Two-loop Master Integrals for the mixed QCD$\times$EW corrections to Drell-Yan processes

Stefano Di Vita

based on work with Roberto Bonciani, Pierpaolo Mastrolia and Ulrich Schubert, submitted to JHEP [arXiv:1604.08581]

DESY (Hamburg)

LoopFest XV, University at Buffalo
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I barely have 1 "phenomenological" slide . . . hold on, dinner is close!

Outline

1. Drell-Yan processes: a very (very!) compact introduction
2. Two-loop mixed QCD×EW corrections: what to compute
3. Two-loop mixed QCD×EW corrections: how we computed

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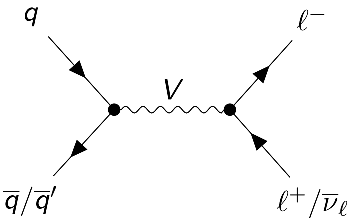
What this is about [Drell, Yan 70; . . . ; Alioli et. al. 16]

“my most phenomenological slide” ☹

- dilepton production at hadron colliders
- proceeds at LO via vector boson exchange in the *s*-channel
- useful:
  1. constrain PDFs
  2. direct determination of m_W

     template fit of $\ell\nu_\ell$ transverse mass distribution
  3. background to BSM



all diagrams drawn with `tikz-feynman` [Ellis 16]
`axodraw` [Vermaseren 94]

- recall: SM relates m_W to m_Z and EW fit is a factor 2 more precise than direct determination (PDG 80.385 ± 0.015 GeV)
- direct measurement limited by stat. (PDFs uncert. ~ 10 MeV)

History of QCD corrections I apologize for any omission

- W,Z total production rate NLO [Altarelli, Ellis, Martinelli 79; + Greco 84]
- W,Z total production rate NNLO [Matsuura, van der Marckm van Neerven 89; Hamberg, van Neerven, Matsuura 91]
- Prod. @ $p_T^{W,Z} \neq 0$ [Ellis, Martinelli, Petronzio 83; Arnold, Reno 89; Gonsalves, Pawlowski, Wai 89; Brandt, Kramer, Nyeo 91; Giele, Glover, Kosower 93; Dixon, Kunszt, Signer 98]
- Fully differential NLO to $\ell\bar{\ell}'$ (`MCFM`) [Campbell, Ellis 99]
- W,Z rapidity distrib NNLO [Anastasiou, Dixon, Melnikov, Petriello 04]
- Fully differential NNLO to $\ell\bar{\ell}'$ (`FEWZ`) [Melnikov, Petriello 06]
- Soft g resummation LL,. . . ,N^{3}LL [Sterman 87; Catani, Trentadue 89; 91; Moch, Vogt 05]
- Resummation LL/NLL in p_T^W/M_W (`RESBOS`) [Balazs, Yuan 97]
- NLO+NLL p_T^W/M_W resummation [Bozzi, Catani, De Florian, Ferrera, Grazzini 09]
- NLO+PS (`MC@NLO`, `POWHEG`) [Frixione, Webber 02; Frixione, Nason, Oleari 07; Alioli et. al. 08]
- NNLO+PS [Karlberg, Re, Zanderighi 14; Hoeche, Li, Prestel 14; Alioli, Bauer, Berggren, Tackmann, Walsh 15]
- NNLO QCD implemented in `DYNNLO` [Catani, Grazzini 07; + Cieri, Ferrera, de Florian 09]

History of EW corrections I apologize for any omission

- W,Z production at non-zero p_T [Kühn, Kulesza, Pozzorini, Schulze 04]
- W production at NLO
  - NWA [Wackeroth, Hollik 97; Baur, Keller, Wackeroth 99]
  - Exact corrections [Zykunov et. al. 01; Dittmaier, Krämer 02; Baur, Wackeroth 04 (`WGRAD2`); Arbuzov et. al. 06 (`SANC`); Carloni Calame et. al. 06 (`HORACE`); Hollik, Kasprzik, Kniehl 08; Bardin et. al. 08 `WINHAC`]
  - γ induced processes [Baur, Wackeroth 04; Dittmaier, Krämer 05; Carloni Calame et. al. 06; Arbuzov et. al. 07]
- Z production at NLO
  - Only QED [Barberio et. al. 91; Baur, Keller, Sakumoto 98; Golonka, Was 06 (`PHOTOS`); Placzek, Jadach 03+13]
  - Exact corrections [Baur et. al. 02+04; Zykunov et. al. 07; Carloni Calame et. al. 07 (`HORACE`); Dittmaier, Huber 12; Arbuzov et. al. 07 (`SANC`)]
  - γ induced processes [Carloni Calame et. al. 07 (`HORACE`)]
- V+j [Denner, Dittmaier, Kasprzik, Muck 09+11; Kallweit, Lindert, Maierhöfer, Pozzorini, Schönherr 14+15]
- 2-loop V+γ [Gehrmann, Tancredi 11]
- NNLO QCD + NLO EW in `FEWZ` [Melnikov, Petriello 06; Li, Petriello 12; + Li, Quackenbush 12]
- NLO QCD/EW `POWHEG` [Barze, Montagna, Nason, Nicrosini, Piccinini, Vicini 12+13; Bernaciak, Wackeroth 12]

NNLO mixed QCD×EW corrections: not yet fully available

- $\mathcal{O}(\alpha_s^2) \sim \mathcal{O}(\alpha)$, i.e. when QCD NNNLO is considered , also $\mathcal{O}(\alpha_s \alpha)$ becomes relevant
- Two-loop $2 \to 2$ with exchange of gluons and $\gamma/Z/W$
- One-loop $2 \to 3$, with 1 unresolved gluon or γ
- Tree-level $2 \to 4$, with 1 unresolved gluon and 1 unresolved γ
- Brief history
  - Two-loop form factors for Z production [Kotikov, Kühn, Veretin 08]
  - QCD×QED [Kilgore, Sturm 11]
  - Expansion around pole in the resonance region [Dittmaier, Huss, Schwinn 14+16]
- Bulk of corrections to inclusive observables comes from resonant region . . .
- . . . but for accurate differential distributions in regions different from resonance (and to check the pole expansion), the full calculation is needed

SHUT
UP
AND
CALCULATE

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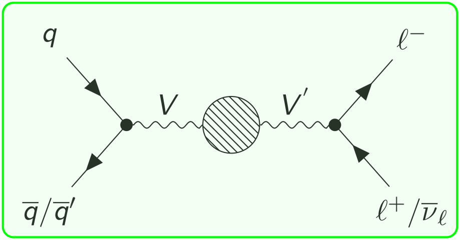
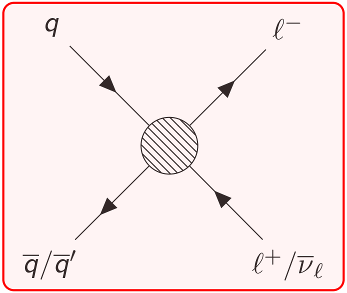
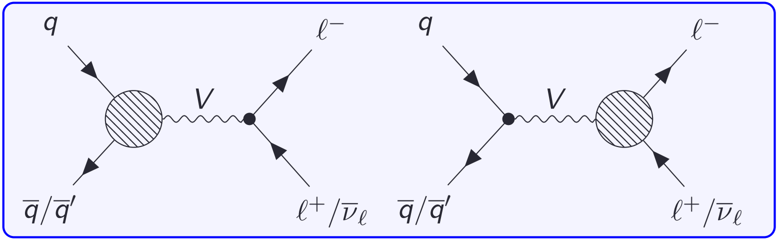
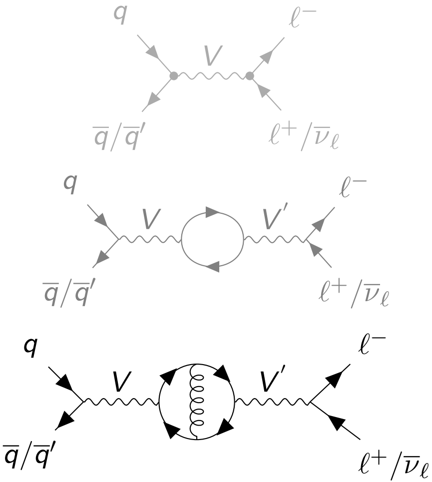
Drell-Yan dilepton production: virtual corrections





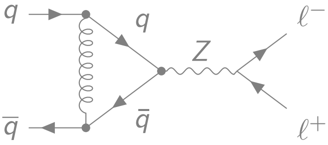


Propagator NNLO QCD×EW corrections: e.g.

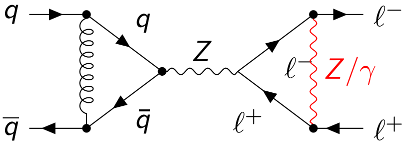


- gauge bosons couple to quarks, and quarks to gluons
- general two-loop self-energies are in principle solved, at least numerically
    - `TSIL` [Martin and Robertson 04]
    - `S2LSE` [Bauberger]
- essential building block of SM renormalization at two loops

Vertex NNLO QCD×EW corrections: e.g.

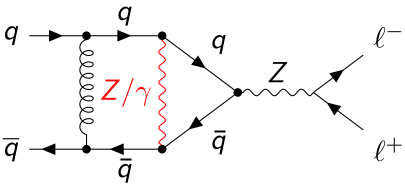


NLO QCD



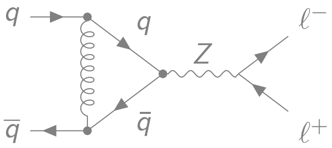
NNLO QCD×EW, factorizable, (1-loop)^2

- quarks in the initial state
- leptons in the final state
  - no QCD corrections there at 1- and 2-loops
  - no gluon exchange with initial state at 1- and 2-loops

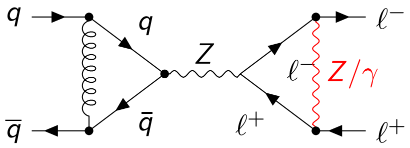


NNLO QCD×EW, factorizable, 1PI

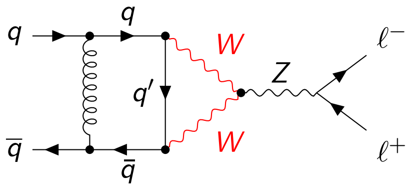
Vertex NNLO QCD×EW corrections: e.g.



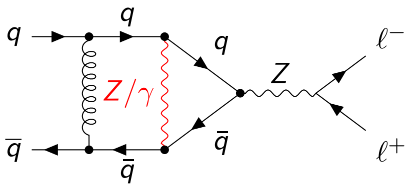
NLO QCD



NNLO QCD×EW, factorizable, (1-loop)^2

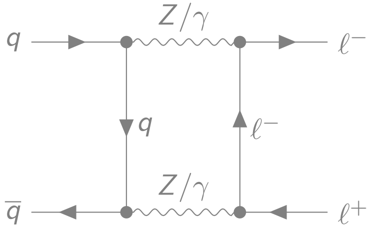


[Kotikov, Kühn, Veretin 08]

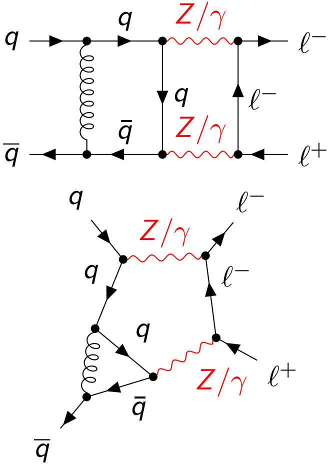


NNLO QCD×EW, factorizable, 1PI

Box NNLO QCD×EW corrections: e.g.



NLO EW, non-factorizable



NNLO QCD×EW, non-factorizable

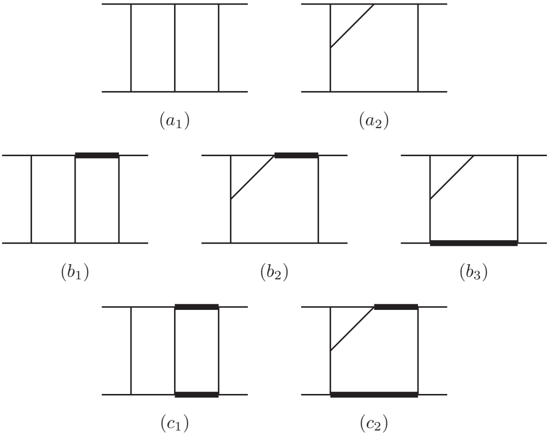
leptons in the final state

- no QCD corrections at 1-loop
- no gluon exchange with initial state
- can get boxes only by dressing the non-factorizable NLO EW

Two-loop mixed QCD×EW corrections: $q\bar{q} \to \ell^+\ell^-$

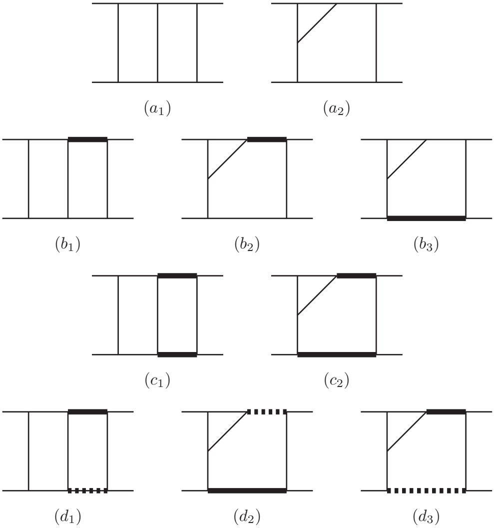
- Do it carefully (`FeynArts` [Hahn 01])
- One can map all the Feynman diagrams onto 3 families
- The corrections to the neutral current DY process never involve W and Z at the same time
- Topology A well known

  [Smirnov 99; Gehrmann, Remiddi 99]
- Topologies B-C unknown so far



Two-loop mixed QCD×EW corrections: $q\bar{q}' \to \ell^- \overline{\nu}_\ell$

- Do it carefully (`FeynArts` [Hahn 01])
- One can map all the Feynman diagrams onto 4 families
- The corrections to the charged current DY process also involve W and Z at the same time
- Topology A well known [Smirnov 99; Gehrmann, Remiddi 99]
- Topologies B-C-D unknown so far



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Let's make life a bit simpler

- Families with 1 or 2 degenerate massive propagators $\Rightarrow (s, t, m^2_{W,Z})$
- Family with 2 different massive propagators $\Rightarrow (s, t, m^2_W, m^2_Z)$
- We exploit $\Delta m^2 \equiv m^2_Z - m^2_W \ll m^2_Z$
- Expanding for instance the Z propagators around m_W

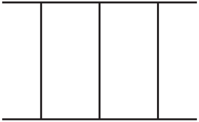
$$\frac{1}{p^2 - m^2_Z} = \frac{1}{p^2 - m^2_W - \Delta m^2} \approx \frac{1}{p^2 - m^2_W} + \frac{m^2_Z}{(p^2 - m^2_W)^2}\,\xi + \ldots$$

where

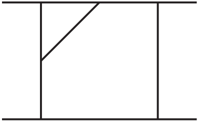
$$\xi = \frac{\Delta m^2}{m^2_Z} = \frac{m^2_Z - m^2_W}{m^2_Z} \sim \frac{1}{4}$$

- The coefficients of the series in ξ are Feynman diagrams with 3 scales
- The expanded denominators will appear raised to powers $> 1 \Rightarrow$ IBP

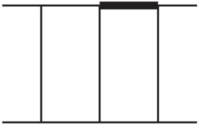
So this is what we computed Bonciani, Mastrolia, Schubert, **DV** 16



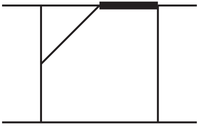
(a_1)



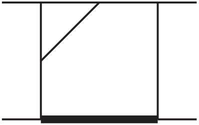
(a_2)



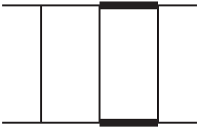
(b_1)



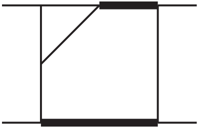
(b_2)



(b_3)



(c_1)



(c_2)

Differential equations for Master Integrals [Kotikov 91; Remiddi 97]

Integration by parts identities

Loop integrals in d dimensions satisfy linear identities (IBPs + other)

$$\int \frac{d^d k}{(k^2-m^2)^2[(k-p)^2-m^2]} \equiv \int \frac{d^d k}{D_1^2\, D_2}$$

$$= \frac{d-3}{(p^2-4m^2)} \int \frac{d^d k}{D_1\, D_2} - \frac{d-2}{2m^2(p^2-4m^2)} \int \frac{d^d k}{D_1}$$

Only a finite number of them are independent (MIs)! ☺

- `AIR` [Anastasiou, Lazopoulos 04], `FIRE` [Smirnov 08], `REDUZE` [Studerus 10; + von Manteuffel 12], `LiteRed` [Lee 12]
- Take derivatives wrt external p_{ij}^2's and m_i^2's → use IBPs → obtain system of linear differential equations for the MIs (ODEs or PDEs)

$\mathbf{F} \equiv$ vector of MIs

$\mathbb{K} \equiv$ coeff. matrix

$$\boxed{d\mathbf{F}(\vec{x},\epsilon) = d\mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon)}$$

$\epsilon = (4-d)/2$

Canonical DEs systems and iterated integrals

A smart change of basis can bring to big simplifications [Henn 13]

$$\boxed{\mathbf{F}(\vec{x},\epsilon) = \mathbb{B}(\vec{x},\epsilon)\,\mathbf{I}(\vec{x},\epsilon)}$$

bad basis ☹

$$d\mathbf{F}(\vec{x},\epsilon) = \mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon)$$

good basis ☺

$$d\mathbf{I}(\vec{x},\epsilon) = \epsilon\, d\mathbb{A}(\vec{x})\,\mathbf{I}(\vec{x},\epsilon)$$

Solution order by order in ϵ

$$\mathbf{I}(\epsilon,\vec{x}) = \mathcal{P}\exp\left\{\epsilon\int_\gamma d\mathbb{A}\right\}\mathbf{I}(\epsilon,\vec{x}_0) \qquad \mathbf{I}(\epsilon,\vec{x}_0) \equiv \text{ boundary constants}$$

$$\mathcal{P}\exp\left\{\epsilon\int_\gamma d\mathbb{A}\right\} = \mathbb{1} + \epsilon\int_\gamma d\mathbb{A} + \epsilon^2\int_\gamma d\mathbb{A}\,d\mathbb{A} + \epsilon^3\int_\gamma d\mathbb{A}\,d\mathbb{A}\,d\mathbb{A} + \dots$$

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γ is a path from $\vec{x}_0$ to $\vec{x}$ (that does not cross branch cuts and singularities of the integrand)

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It follows from Chen's theorem …

… that the matrices

$$\int_\gamma \underbrace{d\mathbb{A}\dots d\mathbb{A}}_{\text{k times}}$$

are invariant under smooth deformations of the path γ (provided branch cuts and singularities are avoided)! A lot of freedom ☺

Canonical DEs systems and iterated integrals

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Achieving a “canonical” basis

No general algorithm devised yet, mathematical status of a “conjecture”. Some ideas and special cases (constant leading singularity, ϵ-linear DEs, triangular DEs for $\epsilon \to 0$, Moser algorithm, . . .) [Henn 13; Argeri et. al. 14; Bern et. al. 14; Lee 14; Höschele et. al. 14; Gehrmann et. al. 14; Tancredi 15]

Chen's iterated integrals [Chen 77]

In our case the "canonical" coefficient matrix is a $d\log$ form

$$d\mathbb{A} = \sum_{i=1}^{n} \mathbb{M}_i \; d\log \eta_i(\vec{x}) \qquad \text{where} \begin{cases} \text{the } \mathbb{M}_i \text{ are } \mathbb{Q}\text{-valued matrices} \\ \text{the "letters" } \eta_i \text{ are functions of } \vec{x} \end{cases}$$

Therefore the entries of

$$\int_\gamma \underbrace{d\mathbb{A} \dots d\mathbb{A}}_{\text{k times}}$$

are linear combinations of Chen's iterated integrals of the form

$$\underbrace{\int_\gamma d\log \eta_{i_k} \dots d\log \eta_{i_1}}_{\equiv \mathcal{C}^{[\gamma]}_{i_k,\dots,i_1}} \equiv \int_{0 \le t_1 \le \dots \le t_k \le 1} g^\gamma_{i_k}(t_k) \dots g^\gamma_{i_1}(t_1)\, dt_1 \dots dt_k$$

where, given a parametrization $\gamma(t)$, $t \in [0,1]$, $g^\gamma_i(t) = \frac{d}{dt} \log \eta_i(\gamma(t))$

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Recall GPLs

$$G_{i_k,\ldots,i_1}(1) \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} \frac{1}{t_k - i_k} \ldots \frac{1}{t_1 - i_1} \, dt_1 \ldots dt_k$$

where, given a parametrization $\gamma(t)$, $t \in [0,1]$, $g_i^\gamma(t) = \frac{d}{dt} \log \eta_i(\gamma(t))$

Chen's iterated integrals: properties

- Invariance under path reparametrization
- Reverse path formula: $\mathcal{C}^{[\gamma^{-1}]}_{i_k,\dots,i_1} = (-1)^k \mathcal{C}^{[\gamma]}_{i_k,\dots,i_1}$
- Recursive structure: ($\gamma^s(t) \equiv \gamma(s\,t)$, with $s \in [0,1]$)

$$\mathcal{C}^{[\gamma]}_{i_k,\dots,i_1} = \int_0^1 g^{\gamma}_{i_k}(s)\,\mathcal{C}^{[\gamma_s]}_{i_{k-1},\dots,i_1}\,ds \qquad \frac{d}{ds}\,\mathcal{C}^{[\gamma_s]}_{i_k,\dots,i_1} = g^{\gamma}_{i_k}(s)\,\mathcal{C}^{[\gamma_s]}_{i_{k-1},\dots,i_1}$$

- Shuffle algebra:

$$\mathcal{C}^{[\gamma]}_{\vec{m}}\,\mathcal{C}^{[\gamma]}_{\vec{n}} = \sum_{\text{shuffles }\sigma} \mathcal{C}^{[\gamma]}_{\sigma(m_M),\dots,\sigma(m_1),\sigma(n_N),\dots,\sigma(n_1)}$$

- Path composition formula: if $\gamma \equiv \alpha\beta$, i.e. first α, then β

$$\mathcal{C}^{[\alpha\beta]}_{i_k,\dots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\beta]}_{i_k,\dots,i_{p+1}}\,\mathcal{C}^{[\alpha]}_{i_p,\dots,i_1}$$

- Integration-by-parts formula: get rid of outermost integration

$$\mathcal{C}^{[\gamma]}_{i_k,\dots,i_1} = \log \eta_{i_k}(\vec{x})\,\mathcal{C}^{[\gamma]}_{i_{k-1},\dots,i_1} - \int_0^1 \log \eta_{i_k}(\vec{x}(t))\,g_{i_{k-1}}(t)\,\mathcal{C}^{[\gamma_t]}_{i_{k-2},\dots,i_1}\,dt$$

Connection with GPLs

A representation in terms of GPLs can be obtained if the η_i's are *multilinear* in $\vec{x}$. E.g. single letter $\eta = 1 + xy$. Choose $\gamma = \alpha\beta$ with

$$\alpha(t) = (x_0 + t(x_1 - x_0),\, y_0)\,,$$
$$\beta(t) = (x_1,\, y_0 + t(y_1 - y_0))\,,$$

and $t \in [0, 1]$. Then

$$\begin{aligned} \int_{\alpha\beta} d\log(1+xy) &= \int_{\alpha} d\log(1+xy) + \int_{\beta} d\log(1+xy) \\ &= G\left(\tfrac{1+x_0y_0}{y_0(x_0-x_1)};1\right) + G\left(\tfrac{1+x_0y_0}{x_0(y_0-y_1)};1\right) \end{aligned}$$

$$\begin{aligned} \int_{\alpha\beta} d\log(1+xy)\ d\log(1+xy) &= \int_{\alpha} d\log(1+xy)\ d\log(1+xy) + \int_{\alpha} d\log(1+xy) \times \\ &\quad \times \int_{\beta} d\log(1+xy) + \int_{\beta} d\log(1+xy)\, d\log(1+xy) \\ &= G\left(\tfrac{1+x_0y_0}{y_0(x_0-x_1)}, \tfrac{1+x_0y_0}{y_0(x_0-x_1)};1\right) + G\left(\tfrac{1+x_0y_0}{x_0(y_0-y_1)}, \tfrac{1+x_0y_0}{y_0(x_0-x_1)};1\right) \\ &\quad + G\left(\tfrac{1+x_0y_0}{x_0(y_0-y_1)}, \tfrac{1+x_0y_0}{x_0(y_0-y_1)};1\right) \end{aligned}$$

Mixed Chen-Goncharov representation

Exploiting the recursive structure, the weight k coefficient is

$$\mathbf{I}^{(k)}(\vec{x}) = \mathbf{I}^{(k)}(\vec{x}_0) + \int_0^1 \left[\frac{d\mathbb{A}(t)}{dt}\mathbf{I}^{(k-1)}(\vec{x}_t)\right] dt\,,$$

where $\vec{x}_t$ is the point $(x(t), y(t))$ along the curve identified by γ.

- Need weight-$(k-1)$ coefficient, which is independent of the path
- Rational alphabet → factorize over $\mathbb{C}$ → GPLs
- Square roots → path integration over GPLs
- Exploit IBP to perform always only 1 path integration

$$\mathcal{C}^{[\gamma]}_{a|\vec{m}|\vec{n}} \equiv \int_0^1 g_a^{\gamma}(t)\, G^{\gamma}_{\vec{m}}(x)\, G^{\gamma}_{\vec{n}}(y)\, dt\,,$$

$$\mathcal{C}^{[\gamma]}_{a|\vec{m}|\varnothing} \equiv \int_0^1 g_a^{\gamma}(t)\, G^{\gamma}_{\vec{m}}(x)\, dt\,,$$

$$\mathcal{C}^{[\gamma]}_{a|\varnothing|\vec{n}} \equiv \int_0^1 g_a^{\gamma}(t)\, G^{\gamma}_{\vec{n}}(y)\, dt\,,$$

$$\mathcal{C}^{[\gamma]}_{a,\vec{b}|\vec{m}|\vec{n}} \equiv \int_0^1 g_a^{\gamma}(t)\, \mathcal{C}^{[\gamma_t]}_{\vec{b}|\vec{m}|\vec{n}}\, dt\,,$$

where $G^{\gamma}_{\vec{m}}(x)$ and $G^{\gamma}_{\vec{n}}(y)$ stand for the GPLs $G_{\vec{m}}(x)$ and $G_{\vec{n}}(y)$ evaluated at $(x, y) = (\gamma^1(t), \gamma^2(t))$.

Integrating ϵ-linear DE's [Argeri, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, **DV** 14]

1. start with DE linear in ϵ (may need a bit of trial and error + expertise)

$$\partial_x \mathbf{F}(\epsilon, x) = A(\epsilon, x)\mathbf{F}(\epsilon, x)\,, \quad A(\epsilon, x) = A_0(x) + \epsilon\, A_1(x)$$

2. basis change with Magnus's exponential: $\mathbf{F}(\epsilon, x) = B_0(x)\ \mathbf{I}(\epsilon, x)$

$$B_0(x) \equiv e^{\Omega[A_0](x,x_0)} \quad \leftrightarrow \quad \partial_x B_0(x) = A_0(x)B_0(x)$$

3. obtain a canonical system for the $\mathbf{I}$'s

$$\partial_x \mathbf{I}(\epsilon, x) = \epsilon\, \hat{A}_1(x)\mathbf{I}(\epsilon, x)\,, \quad \hat{A}_1(x) = B_0^{-1}(x)A_1(x)B_0(x)$$

4. obtain the solution with Magnus (or Dyson)

$$\mathbf{I}(\epsilon, x) = B_1(\epsilon, x)g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon\, \hat{A}_1](x,x_0)}$$

5. ϵ-expansion of g's will have uniform weight ("transcendentality")
(if $\mathbf{I}(0)$'s are chosen wisely)

In two-dimensions [Mastrolia, Schubert, Yundin, **DV** 14]

- the **F**'s obey an ϵ-linear DE system ($x = \frac{s}{m^2}$, $y = \frac{t}{m^2}$)

$$\partial_x \mathbf{F}(x,y,\epsilon) = \left(A_{1,0}(x,y) + \epsilon\, A_{1,1}(x,y)\right) \mathbf{F}(x,y,\epsilon)$$
$$\partial_y \mathbf{F}(x,y,\epsilon) = \left(A_{2,0}(x,y) + \epsilon\, A_{2,1}(x,y)\right) \mathbf{F}(x,y,\epsilon)$$

- After getting rid of $A_{i,0}$'s with Magnus (one variable at the time), the g's obey a canonical DE

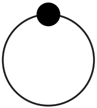
$$\partial_x \mathbf{I}(x,y,\epsilon) = \epsilon\, \hat{A}_x(x,y) \mathbf{I}(x,y,\epsilon)$$
$$\partial_y \mathbf{I}(x,y,\epsilon) = \epsilon\, \hat{A}_y(x,y) \mathbf{I}(x,y,\epsilon)$$

- which can be cast in *dlog* form

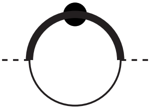
$$d\mathbf{I}(x,y,\epsilon) = \epsilon\, d\mathbb{A}(x,y)\, \mathbf{I}(x,y,\epsilon)$$

- with some *alphabet* $\{\eta_1, \ldots, \eta_n\}$

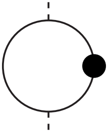
One-mass MIs: 1-loop



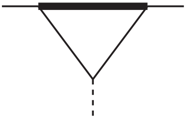
$(\mathcal{T}_1)$



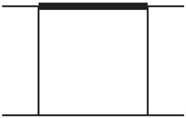
$(\mathcal{T}_2)$



$(\mathcal{T}_3)$



$(\mathcal{T}_4)$



$(\mathcal{T}_5)$

$$\mathsf{F}_1 = \epsilon\,\mathcal{T}_1\,, \qquad \mathsf{F}_2 = \epsilon\,\mathcal{T}_2\,, \qquad \mathsf{F}_3 = \epsilon\,\mathcal{T}_3\,,$$
$$\mathsf{F}_4 = \epsilon^2\,\mathcal{T}_4\,, \qquad \mathsf{F}_5 = \epsilon^2\,\mathcal{T}_5$$

The vector **F** obeys an ϵ-linear DE: we obtain the canonical MIs with the Magnus procedure

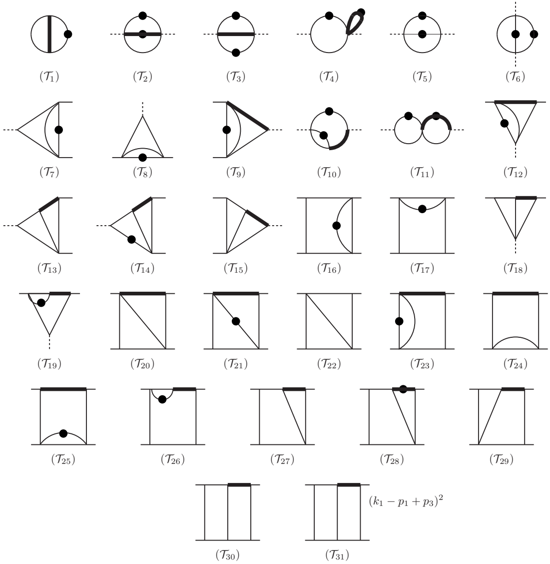
$$\mathsf{I}_1 = \mathsf{F}_1\,, \qquad \mathsf{I}_2 = -s\,\mathsf{F}_2\,, \qquad \mathsf{I}_3 = -t\,\mathsf{F}_3\,,$$
$$\mathsf{I}_4 = -t\,\mathsf{F}_4\,, \qquad \mathsf{I}_5 = (s - m^2)\,t\,\mathsf{F}_5$$

The alphabet of the corresponding $d\log$-form is ($x \equiv -s/m^2$, $y \equiv -s/m^2$)

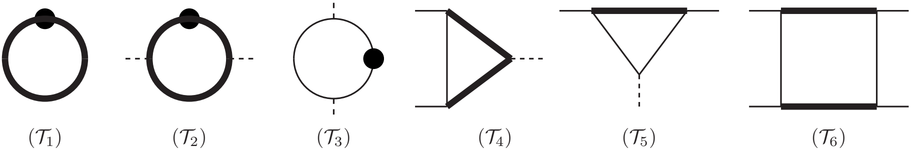
$$\eta_1 = x\,, \qquad \eta_2 = 1 + x\,, \qquad \eta_3 = y\,, \qquad \eta_4 = 1 - y\,, \qquad \eta_5 = x + y$$

One-mass MIs: 2-loop

- 1 extra letter
  $\eta_6 = x + y + xy$
- alphabet multilinear in $x, y \Rightarrow$ GPLs
- boundary conditions
    - regularity at pseudo-thresholds
    - zero momentum limits
    - direct integration
- analytic continuation straightforward $\Rightarrow$ complex (s, t, m^2)
- Checked against `SecDec` (Euclidean and in the physical regions)



Two-mass MIs: 1-loop

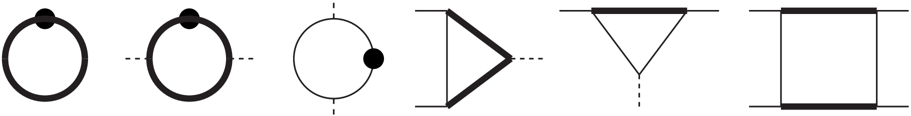


$$\mathsf{F}_1 = \epsilon\,\mathcal{T}_1\,, \qquad \mathsf{F}_2 = \epsilon\,\mathcal{T}_2\,, \qquad \mathsf{F}_3 = \epsilon\,\mathcal{T}_3\,,$$
$$\mathsf{F}_4 = \epsilon^2\,\mathcal{T}_4\,, \qquad \mathsf{F}_5 = \epsilon^2\,\mathcal{T}_5\,, \qquad \mathsf{F}_6 = \epsilon^2\,\mathcal{T}_6$$

Canonical basis

$$\mathsf{I}_1 = \mathsf{F}_1\,, \qquad \mathsf{I}_2 = -s\sqrt{1-\frac{4m^2}{s}}\,\mathsf{F}_2\,, \quad \mathsf{I}_3 = -t\,\mathsf{F}_3\,,$$
$$\mathsf{I}_4 = -s\,\mathsf{F}_4\,, \quad \mathsf{I}_5 = -t\,\mathsf{F}_5\,, \qquad \mathsf{I}_6 = s\,t\sqrt{1-4\frac{m^2}{s}\left(1+\frac{m^2}{t}\right)}\,\mathsf{F}_6$$

Two-mass MIs: 1-loop



Four square roots appear

$$\sqrt{-s},\ \sqrt{4m^2-s}, \sqrt{-t},\ \text{and}\ \sqrt{1-\frac{4\,m^2}{s}\left(1+\frac{m^2}{t}\right)}$$

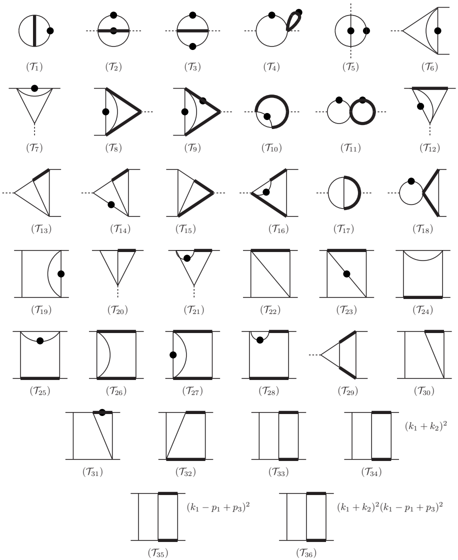
A change of variables gets rid of them

$$-\frac{s}{m^2}=\frac{(1-w)^2}{w}\,, \qquad -\frac{t}{m^2}=\frac{w}{z}\frac{(1+z)^2}{(1+w)^2}\,.$$

$$\eta_1=z\,,\qquad \eta_2=1+z\,,\qquad \eta_3=1-z\,,\qquad \eta_4=w\,,$$
$$\eta_5=1+w\,,\qquad \eta_6=1-w\,,\qquad \eta_7=z-w\,,\qquad \eta_8=z+w^2\,,$$

Two-mass MIs: 2-loop

- one extra sqrt $\sqrt{1+\frac{m^4}{t^2}-\frac{2\,m^2}{s}\left(1-\frac{u}{t}\right)}$
  - in DE for I_{32} at weight 3,4
  - in DEs for $\mathrm{I}_{33,\ldots,36}$ at weight 4
  - all the rest → GPLs
- boundary conditions
  - regularity at pseudo-thresholds
  - zero momentum limits
  - direct integration
- analytic continuation
  - straightforward for $\mathrm{I}_{1,\ldots,31}$
  - requires care for $\mathrm{I}_{32,\ldots,36}$
- checks against `SecDec`
  - $\mathrm{I}_{1,\ldots,31}$ (Eucl./phys.)
  - $\mathrm{I}_{32,\ldots,36}$ (Eucl.)



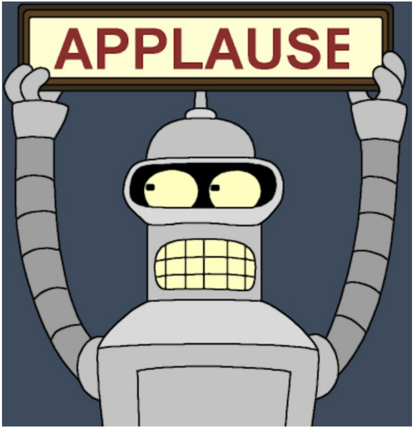
Summary and perspectives

- We computed the MIs for the virtual QCD×EW two-loop corrections to the Drell-Yan scattering processes (for massless external particles)

$$q + \bar{q} \to l^- + l^+ \ , \qquad q + \bar{q}' \to l^- + \overline{\nu}$$

- We exploited $\Delta m^2 \equiv m_Z^2 - m_W^2 \ll m_Z^2$ to reduce the number of scales to 3
- We identified 49 canonical MIs (8 fully massless, 24 one-mass, 17 two-mass) with the help of the Magnus exponential
- The result is given as a Taylor series around $d = 4$ space-time dimensions in terms of iterated integrals up to weight four
- We adopted a mixed representation in terms of Chen-Goncharov iterated integrals, suitable for numerical evaluation.
- Future work:
  - Analytic continuation of Chen's iterated integrals
  - Optimization of numerical evaluation
  - Amplitudes and cross-section

(canonical)



Thanks for your attention!

A convenient tool: the Magnus series expansion [Magnus 54]

- a generic matrix linear system of 1st order ODE

$$\partial_x Y(x) = A(x) Y(x) \; , \quad Y(x_0) = Y_0$$

- in the general non-commutative case, the Magnus theorem tells us that

$$Y(x) = e^{\Omega(x,x_0)} \; Y(x_0) \equiv e^{\Omega(x)} \; Y_0$$

- with $\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x)$ and

$$\begin{aligned}
\Omega_1(x) &= \int_{x_0}^{x} d\tau_1 A(\tau_1) \; , \\
\Omega_2(x) &= \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)] \\
\Omega_3(x) &= \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \\
&\dots
\end{aligned}$$

Relation with Dyson series [Blanes, Casas, Oteo and Ros 09]

Magnus ↔ Dyson series. Dyson expansion of the solution Y in terms of the *time-ordered* integrals Y_n

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x)$$
$$Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \; A(\tau_1) A(\tau_2) \cdots A(\tau_n) \;,$$

Then

$$Y(x) = e^{\Omega(x)} Y_0 \quad \Rightarrow \quad \sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right)$$

and

$$\begin{aligned} Y_1 &= \Omega_1 \;, \\ Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2 \;, \\ Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3 \end{aligned}$$