 0.9282 \pm 0.0445\,, \\
z_m^{LN_c^0} &= 0.28392 \pm 0.00005\,, \\
z_m^{LN_c^1} &= -32.7991 \pm 0.0109\,, \\
z_m^{LN_c^2} &= -0.10316 \pm 0.00005\,, \\
z_m^{LN_c^3} &= 31.69215 \pm 0.00124\,,
\end{aligned}
$$

$$
\begin{aligned}
z_m^{1/N_c^4} &= -0.4364 \pm 0.0503\,, \\
z_m^{1/N_c^3} &= 0.821 \pm 0.121\,, \\
z_m^{1/N_c^2} &= 0.1739 \pm 0.0738\,, \\
z_m^{1/N_c^1} &= 0.645 \pm 0.161\,, \\
z_m^{N_c^0} &= -0.614 \pm 0.175\,, \\
z_m^{N_c^1} &= -2.6228 \pm 0.0415\,, \\
z_m^{N_c^2} &= 52.0579 \pm 0.0808\,, \\
z_m^{N_c^3} &= 1.15654 \pm 0.00424\,, \\
z_m^{N_c^4} &= -51.1812 \pm 0.0161
\end{aligned}
$$

n_l dependence

$$
\begin{aligned}
z_m^{(4)} &= -3654.15 \pm 1.64 + (756.942 \pm 0.040) n_l \\
&\quad -43.4824 n_l^2 + 0.678141 n_l^3 \,.
\end{aligned}
$$



Beyond 4-loops

$$m_P = m(\mu)\left(1 + \sum_{n=1}^{\infty} c_n(\mu, \mu_m, m(\mu))\, \alpha_s^n(\mu)\right)$$

for large n

$$c_n(\mu, \mu_m, m(\mu_m)) \underset{n\to\infty}{\longrightarrow} N c_n^{(\mathrm{as})}(\mu, m(\mu_m)) \equiv N \frac{\mu}{m(\mu_m)}\, \tilde{c}_n^{(\mathrm{as})}\ ,$$

[Beneke '94 '99]

where

$$\tilde{c}_{n+1}^{(\mathrm{as})} = (2b_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \cdots\right)$$

b_0, b, s_1, s_2: Combinations of coefficients of the β-function.

Beyond 4-loops

Fit N to 4-loop term and take higher orders from asymptotic formula

j	$\tilde{c}_j^{(\mathrm{as})}$	$\tilde{c}_j^{(\mathrm{as})}\alpha_s^j$
5	0.985499×10^2	0.001484
6	0.641788×10^3	0.001049
7	0.495994×10^4	0.000880
8	0.443735×10^5	0.000854
9	0.451072×10^6	0.000942
10	0.513535×10^7	0.001164

$$\begin{aligned}\delta^{(5+)} m_P &= 0.272^{+0.016}_{-0.041}\,(N) \pm 0.001\,(c_4) \pm 0.011\,(\alpha_s) \\ &\quad \pm 0.066\ (\text{amb})\ \text{GeV}\end{aligned}$$

[Beneke, PM, Nason, Steinhauser '16]

Conclusions

- presented 4-loop corrections to the $\overline{\text{MS}}$–on-shell relation
- full dependence on the number of light quarks
- full color structure, some coefficients not known very precise
- conversion between threshold masses and $\overline{\text{MS}}$ mass well under control
- for the top-quark case
  - four-loop contribution $\approx 200\,\text{MeV}$
  - five-loop and beyond add $\approx 250\,\text{MeV}$ with an intrinsic ambiguity of $\approx 66\,\text{MeV}$