QFT Scattering Amplitudes from Riemann Surfaces

LoopFest XV, University at Buffalo, NY

Freddy Cachazo
Perimeter Institute for Theoretical Physics
Outline

• Part I:  A brief history of the “S-Matrix” program since 2003

• Part II:  Unification of Theories via Riemann Surfaces (Tree Level)

• Part III:  Loop Level Constructions
Part I: History

• In 2003, motivated by the AdS/CFT duality and by work of Nair, Witten introduced a “string dual” of weakly coupled N=4 super Yang-Mills called **Twistor String Theory**.
Part I: Twistor String Theory

A closed string theory whose target space is Penrose’s twistor space (supersymmetrized). D-instanton computations are dual to scattering amplitudes of N=4 super Yang-Mills.
Part I: Witten-RSV Formula

Amplitudes in the $k$-sector are constructed as an integral over the moduli space of maps of degree $k-1$ from an $n$-punctured sphere into momentum space. The integral localizes (it is really a contour integral that computes residues).
Part I: **Witten-RSV Formula**

Amplitudes in the $k$-sector are constructed as an integral over the moduli space of maps of degree $k-1$ from an $n$-punctured sphere into momentum space. The integral localizes (it is really a contour integral that computes residues)

$$A_n = \int \frac{dV_a}{\prod_{a=1}^n} \frac{\text{Tr} (T^{a_1} \cdots T^{a_n})}{\Gamma_1 \Gamma_2 \cdots \Gamma_{n-1} \Gamma_n} + \text{perm}$$

$T^a \in su(N)$

$\text{d}M$ a measure over $M(CP^{1\rightarrow 3|4})$

$\Gamma_{ab} = \Gamma_a - \Gamma_b$
Part I: Witten-RSV Formula (Uses)

Partial Amplitudes

\[ A_{n,k} = \text{tr}(T^{a_1} \cdots T^{a_n}) A_{n,k}(123 \cdots n) +_{\text{perm.}} (n-1)_0 \]
Part I: Witten-RSV Formula (Uses)

Partial Amplitudes

\[ \mathcal{A}_{n,k} = \text{tr}(T^{a_1} \cdots T^{a_n}) A_{n,k}(123 \cdots n) + \text{perm.} \]

Kleiss-Kuijf 1989 (KK relations)

\[ A(1\{\alpha\} n \{\beta\}) = \sum_{\omega \in \alpha \omega \beta} A(1 \{\omega\} n) \]

\((n-2)^!\)
Part I: Witten-RSV Formula (Uses)

Partial Amplitudes

\[ \mathcal{A}_{n,k} = \text{Tr}(T^{a_1} \cdots T^{a_n}) A_{n,k}(1 \, 2 \, 3 \, \ldots \, n) + \text{perm.} \]

Kleiss-Kuijf 1989 (KK relations) Proof is trivial using Witten-RSV

Bern-Carrasco-Johansson 2008 (BCJ relations)?

\[ A(1 \{\kappa\} 2 \{\rho\} n) = \sum_{\omega \in S_{n-3}} F(\omega) A(1 \, 2 \, \{\omega\} \, n) \]  
\[ (n-3)! \]
Part I: Witten-RSV Formula (Uses)

BCJ Relations proven in 2012 using a curious set of equations. (FC. 2012)

\[
\int_{a=1}^{n} \prod d^4 \nu_a \, dM \left[ \sum_{b=1}^{n} \frac{S_{ab}}{\nu_a - \nu_b} \right] = 0
\]

\[S_{ab} = 2 \, k_a \cdot k_b\]

Obs: BCJ is valid in any number of dimensions (e.g. doesn’t rely on SUSY or the magic of four dimensional kinematics a.k.a. Spinor-Helicity)
Part II: Unification of Theories via Riemann Surfaces
Scattering Equations

Connect the space of kinematic invariants for the scattering of $n$-massless particles to the moduli space of $n$-punctured spheres.

Ingredients:

$S_{3n} \xrightarrow{K_n} M_n$

$S_{12}$

$S_{45}$

$S_{ab} = \left( K_a + K_b \right)^2$

Fairlie-Roberts ’72 (Unpublished), Gross-Mende ’88, Witten ’04, Fairlie ’08, Makeenko-Olesen ’09, F.C. ’12. F.C-He-Yuan ’13
Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

\[ F(\sigma) = \sum_{a,b=1}^{n} S_{ab} \log |\sigma_a - \sigma_b| \]

\[ S_{ab} = (K_a + K_b)^2 \]

\[ \frac{\partial F(\sigma)}{\partial \sigma_a} = 0 \quad \forall a \]

Fairlie-Roberts ’72 (Unpublished), Gross-Mende ’88, Witten ’04, Fairlie ’08, Makeenko-Olesen ’09, F.C. ’12. F.C-He-Yuan ’13
Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

\[ \frac{\partial F(a)}{\partial a} = 0 \quad \forall a \]

Fairlie-Roberts ‘72 (Unpublished), Gross-Mende ’88, Witten ‘04, Fairlie ‘08, Makeenko-Olesen ‘09, F.C. ’12, F.C-He-Yuan ‘13
Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

\[ K_n \rightarrow M_n \]

\[ \Omega_a = \text{puncture locations} \]

\[ \frac{\partial F(g)}{\partial \Omega_a} = \sum_{b=1}^{n} \frac{S_{ab}}{\Omega_a - \Omega_b} = 0 \quad \forall \ a \]

\[ S_{ab} = (k_a + k_b)^2 = 2 k_a \cdot k_b \]
Constructing Yang-Mills:
Poincare covariance + Polarization vectors = Gauge invariance

• Consider Massless particles of helicity +1 or -1 (e.g. gluons)

• Scattering Data:

For each particle \( \{ K_a^\mu, \epsilon_a^\mu \} \)

Under a general Lorentz transformation

\[
\epsilon_{(\lambda K, \pm 1)}^\mu = e^{-i \theta^{(K, \lambda)}} \left( D_{\nu}^\mu(\lambda) \epsilon_{(K, \pm 1)}^\nu + \Omega^{(K, \lambda)} K^\mu \right)
\]
CHY Construction: Yang-Mills

• Integral over the moduli space of n-punctured spheres.
• Integrand must make gauge invariance manifest.
• $U(N)$ color structure.

F.C., Song He and Ellis Yuan arXiv: 1307.2199
CHY Construction: Yang-Mills

- Integral over the moduli space of n-punctured spheres.
- Integrand must make gauge invariance manifest.
- U(N) color structure.

\[
A_n = \int \prod_{a=1}^{n} d\sigma_a \left[ \delta \left( \frac{\partial F(\sigma)}{\partial \sigma_a} \right) \right] P_F \Psi_{(K, \ell, \sigma)} \left( \frac{\text{Tr} (T^a_1 \ldots T^a_n)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\ldots} + \ldots \right)
\]

Tree-Level

F.C., Song He and Ellis Yuan arXiv: 1307.2199
CHY Construction: Gauge Invariance

\[ P_F \Psi_{(k,e,s)} = P_F \]

\[(P_{\text{pfaffian}})^2 = \det \]

\[ \frac{K_a \cdot K_b}{\sigma_a - \sigma_b} \]

\[ \frac{K_a \cdot E_b}{\sigma_a - \sigma_b} \]

\[ \frac{E_a \cdot K_b}{\sigma_a - \sigma_b} \]

\[ \frac{E_a \cdot E_b}{\sigma_a - \sigma_b} \]

F.C., Song He and Ellis Yuan arXiv: 1307.2199
CHY Construction: Gauge Invariance

If any polarization vector is replaced by its momentum vector, the matrix reduces its rank and the pfaffian vanishes.

\[ \text{P}_F - \Psi^{(k_a, \epsilon_a, \nu_a)} \xrightarrow{\epsilon_1^a \rightarrow k_1^a} 0 \]
CHY Construction: Gauge Invariance

If any polarization vector is replaced by its momentum vector, the matrix reduces its rank and the pfaffian vanishes.

\[ \begin{align*}
\mathcal{P}_F^{-1} \Psi (k_a, \epsilon_a, \gamma_a) & \quad \varepsilon'_1 \rightarrow k'_1 \\
& \rightarrow 0
\end{align*} \]

The pfaffian is the basic object that transforms correctly under Lorentz transformations in the massless helicity +1 or -1 representation!
We found

\[ P_F \Psi \xrightarrow{\lambda} e^{\sum_{a} h_a \theta(k_a, \lambda)} P_F \Psi \quad (h_a = \pm 1) \]
CHY Construction: Gravity

We found
\[ P_f \Psi \xrightarrow{\Lambda} e^{\sum_a h_a \Theta(k_a, \Lambda)} P_f \Psi \quad (h_a = \pm 1) \]

This means that
\[ \det \Psi \xrightarrow{\Lambda} e^{\sum_a 2h_a \Theta(k_a, \Lambda)} \det \Psi \quad (h_a = \pm 2) \]
CHY Construction: Gravity

- Gauge invariance is manifest again.

\[ A_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \, S(\frac{\partial F(\sigma)}{\partial \sigma_a}) \, \det \Psi_{(k,\varepsilon,\sigma)} \]
CHY Construction: Gravity

- Gauge invariance is manifest again.
- Soft theorems are manifest in both Yang-Mills and Gravity.
- This is now valid in any number of dimensions!

\[ A_n^{\text{Gravitons}} = \sum_{n} \int \prod_{a=1}^{n} \left[ d\gamma_a S \left( \frac{\partial F(\gamma)}{\partial \gamma_a} \right) \right] \det \Psi_{(K,\epsilon,\xi)} \]

Tree-Level
This seems to be a unifying framework!

This is a sample of some of the theories known so far:

\[ \text{Einstein Gravity} \rightarrow \text{Einstein-Maxwell} \rightarrow \text{Einstein-YM} \]
\[ \text{Yang-Mills} \rightarrow \text{Yang-Mills-Scalar} \]
\[ \text{Born-Infeld} \rightarrow \text{DBI} \]
\[ \text{NLSM \ & \ Galileons} \]
Part III: One-Loop Construction
One-Loop Scattering Equations

The most natural idea is to replace the Riemann sphere by
One-Loop Scattering Equations

The most natural idea is to replace the Riemann sphere by a torus!

Adamo, Casali, Geyer, Mason, Monteiro, Skinner, Tourkine ‘13,’14,’15
One-Loop Scattering Equations

But a torus is too complicated. It leads to elliptic functions while we expect dilogs!

Geyer, Mason, Monteiro, Tourkine ’15
One-Loop Scattering Equations: A Trick

One way to reproduce the results of GMMT directly is to start with tree-level scattering of n+2 massless particles in 5 dimensions!

\[ K^m_a = (K^m_a, 0) \quad L^m_+ = (l^m_+, e_+) \quad L^m_- = (l^m_-, e_-) \]

\[ (L^m_+)^2 = (L^m_-)^2 = 0 \quad \& \quad L^m_+ = -L^m_- = (l^m, e) \]

\[
\sum_{b=1}^{n} \frac{S_{ab}}{\sqrt{a} - \sqrt{b}} + \frac{2K_a \cdot l}{\sqrt{a} - \sqrt{+}} - \frac{2K_a \cdot l}{\sqrt{a} - \sqrt{-}} = 0
\]

F.C., He, Yuan ’15
One-Loop Amplitudes

\[ M_{\text{tree}}^{n+2} = \int \frac{d^4l}{l^2} \prod_{a=1}^{n+2} \left[ d\Gamma_a S(\frac{\partial F}{\partial \Gamma_a}) \right] \det \Psi_{(k, \varepsilon, \sigma)} \]

\[ M_{n}^{1-\text{loop}} = \int \frac{d^4l}{l^2} \prod_{a=1}^{n} \left[ d\Gamma_a S(\frac{\partial F(\sigma)}{\partial \Gamma_a}) \right] \det \Psi_{(k, \varepsilon, \sigma)} \]

Geyer, Mason, Monteiro, Tourkine ’15
An Aside: One-Loop Integrands

• In theories with color one can define a natural notion of an integrand in the planar limit at any loop order.

• In gravity (or any colorless theory) there seems to be no natural way of combining different integrals into a single one!

\[ M_{\text{4-loop}} = M_{\text{4-tree}} \left( \begin{array}{c c c c}
2 & 3 & 3 & 2 \\
1 & 4 & 1 & 4 \\
\end{array} \right) + \left( \begin{array}{c c c c}
2 & 2 & 2 & 2 \\
3 & 4 & 1 & 3 \\
\end{array} \right) \]

\[ \text{e.g. } N=8 \text{ SUGRA} \]
A New One-Loop Integrand

\[ M^{1 \text{-loop}}_n = \int \frac{d^D l}{l^2} \left( \prod_{a=1}^n \left[ \frac{d}{d G_a} S (\frac{\partial F}{\partial G_a}) \right] \right) \text{det} \Psi (k_a k_b, k_{a l}, \epsilon, \Sigma) \]

\[ M^{1 \text{-loop}}_n = \int \frac{d^D l}{l^2} I \left( k_a k_b, k_{a l} \right) \]
A New One-Loop Integrand

This integrand can be obtained from standard ones by partial fractions. However, I believe that

• This can be taken as a new starting point for the definition of loop amplitudes.

• One can use reduction techniques (P-V or vN-V) to bring any formula to a sum over a basis of new integrals.

• Only simple ones are known. The basis has to be computed!
Higher Loops? (Some Numerology)

\[ M_{\text{loop}}^{\text{L-loop}} \leftrightarrow M_{\text{tree}}^{n+2L} \]

The best understood theory in N=8 SUGRA.

<table>
<thead>
<tr>
<th>Loops</th>
<th>Dimension</th>
<th>Particles</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>n</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>n+2</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>n+4</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>n+6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>n+8</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>n+10</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>n+12</td>
</tr>
</tbody>
</table>

F.C., He, Yuan 2013
Geyer, Mason, Monteiro, Tourkine 2015, F.C. He, Yuan 2015
Geyer, Mason, Monteiro, Tourkine 2016
Higher Loops? (Some Numerology)

If these constructions are related to string theory in anyway then it is tempting to suggest that something special happens at 7 loops...

The best understood theory in $N=8$ SUGRA.

<table>
<thead>
<tr>
<th>Loops</th>
<th>Dimension</th>
<th>Particles</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>$n$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$n+2$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>$n+4$</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>$n+6$</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>$n+8$</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>$n+10$</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>$n+12$</td>
</tr>
</tbody>
</table>
Concluding Remarks:

• The moduli space of punctured Riemann surfaces can be used to encode locality and unitarity of a large collection of theories. There are extensions to massive theories. (Massive: Goddard, Naculich, 2013)

• Could there be a relation between symmetries of null infinity, i.e. extensions of BMS (Strominger et.al.) and the CHY formulation? Perhaps ambitwistor string ideas will make the connection clear. (Mason, Skinner, et.al 2014)

• Developments at loop level are in their infancy but they could lead to new techniques and ways of thinking!
Bonus Material: Extension of Theories
Extension of Theories via Soft Limits

• Consider the effective theory of $U(N)$ (massless) pions (NLSM):

$$L = \text{tr} \, \partial_\mu \mathbf{U} \, \partial^\mu \mathbf{U}^\dagger$$

$$\mathbf{U}(x) = e^{i \mathbf{T}_a(x) \mathbf{T}^a}$$

• Adler’s zero: When a single pion becomes soft the amplitude vanishes
Extension of Theories via Soft Limits

The CHY formula is given by:

\[ A_n(12\ldots n) = \int \frac{d^n \mathbf{q}}{\prod_{a=1}^{n} \mathbf{q}_a} \left( \sum_{b} \frac{S_{ab}}{\mathbf{q}_{ab}} \right) \mathcal{P}_F \frac{A_n}{\mathbf{q}_{12} \mathbf{q}_{23} \ldots \mathbf{q}_{n-1 \! n} \mathbf{q}_{n1}} \]

\[ [A_n]_{ab} = \begin{cases} 
\frac{S_{ab}}{\mathbf{q}_{ab}} & \text{if } a \neq b \\
0 & \text{if } a = b 
\end{cases} \]
Extension of Theories via Soft Limits

In the soft limit it is easy to write it as:

\[ A_n(1_2 \ldots n) = \mathcal{Z} \sum_{a=2}^{n-2} \lambda_a \cdot q \int_{c=1}^{n-1} \frac{d\Sigma_c}{V_{C_b}} \left( \sum_b \frac{S_{cb}}{V_{C_b}} \right) \frac{P_F \mathcal{A}_{n-1, a, 1}^{n-1, a, 1}}{(\Sigma_{12} \ldots \Sigma_{n-1}) (\Sigma_{n-1} \Lambda_{a 1} \Sigma_{1 n-1})} \]

(FC, P. Cha, S. Mizera 2016)
Extension of Theories via Soft Limits

In the soft limit it is easy to write it as:

\[ A_n(12\ldots n) = \sum_{a=2}^{n-2} \kappa_a \cdot q \int d\sigma_c \sum_{c=1}^{n-1} \left( \sum_{b} \frac{S_{cb}}{\sqrt{v_{cb}}} \right) \mathcal{P}_F A_{n-1,a,1}^{n-1,a,1} \left( \frac{1}{v_{12}\ldots v_{n-1}} \right) \left( \frac{1}{v_{n-1} a d_1 v_{1n-1}} \right) \]

Could these new objects be amplitudes of another theory?

(FC, P. Cha, S. Mizera 2016)
Extension of Theories via Soft Limits

Hints:

\[ A_n(i_1 \ldots i_n | i_1 \ldots i_m) = \int \prod d\sigma \, S \left( \frac{\partial F}{\partial \sigma} \right) \frac{P_F}{(i_{12} \ldots i_{n1})(i_{1i2} \ldots i_{i1i})} \]

\[ \text{ If } m = n \text{ the numerator is } 1 \]

\[ \Rightarrow \text{ No derivative interactions} \]

\( \cup (N) \)

(FC, P. Cha, S. Mizera 2016)
Extension of Theories via Soft Limits

NLSM + Biadjoint scalar!

\[ L = \text{tr} \partial_{\mu} U U^{\dagger} + \partial_{\mu} \phi^{a\bar{b}} \partial^{\mu} \phi_{a\bar{b}} + f_{abc} \phi^{a\bar{b}\bar{c}} + L_{\text{mix}} (\Pi^a, \phi^{a\bar{a}}) \]

The biadjoint scalar naturally emerged from the NLSM. Note that a new flavor group also emerged! This is what we call the extension of the NLSM.

(FC, P. Cha, S. Mizera 2016)