Dispersion relations and differential equations for Feynman Integrals

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Based on collaboration with *A. von Manteuffel* and *E. Remiddi*

[arXiv:1602.01481], [arXiv:1609.xxxxx]

Dimensionally regularised Feynman Integrals fulfil differential equations!
[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,...]

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Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

Reduced to N master integrals, $I_i(d; x_k)$ with $i = 1, \dots, N$.

$\Downarrow$

Differentiating the masters and using the **IBPs** we get a system of
N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, I_j(d; x_k)\,.$$

(*Relatively*) Simple standard case: multiple polylogarithms!

- In this case, one can find a **canonical basis** [Henn '13]

$$\frac{\partial}{\partial x_k} I_i(d;x_k) = (d-4)\sum_{j=1}^{N} c_{ij}(x_k)\, I_j(d;x_k)\,, \quad c_{ij}(x_k) \;\; \text{in } d\text{-}log \text{ form}\,.$$

- Existence of such a basis related to **decoupling** of diff. eqs. for $d \to 4$.

  Decoupling due to degeneracy of IBPs in **even integer numbers of dimensions**, i.e. number of master integrals in $d = 2\,n$ is smaller than for generic d! [E.Remiddi, L.T. '13; L.T. '15]

$\Downarrow$

Let's talk about what happens when this is not possible

As we'll see, it's enough to start putting some masses in the loops!

Interesting because:

1- LHC is pushing precision beyond 5%

2- High energies and High p_T $\rightarrow$ probe massive particles in the loops

a- Top quark corrections to Hj, HH, $\gamma\gamma$, jj, ...

b- New massive states?

Let's look more in detail - *in reality we have*

$$I_j(d;x_k) = (m_j(d;x_k)\,, \quad sub_j(d;x_k))$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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homogeneous piece is first source of complexity - whether differential equations are coupled

$$\Downarrow$$

No way to solve this in general.
Need to do some "statistics"!

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non-homogeneous piece is second source of complexity - we must integrate over it!

$$\Downarrow$$

Systematized using **differential equations** and **dispersion relations**

Let's have a look at two completely unrelated examples

$S(d;p^2) =$

$T(d;p^2) =$

- $p^2 \neq 0$, three massive lines
- 2 master integrals
- Satisfy 2 coupled diff. eqs.
- Needed for NNLO $t\bar{t}$

$\Downarrow$

massive 3-particle cut
Integrals over elliptic integrals!

- $p_1^2 = p_2^2 = 0$, four massive lines
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SAME Integrals over elliptic integrals!

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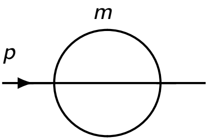
$S(d;p^2) =$ 

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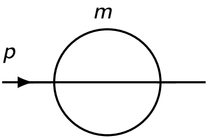
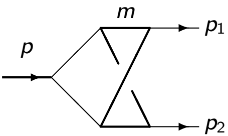
$S(d;p^2) =$ 

$T(d;p^2) =$ 

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In both cases diff. eqs. can be written as ($p^2 = u$, $m^2 = 1$ for simplicity)

$$\frac{d}{du}\begin{pmatrix} h_1(d;u) \\ h_2(d;u) \end{pmatrix} = B(u)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (d-4)\, D(u)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} N_1(d;u) \\ N_2(d;u) \end{pmatrix}.$$

- Matrices $B(u)$ and $D(u)$ different for the two cases $\rightarrow$ but don't depend on d!

$\Downarrow$

Matrices may have only regular singular points $1/(u-a)$
Fuchsian differential equations! [Lee, '14]

- $N_j(d;u)$ trivial for the sunrise $\rightarrow$ **Tadpole**!

- $N_j(d;u)$ non-trivial for triangle, contains **multiple polylogarithms**!

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Start by looking for **two independent solutions** of the homogeneous equation

$$G(u) = \begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix}$$

such that

$$\frac{d}{du} G(u) = B(u)\, G(u)\,.$$

Fundamental properties (in both cases):

1- $I_k(u)$ and $J_k(u)$ are linear combinations of **complete elliptic integrals**

2- **The Wronskian** of the solutions is trivial $\to$ matrix $B(u)$ is traceless

$$W(u) = \det G(u) = I_1(u) J_2(u) - J_1(u) I_2(u) = \pi$$

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$$\begin{pmatrix} h_1(d;u) \\ h_2(d;u) \end{pmatrix} = G(u) \begin{pmatrix} m_1(d;u) \\ m_2(d;u) \end{pmatrix}, \quad \text{with} \quad G^{-1}(u) = \frac{1}{\pi} \begin{pmatrix} J_2(u) & -J_1(u) \\ -I_2(u) & I_1(u) \end{pmatrix}.$$

And the new functions satisfy the equations

$$\frac{d}{du} \begin{pmatrix} m_1(d;u) \\ m_2(d;u) \end{pmatrix} = (d-4)\,\frac{1}{\pi} M(u) \begin{pmatrix} m_1(d;u) \\ m_2(d;u) \end{pmatrix} + G^{(-1)}(u) \begin{pmatrix} N_1(d;u) \\ N_2(d;u) \end{pmatrix}$$

with $M(u) = G^{(-1)}(u)\, D(u)\, G(u)$, which **does not depend on** d !!!!

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Let's see example of the **two-loop sunrise graph** [E.Remiddi, L.T., '16]

The differential equations can be put in the form above

$$\frac{d}{du}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = B(u)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (d-4)\,D(u)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

where the two matrices $B(u), D(u)$ are defined as

$$B(u) = \frac{1}{6\,u(u-1)(u-9)}\begin{pmatrix} 3(3+14u-u^2) & -9 \\ (u+3)(3+75u-15u^2+u^3) & -3(3+14u-u^2) \end{pmatrix}$$

$$D(u) = \frac{1}{6\,u(u-9)(u-1)}\begin{pmatrix} 6\,u(u-1) & 0 \\ (u+3)(9+63u-9u^2+u^3) & 3(u+1)(u-9) \end{pmatrix}$$

Four **regular singular points**: $u = 0, 1, 9, \pm\infty$

Matrix of solutions of homogeneous eq. obtained studying the **imaginary part**

$$G(u) = \begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix}$$

$$I_1(u) = \int_4^{(\sqrt{u}-1)^2} db \, \frac{1}{\sqrt{R_4(b,u)}}, \quad I_2(u) = \int_4^{(\sqrt{u}-1)^2} db \, \frac{b^2}{\sqrt{R_4(b,u)}}$$

$$J_1(u) = \int_0^4 db \, \frac{1}{\sqrt{-R_4(d,u)}}, \quad J_2(u) = \int_0^4 db \, \frac{b^2}{\sqrt{-R_4(d,u)}} + \frac{\pi}{3}(u+3)$$

$$R_4(b,u) = b(b-4)((\sqrt{u}-1)^2 - b)((\sqrt{u}+1)^2 - b)$$

$I_k(u), J_k(u)$ are linear combinations of complete elliptic integrals

After rotation to solve homogeneous part defining

$$G(u) = \begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix}$$

and

$$\begin{pmatrix} h_1(d;u) \\ h_2(d;u) \end{pmatrix} = G(u) \begin{pmatrix} m_1(d;u) \\ m_2(d;u) \end{pmatrix}$$

Equations become as we saw

$$\frac{d}{du} \begin{pmatrix} m_1(d;u) \\ m_2(d;u) \end{pmatrix} = (d-4)\,\frac{1}{\pi} M(u) \begin{pmatrix} m_1(d;u) \\ m_2(d;u) \end{pmatrix} + \frac{1}{\pi} \begin{pmatrix} -J_1(u) \\ I_1(u) \end{pmatrix}$$

And the matrix $M(u)$ can be written as a **total differential**!!!!

$$M_{11}(u) = -\frac{d}{d\,u}\left[\left(\frac{(u+3)^2}{6} I_1(u)\,J_1(u)\right) + \frac{\pi}{4}\left(2\ln(u-9) + 2\ln(u-1) - \ln(u)\right)\right],$$

$$M_{12}(u) = -\frac{d}{d\,u}\left(\frac{(u+3)^2}{6} I_1(u)\,I_1(u)\right),$$

$$M_{21}(u) = +\frac{d}{d\,u}\left(\frac{(u+3)^2}{6} J_1(u)\,J_1(u)\right),$$

$$M_{22}(u) = +\frac{d}{d\,u}\left[\left(\frac{(u+3)^2}{6} I_1(u)\,J_1(u)\right) + \frac{\pi}{4}\left(2\ln(u-9) + 2\ln(u-1) - \ln(u)\right)\right],$$

Derivatives of logs and of products of elliptic integrals and rational functions!

It looks like a natural **generalization** of a “simple” matrix in d-log form!

Integral representation of solution for the master integrals at order zero

$$h_1^{(0)}(u) = \frac{1}{\pi} \left[J_1(u) \int_0^u dt\, I_1(t) - I_1(u) \left(\int_0^u dt\, J_1(t) - \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \right) \right],$$
$$h_2^{(0)}(u) = \frac{1}{\pi} \left[J_2(u) \int_0^u dt\, I_1(t) - I_2(u) \left(\int_0^u dt\, J_1(t) - \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \right) \right].$$

Similarly simple (one-fold integral) representation one order higher!

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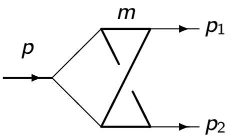
See [arXiv:1602.01481] for details!

Same approach can be applied to master integrals of non-planar crossed triangle
To appear soon... [A.von Manteuffel, L.T.]!



$$, \qquad x = -\frac{p^2}{m^2}$$

All subtopologies can be written in terms of (*not trivial!*)

$$\ln\left(f(l_i)\right), \quad \mathrm{Li}_n(f(l_i)), \quad \mathrm{Li}_{2,2}(f(l_i), g(l_j)),$$

with

$$l_i = \{\sqrt{x}, \tfrac{1}{2}(\sqrt{x}+\sqrt{x+4}), \sqrt{x+4}, \tfrac{1}{2}(\sqrt{x}+\sqrt{x-4}), \sqrt{x-4}\}$$

Also in this case, analytical result for two master integrals can be written as

$$T_1^{(0)}(x) = \frac{2}{\pi}\left[J_1(x)\int_0^x dt\, I_1(t)\, Q(t) - I_1(x)\int_0^x dt\, J_1(t)\, Q(t)\right],$$
$$T_2^{(0)}(x) = \frac{2}{\pi}\left[J_2(x)\int_0^x dt\, I_1(t)\, Q(t) - I_2(x)\int_0^x dt\, J_1(t)\, Q(t)\right].$$

$I_k(t), J_k(t)$ are again linear combinations of **complete elliptic integrals**

$Q(t)$ subtopologies $\rightarrow$ linear combination of weight 2 polylogs

In particular we have

$$Q(x) = 5\ln^2(I_2) - I_1\ \frac{3/2\,\zeta_2 + 3\ln^2(I_4) + 3\mathrm{Li}_2(-1/I_4^2)}{I_5}$$

And

$$I_1(t) = \sqrt{x}\ \mathrm{K}\left(\frac{x}{16}\right)\,, \qquad J_1(x) = \sqrt{x}\ \mathrm{K}\left(1-\frac{x}{16}\right)\,.$$

$$I_2(x) = -\sqrt{x}\ \mathrm{E}\left(\frac{x}{16}\right)\,, \qquad J_2(x) = \sqrt{x}\left[\mathrm{E}\left(1-\frac{x}{16}\right) - \mathrm{K}\left(1-\frac{x}{16}\right)\right]\,,$$

The integral representation of the solution is also in this case suited for analytic continuation and, **very importantly**, fast and precise numerical evaluation

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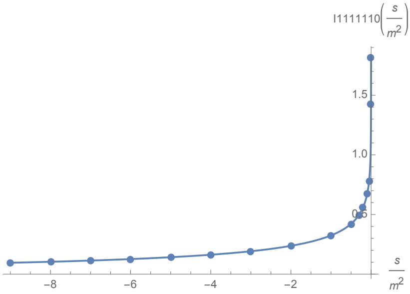
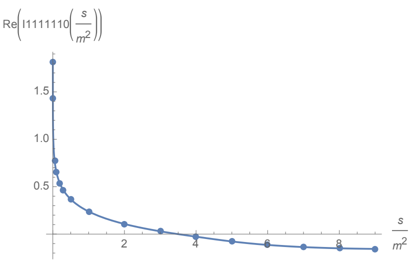
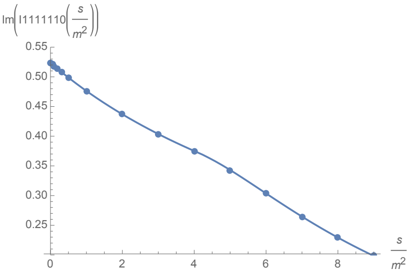
Comparison against SecDec3 [S.Borowka et al., '15] for scalar triangle



Euclidean kinematics



ℜ Minkoswki kinematics



ℑ Minkowski kinematics

How do we go further?

What do we do if these integrals appear in turn as **inhomogeneous** term of more complicated graphs?

$\rightarrow$ **Dispersion relations** together with **differential equations** allow to simplify this step.

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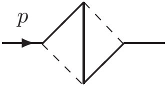
Worked out explicitly case of the Kite integral [arXiv:1602.01481]

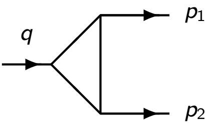
$$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) =$$



$$= \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}}$$

$$D_1 = k^2 + m^2 \,, \qquad D_2 = l^2 \,, \qquad D_3 = (k-l)^2 + m^2 \,,$$
$$D_4 = (k-p)^2 \,, \qquad D_5 = (l-p)^2 + m^2 \,,$$

Let's see how this works for a simple 1-loop example

$\mathrm{Tri}(s) =$ 

with $q^2 = s, \quad p_1^2 = p_2^2 = 0$

$$\mathrm{Tri}(s) = \int \mathfrak{D}^d k \frac{1}{(k^2+m^2)((k-p_1)^2+m^2)((k-p_1-p_2)^2+m^2)}$$

It is easy to derive **differential equation** in s for 1-loop triangle $\mathrm{Tri}(s)$

$$\frac{d}{ds}\mathrm{Tri}(s) = -\frac{1}{s}\mathrm{Tri}(s) + \frac{(d-2)}{8m^4}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathrm{Tad}(m) + \frac{(d-3)}{4\,m^2}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathrm{Bub}(s)\ ,$$

and

$$\mathrm{Tad}(m) = \int \mathfrak{D}^{\,d}k\ \frac{1}{k^2+m^2} = \frac{m^{d-2}}{(d-2)(d-4)}$$

In order to solve it:

1- Solve **homogeneous** equation

2- Use Euler's variation of constants to find **inhomogeneous** solution

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Homogeneous equation is very simple in this case

$$\frac{d}{ds} h(s) = -\frac{1}{s} h(s) \qquad \rightarrow \qquad h(s) = \frac{c}{s}$$

$$\Downarrow$$

Euler's method gives then for inhomogeneous solution

$$\begin{aligned}\mathrm{Tri}(s) = \frac{c}{s} &+ \frac{(d-2)}{2\,s} \int_0^s \frac{du}{u-4m^2} \mathrm{Tad}(m) \\ &+ \frac{(d-3)}{s} \int_0^s \frac{du}{u-4m^2} \, \mathrm{Bub}(u)\,,\end{aligned}$$

Until here nothing new. In order to proceed, **include explicitly** subtopologies

Let us include Tadpole (trivial) and **dispersive representation** for the Bubble

$$\mathrm{Bub}(u) = \frac{1}{\pi} \int_{4\,m^2}^{\infty} \frac{dt}{t - u - i\,\epsilon} \, \mathrm{Im}(\mathrm{Bub}(t))$$

$$\mathrm{Tri}(s) = \frac{m^{d-2}}{2\,s\,(d-4)} \int_0^s \frac{du}{u - 4m^2} + \frac{(d-3)}{s} \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t - 4m^2} \mathrm{Im}(\mathrm{Bub}(t)) \int_0^s du \left(\frac{1}{u - 4m^2} + \frac{1}{t - u - i\,\epsilon} \right)$$

Note that $c = 0$ since the triangle must be regular as $s \to 0$

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Note that $c = 0$ since the triangle must be regular as $s \to 0$

Integration in du is trivial – the relation that we find is **independent** from the explicit value of the **imaginary part of** $\mathrm{Bub}(t)$**!**

$$\mathrm{Tri}(s) = \frac{m^{d-2}}{2\,s\,(d-4)} \ln\left(1 - \frac{s}{4m^2}\right) + \frac{(d-3)}{s}\frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dt}{t-4m^2}\,\mathrm{ImBub}(t)\left[\ln\left(1-\frac{s}{4m^2}\right) - \ln\left(1-\frac{s}{t}\right)\right].$$

- Extracting the **imaginary part** for $s > 4m^2$ is straightforward
- Numerical evaluation is simple once $\mathrm{Im}(Bub(t))$ is known

$\Downarrow$

Worked out for **Kite integral**. Provides one-fold integral representation suitable for numerical evaluation, analytic continuation, ...

SUMMARY (work in progress....!)

- When Feynman Integrals are not multiple polylog, still a lot to understand

- First, we need to *collect statistics*. At two-loops all known examples require integrals over complete elliptic integrals.

  $\rightarrow$ how **general** is this?

- First step, obtain analytical expressions suitable for phenomenology!

$$\Downarrow$$

- **New idea**: When results beyond polylogs, solution of differential equations can be simplified using **dispersion relations**

Thanks!