QFT Scattering Amplitudes from Riemann Surfaces

LoopFest XV, University at Buffalo, NY

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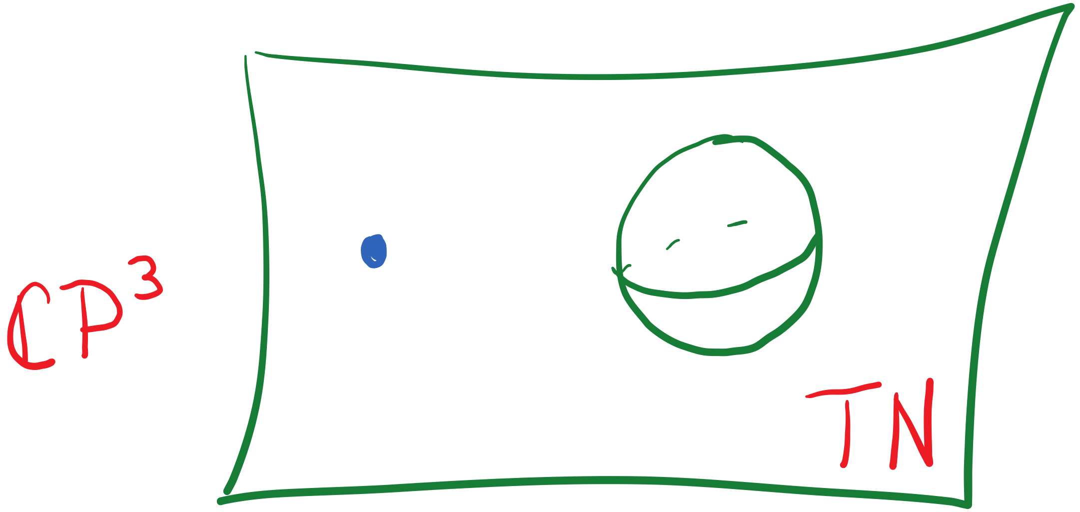
Outline

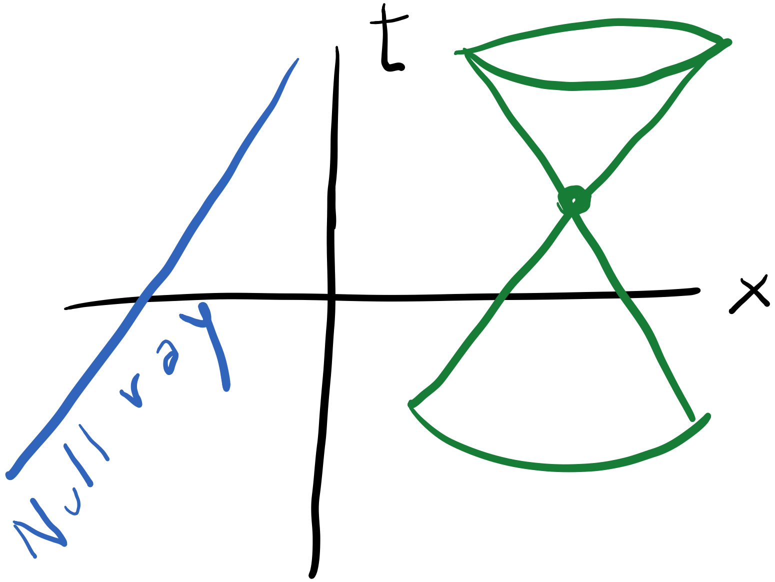
- Part I: A brief history of the “S-Matrix” program since 2003
- Part II: Unification of Theories via Riemann Surfaces (Tree Level)
- Part III: Loop Level Constructions

Part I: History

- In 2003, motivated by the AdS/CFT duality and by work of Nair, Witten introduced a “string dual” of weakly coupled N=4 super Yang-Mills called **Twistor String Theory.**



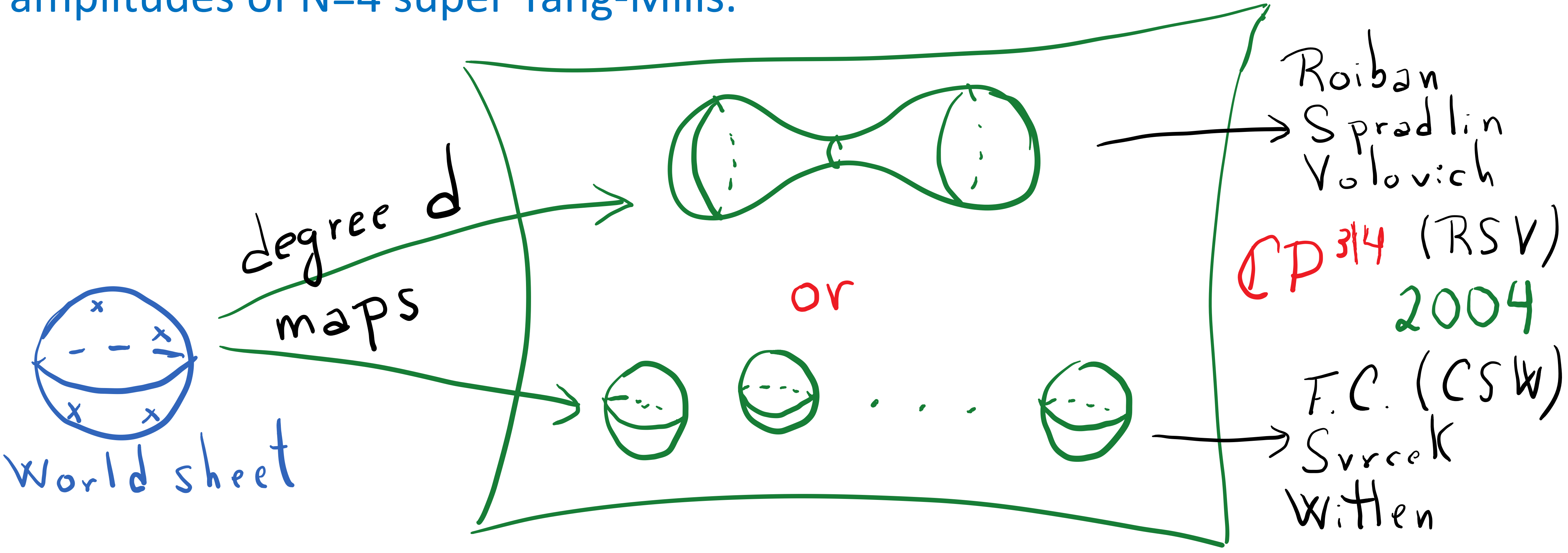
Twistor Space



Space Time

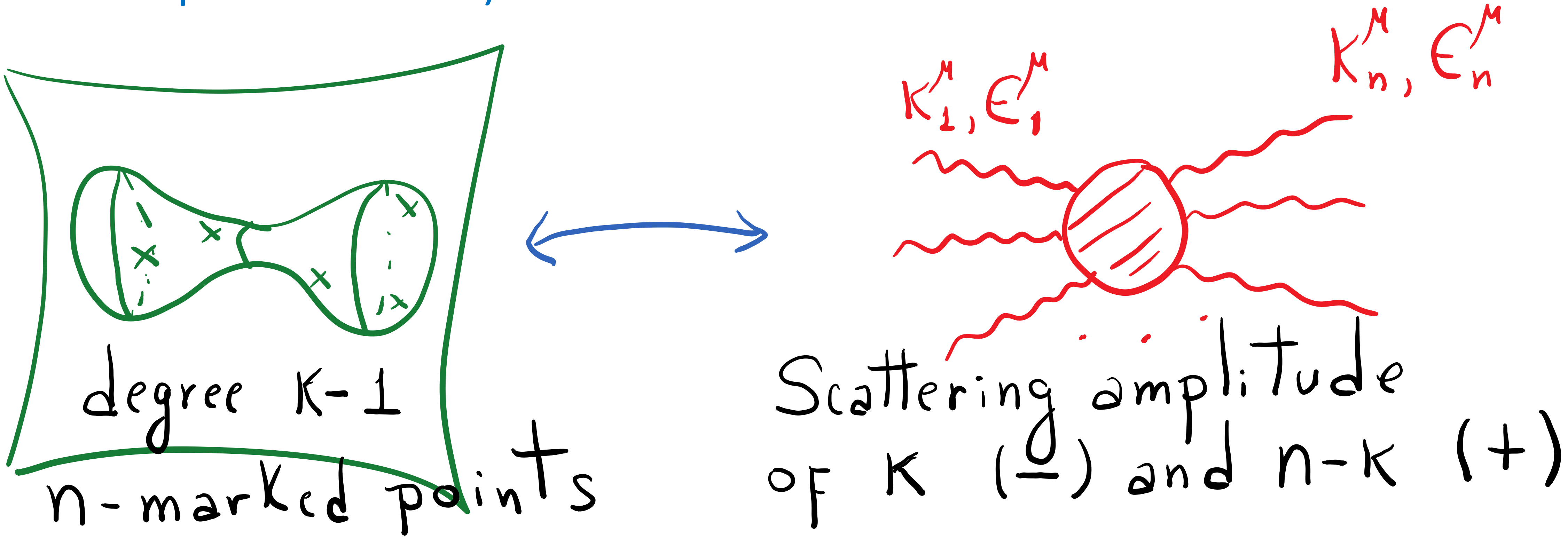
Part I: Twistor String Theory

A closed string theory whose target space is Penrose's twistor space (supersymmetrized). D-instanton computations are dual to scattering amplitudes of N=4 super Yang-Mills.



Part I: Witten-RSV Formula

Amplitudes in the k-sector are constructed as an integral over the moduli space of maps of degree k-1 from an n-punctured sphere into momentum space. The integral localizes (it is really a contour integral that computes residues)



Part I: Witten-RSV Formula

Amplitudes in the k-sector are constructed as an integral over the moduli space of maps of degree k-1 from an n-punctured sphere into momentum space. The integral localizes (it is really a contour integral that computes residues)

$$\mathcal{A}_n = \int \prod_{a=1}^{n} d\sigma_a \, d\mu \left[\frac{\mathrm{tr}(T^{a_1} \cdots T^{a_n})}{\sigma_{12}\sigma_{23}\cdots\sigma_{n-1\,n}\sigma_{n1}} + \text{perm} \right]$$

$T^a \in su(N)$

$d\mu$ a measure over $\mathcal{M}(\mathbb{CP}^1 \xrightarrow[d=k-1]{} \mathbb{CP}^{3|4})$

$\sigma_{ab} \equiv \sigma_a - \sigma_b$

Part I: Witten-RSV Formula (Uses)

Partial Amplitudes

$$\mathcal{A}_{n,k} = \mathrm{tr}(T^{a_1} \cdots T^{a_n}) A_{n,k}(1 2 3 \dots n) + \text{perm.}$$

$(n-1)!$

Part I: Witten-RSV Formula (Uses)

Partial Amplitudes

$$\mathcal{A}_{n,k} = \mathrm{tr}\left(T^{a_1} \cdots T^{a_n}\right) A_{n,k}(1\,2\,3 \ldots n) + \text{perm.}$$

Kleiss-Kuijf 1989 (KK relations)

$$A(1\{\alpha\}n\{\beta\}) = \sum_{\omega \in \alpha \sqcup\!\sqcup \beta} A(1\,\{\omega\}\,n)$$

$(n-2)!$

Part I: Witten-RSV Formula (Uses)

Partial Amplitudes

$$\mathcal{A}_{n,k} = \mathrm{Tr}\left(T^{a_1} \cdots T^{a_n}\right) A_{n,k}(1\,2\,3\ldots n) + \text{perm.}$$

Kleiss-Kuijf 1989 (KK relations) Proof is trivial using Witten-RSV

Bern-Carrasco-Johansson 2008 (BCJ relations)?

$$A(1\{\alpha\}2\{\beta\}n) = \sum_{\omega \in S_{n-3}} F(\omega)\, A(1\,2\,\{\omega\}\,n)$$

$(n-3)!$

Part I: Witten-RSV Formula (Uses)

BCJ Relations proven in 2012 using a curious set of equations. (FC. 2012)

$$\int \prod_{a=1}^{n} d\sigma_a \, d\mu \underbrace{\left[\sum_{b=1}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} \right]}_{=0} = 0$$

$$s_{ab} = 2\, k_a \cdot k_b$$

Obs: BCJ is valid in any number of dimensions (e.g. doesn’t rely on SUSY or the magic of four dimensional kinematics a.k.a. Spinor-Helicity)

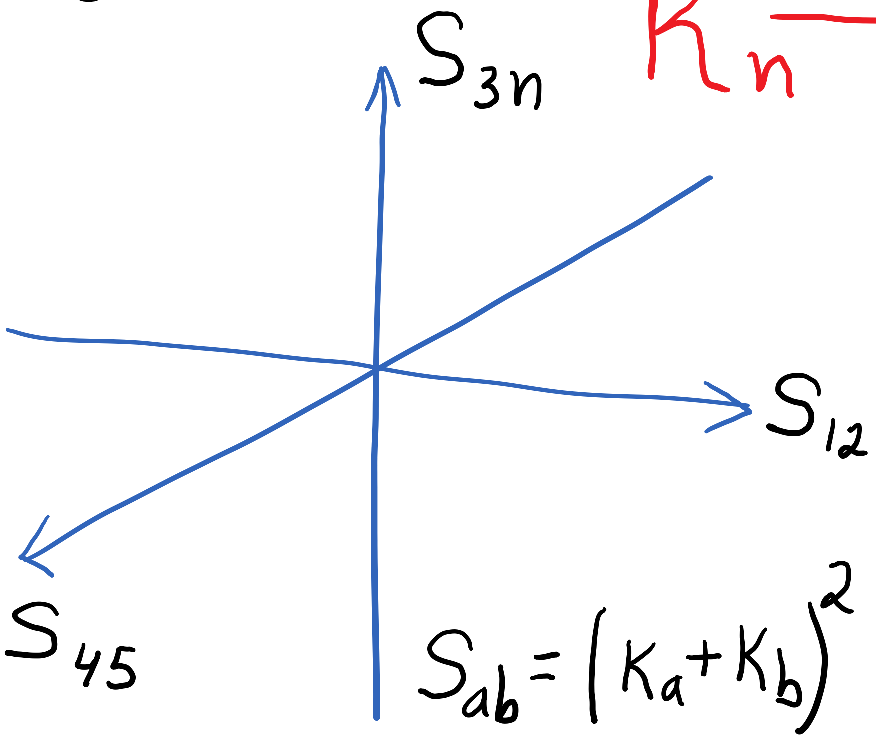
Part II : Unification of Theories via Riemann Surfaces

Scattering Equations

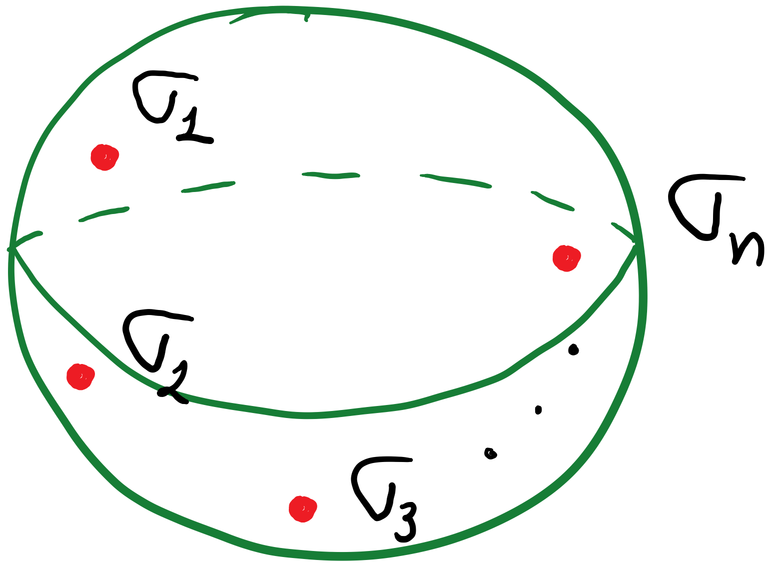
Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

$$\mathcal{K}_n \longrightarrow \mathcal{M}_n$$



$$s_{ab} = (k_a + k_b)^2$$



Fairlie-Roberts '72 (Unpublished), Gross-Mende '88, Witten '04, Fairlie '08, Makeenko-Olesen '09, F.C. '12. F.C-He-Yuan '13

Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

$$\mathcal{K}_n \longrightarrow \mathcal{M}_n$$

$$F(\sigma) = \sum_{a,b=1}^{n} s_{ab} \log|\sigma_a - \sigma_b|$$

$$\boxed{\frac{\partial F(\sigma)}{\partial \sigma_a} = 0 \quad \forall a}$$

$$s_{ab} = (k_a + k_b)^2$$

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Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

Unitarity & Locality

$$\mathcal{K}_n \rightarrow \mathcal{M}_n$$

$\overline{\mathcal{M}}_n$

Deligne-Mumford

$$\boxed{\frac{\partial F(\sigma)}{\partial \sigma_a} = 0 \quad \forall a}$$

Fairlie-Roberts '72 (Unpublished), Gross-Mende '88, Witten '04, Fairlie '08, Makeenko-Olesen '09, F.C. '12. F.C-He-Yuan '13

Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

$$K_n \longrightarrow \mathcal{M}_n$$

σ_a = puncture locations

$$\frac{\partial F(\sigma)}{\partial \sigma_a} = \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0 \quad \forall a$$

$$s_{ab} = (k_a + k_b)^2 = 2\, k_a \cdot k_b$$

Constructing Yang-Mills:

Poincare covariance + Polarization vectors = Gauge invariance

- Consider Massless particles of helicity +1 or -1 (e.g. gluons)
- Scattering Data:

For each particle $\{K_a^\mu, \epsilon_a^\mu\}$

Under a general Lorentz transformation

$$\epsilon^\mu_{(\Lambda K, \pm 1)} = e^{\mp i\theta(K,\Lambda)}\left(D^\mu{}_\nu(\Lambda)\,\epsilon^\nu_{(K,\pm 1)} + \Omega(K,\Lambda)\,K^\mu\right)$$

CHY Construction: Yang-Mills

- Integral over the moduli space of n-punctured spheres.
- Integrand must make gauge invariance manifest.
- U(N) color structure.

F.C., Song He and Ellis Yuan arXiv: 1307.2199

CHY Construction: Yang-Mills

- Integral over the moduli space of n-punctured spheres.
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$$\mathcal{A}_n = \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\left(\frac{\partial F(\sigma)}{\partial \sigma_a} \right) \right] \mathrm{P_F}\, \Psi_{(k,\epsilon,\sigma)} \left(\frac{\mathrm{tr}\left(T^{a_1} \cdots T^{a_n}\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)\cdots} + \cdots \right)$$

Tree-Level

F.C., Song He and Ellis Yuan arXiv: 1307.2199

CHY Construction: Gauge Invariance

$$\mathrm{Pf}\,\Psi_{(k,\epsilon,\sigma)} = \mathrm{Pf}\left[\begin{array}{c|c} \dfrac{k_a\cdot k_b}{\sigma_a-\sigma_b} & \dfrac{k_a\cdot \epsilon_b}{\sigma_a-\sigma_b} \\ \hline \dfrac{\epsilon_a\cdot k_b}{\sigma_a-\sigma_b} & \dfrac{\epsilon_a\cdot \epsilon_b}{\sigma_a-\sigma_b} \end{array}\right]$$

$$(\mathrm{Pfaffian})^2 = \det$$

F.C., Song He and Ellis Yuan arXiv: 1307.2199

CHY Construction: Gauge Invariance

If any polarization vector is replaced by its momentum vector, the matrix reduces its rank and the pfaffian vanishes.

$$\mathrm{Pf}\,\Psi(k_a,\epsilon_a,\sigma_a) \xrightarrow[\epsilon_1^\mu \to k_1^\mu]{} 0$$

CHY Construction: Gauge Invariance

If any polarization vector is replaced by its momentum vector, the matrix reduces its rank and the pfaffian vanishes.

$$\mathrm{Pf}\,\Psi(k_a,\epsilon_a,\sigma_a) \xrightarrow[\epsilon_1^\mu \to k_1^\mu]{} 0$$

The pfaffian is the basic object that transforms correctly under Lorentz tranformations in the massless helicity +1 or -1 representation!

$$\mathrm{Pf}\,\Psi(k_a,\epsilon_a,\sigma_a) \xrightarrow[\Lambda]{} e^{\sum_a h_a \theta(k_a,\Lambda)}\,\mathrm{Pf}\,\Psi$$

CHY Construction: Gravity

We found

$$P_F \Psi \xrightarrow[\Lambda]{} e^{\sum_a h_a \theta(k_a, \Lambda)} P_F \Psi \qquad (h_a = \pm 1)$$

CHY Construction: Gravity

We found

$$\mathrm{Pf}\,\Psi \xrightarrow[\Lambda]{} e^{\sum_a h_a \theta(k_a,\Lambda)}\,\mathrm{Pf}\,\Psi \qquad (h_a = \pm 1)$$

This means that

$$\det \Psi \xrightarrow[\Lambda]{} e^{\sum_a 2h_a \theta(k_a,\Lambda)} \det \Psi$$

$$(h_a = \pm 2)$$

CHY Construction: Gravity

- Gauge invariance is manifest again.

$$\mathcal{A}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\left(\frac{\partial F(\sigma)}{\partial \sigma_a} \right) \right] \det \Psi_{(k,\epsilon,\sigma)}$$

Tree-Level

CHY Construction: Gravity

- Gauge invariance is manifest again.
- Soft theorems are manifest in both Yang-Mills and Gravity.
- This is now valid in any number of dimensions!

$$\mathcal{A}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\left(\frac{\partial F(\sigma)}{\partial \sigma_a} \right) \right] \det \Psi_{(k,\epsilon,\sigma)}$$

Tree-Level

This seems to be a unifying framework!

This is a sample of some of the theories known so far:

Einstein Gravity → Einstein-Maxwell → Einstein-YM

Yang-Mills → Yang-Mills-Scalar

Born-Infeld → DBI

NLSM & Galileons

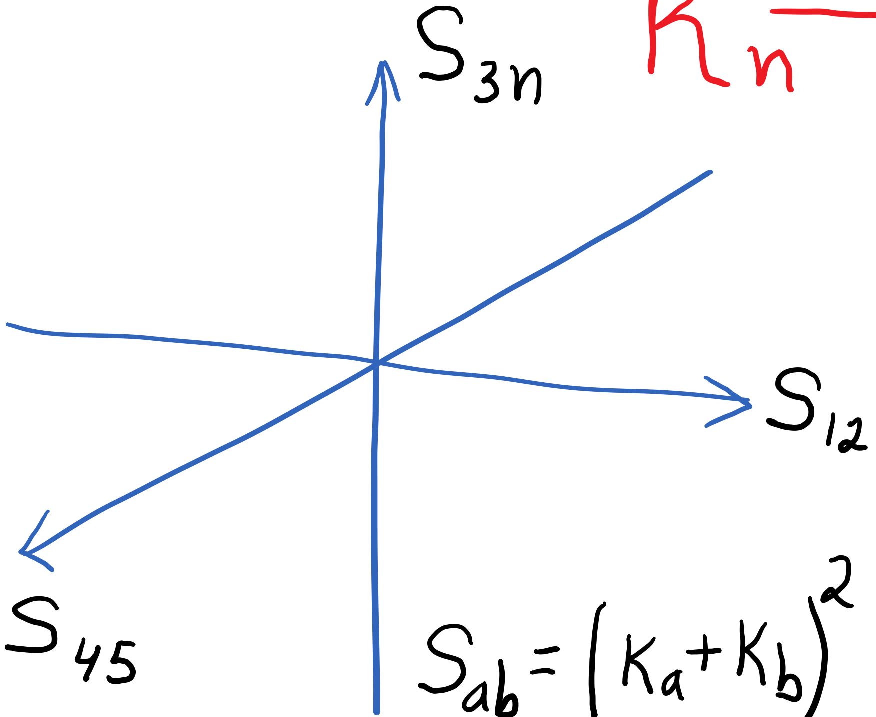
(FC, Song He, Ellis Yuan 2014)

Part III : One-Loop Construction

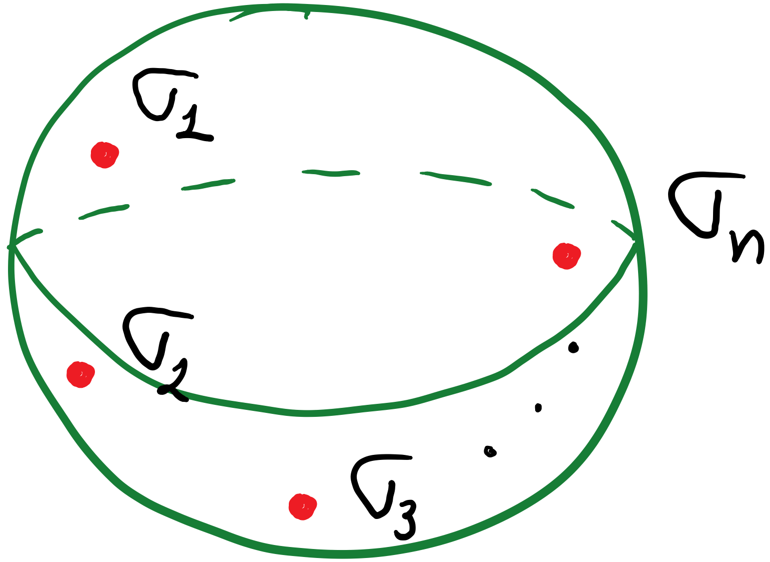
One-Loop Scattering Equations

The most natural idea is to replace the Riemann sphere by

$$\mathcal{K}_n \longrightarrow \mathcal{M}_n$$



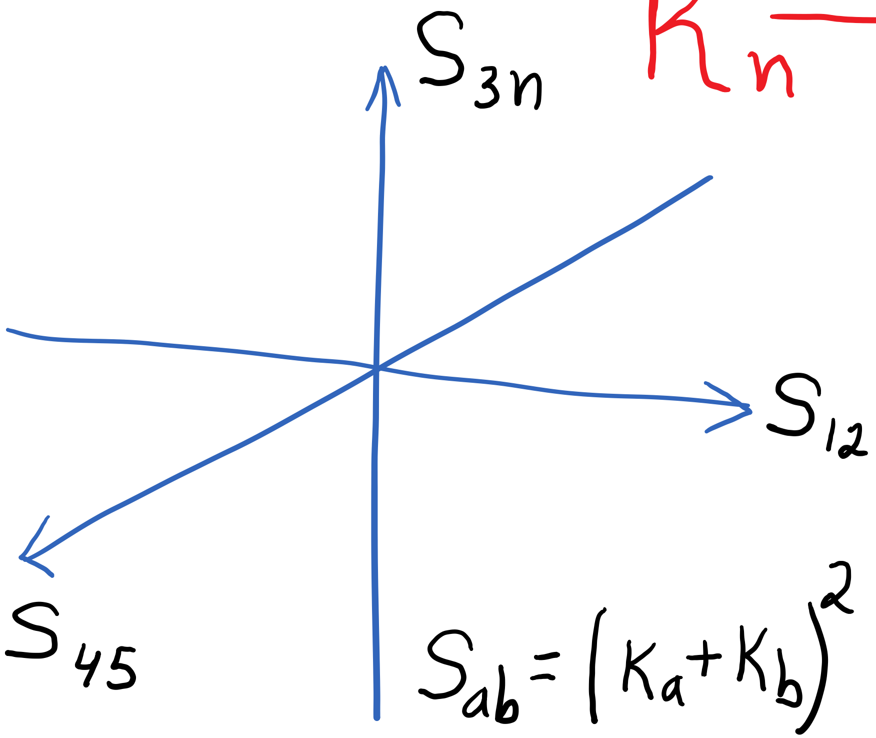
$$s_{ab} = (k_a + k_b)^2$$



One-Loop Scattering Equations

The most natural idea is to replace the Riemann sphere by a torus!

$$\mathcal{K}_n \longrightarrow \mathcal{M}_n$$



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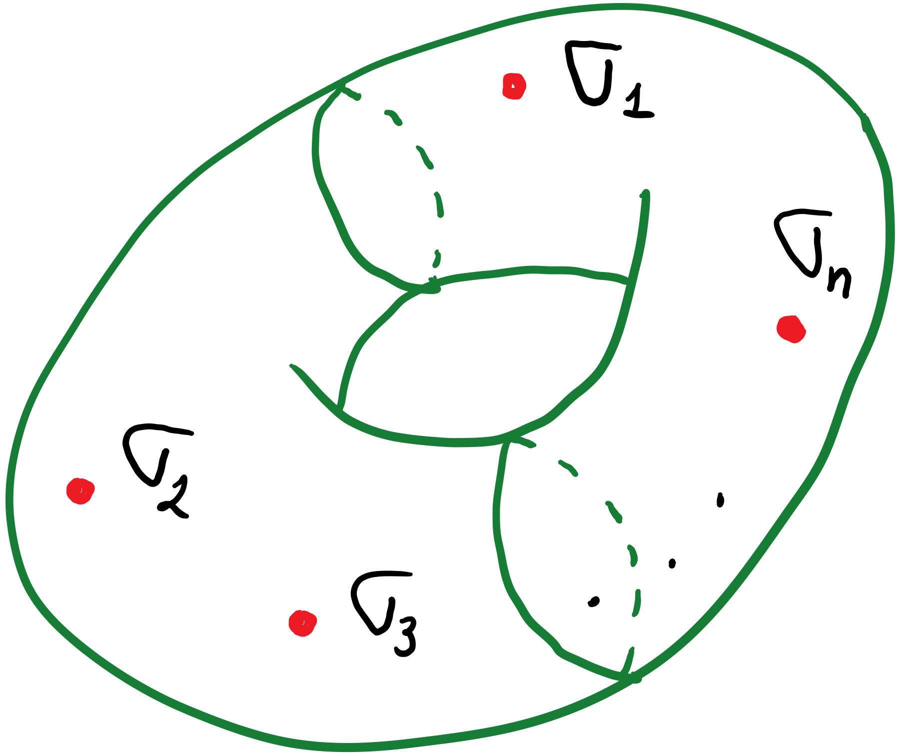
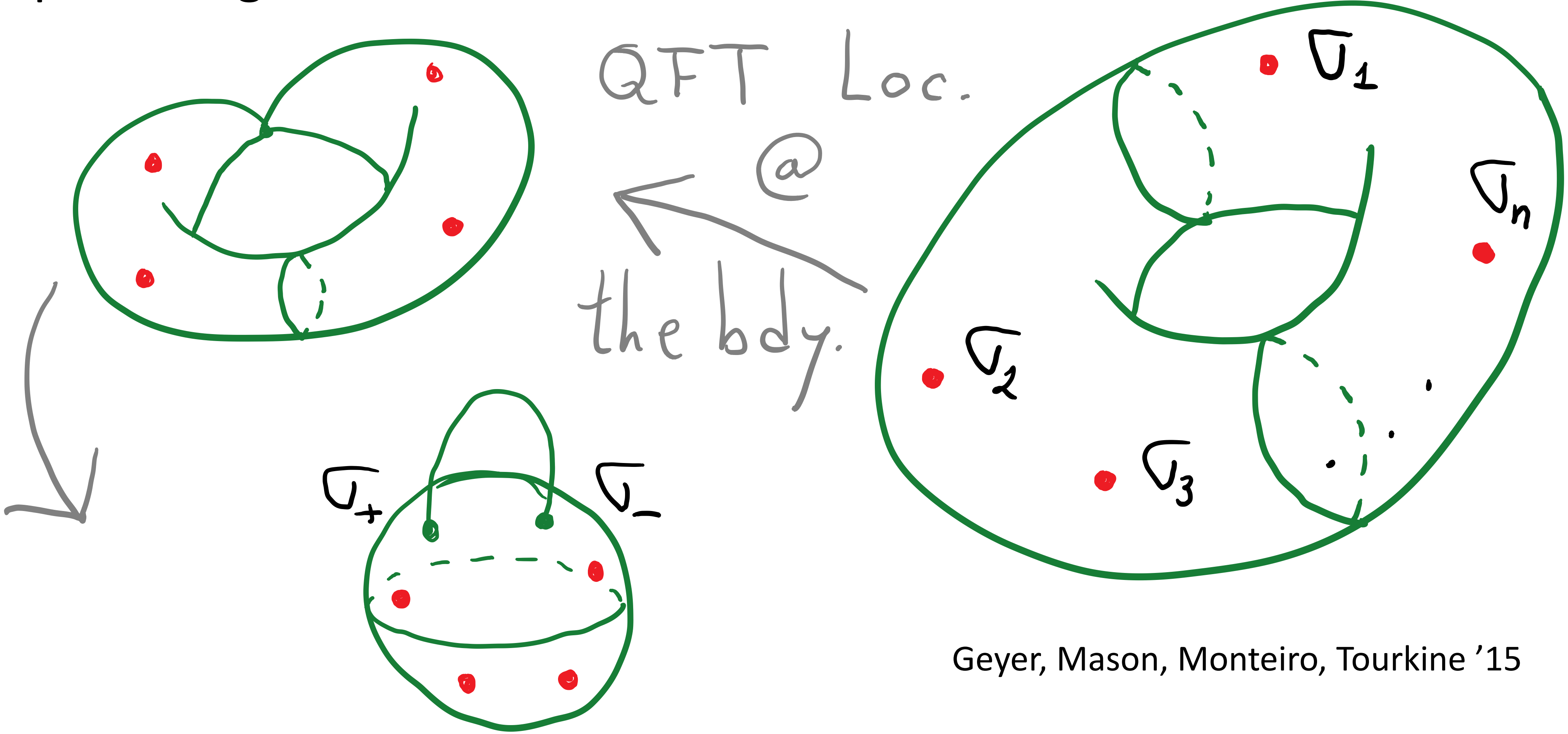


Adamo, Casali, Geyer, Mason, Monteiro, Skinner, Tourkine ‘13,’14,’15

One-Loop Scattering Equations

But a torus is too complicated. It leads to elliptic functions while we expect dilogs!



Geyer, Mason, Monteiro, Tourkine '15

One-Loop Scattering Equations: A Trick

One way to reproduce the results of GMMT directly is to start with tree-level scattering of n+2 massless particles in 5 dimensions!

$$K_a^\mu = (k_a^\mu, 0) \qquad L_+^\mu = (\ell_+^\mu, \ell_+) \qquad L_-^\mu = (\ell_-^\mu, \ell_-)$$

$$(L_+)^2 = (L_-)^2 = 0 \quad \& \quad L_+^\mu = -L_-^\mu = (\ell^\mu, \ell)$$

$$\sum_{b=1}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} + \frac{2k_a \cdot \ell}{\sigma_a - \sigma_+} - \frac{2k_a \cdot \ell}{\sigma_a - \sigma_-} = 0$$

F.C., He, Yuan ’15

One-Loop Amplitudes

$$\mathcal{M}^{\text{tree}}_{n+2} = \int \prod_{a=1}^{n+2} \left[d\sigma_a \, \delta\left(\frac{\partial F}{\partial \sigma_a}\right)\right] \det \Psi_{(k,\epsilon,\sigma)}$$

$$\Downarrow$$

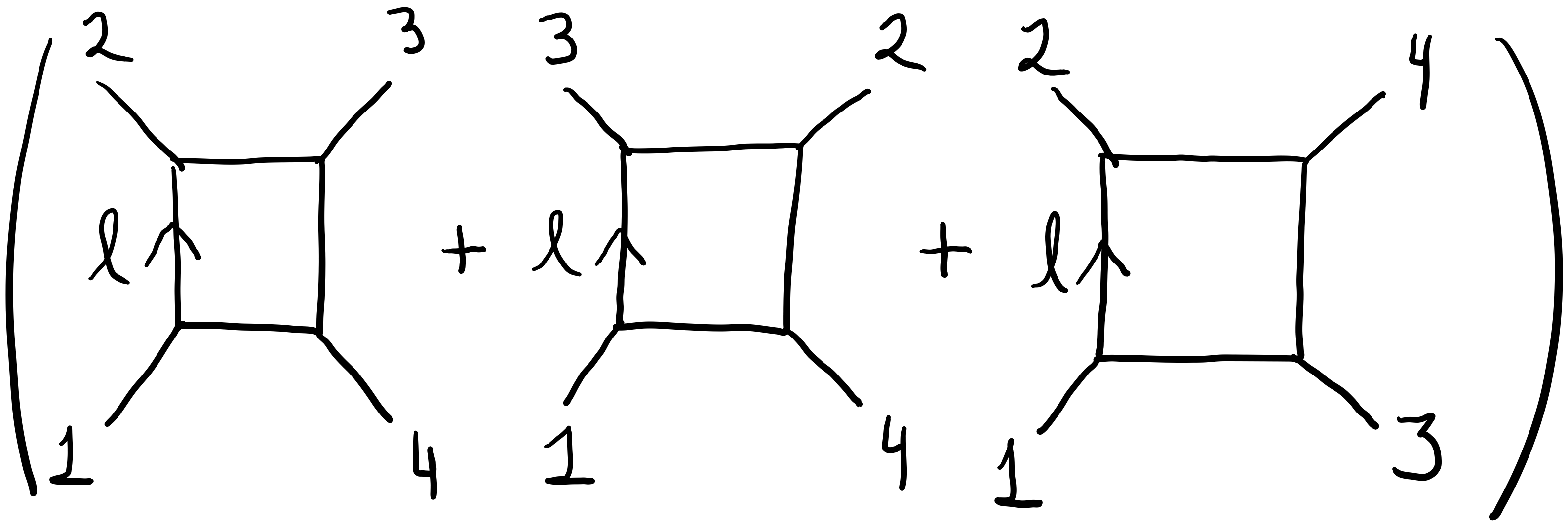
$$\mathcal{M}^{\text{1-loop}}_{n} = \int \frac{d^D \ell}{\ell^2} \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\left(\frac{\partial F(\sigma)}{\partial \sigma_a}\right)\right] \det \Psi_{(k,\epsilon,\sigma)}$$

Geyer, Mason, Monteiro, Tourkine '15

An Aside: One-Loop Integrands

- In theories with color one can define a natural notion of an integrand in the planar limit at any loop order.
- In gravity (or any colorless theory) there seems to be no natural way of combining different integrals into a single one!

$$\mathcal{M}_4^{\text{1-loop}} = \mathcal{M}_4^{\text{Tree}}$$



?

e.g $\mathcal{N}=8$ SUGRA

A New One-Loop Integrand

$$\mathcal{M}_n^{\text{1-loop}} = \int \frac{d^D\ell}{\ell^2} \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\!\left(\frac{\partial F}{\partial \sigma_a}\right)\right] \det \Psi(k_a \cdot k_b, k_a \cdot \ell, \epsilon, \sigma)$$

$$\Downarrow$$

$$\mathcal{M}_n^{\text{1-loop}} = \int \frac{d^D\ell}{\ell^2} \, \mathcal{I}(k_a \cdot k_b, k_a \cdot \ell)$$

No ℓ^2 dependence!

A New One-Loop Integrand

This integrand can be obtained from standard ones by partial fractions. However, I believe that

- This can be taken as a new starting point for the definition of loop amplitudes.
- One can use reduction techniques (P-V or vN-V) to bring any formula to a sum over a basis of new integrals.
- Only simple ones are known. The basis has to be computed!

Higher Loops? (Some Numerology)

$$\underbrace{\mathcal{M}_n^{L\text{-loop}}} \Longleftarrow \underbrace{\mathcal{M}_{n+2L}^{\text{tree}}}$$

The best understood theory is $\mathcal{N}=8$ SUGRA.

Loops	Dimension	Particles
0	4	n
1	5	n+2
2	6	n+4
3	7	n+6
4	8	n+8
5	9	n+10
6	10	n+12

F.C., He, Yuan 2013

Geyer, Mason, Monteiro, Tourkine 2015, F.C. He, Yuan 2015

Geyer, Mason, Monteiro, Tourkine 2016

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If these constructions are related to string theory in anyway then it is tempting to suggest that something special happens at 7 loops…

Concluding Remarks:

- The moduli space of punctured Riemann surfaces can be used to encode locality and unitarity of a large collection of theories. There are extensions to massive theories. (Massive: Goddard, Naculich, 2013)

- Could there be a relation between symmetries of null infinity, i.e. extensions of BMS (Strominger et.al.) and the CHY formulation? Perhaps ambitwistor string ideas will make the connection clear. (Mason, Skinner, et.al 2014)

- Developments at loop level are in their infancy but they could lead to new techniques and ways of thinking!

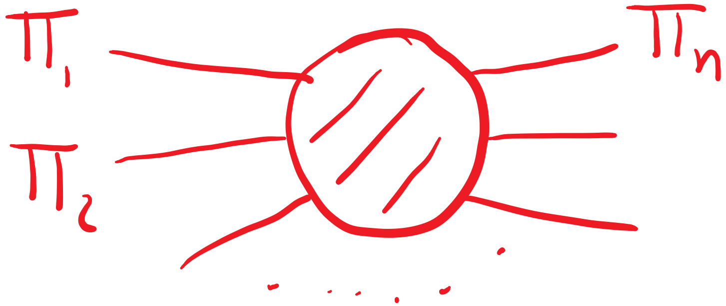
Bonus Material: Extension of Theories

Extension of Theories via Soft Limits

- Consider the effective theory of U(N) (massless) pions (NLSM):

$$\mathcal{L} = \mathrm{tr}\, \partial_\mu U \partial^\mu U^\dagger \qquad U(x) = e^{i\pi_a(x) T^a}$$

- Adler's zero: When a single pion becomes soft the amplitude vanishes



$$\xrightarrow[K_n^\mu \to \tau q^\mu]{} \mathcal{O}(\tau) \qquad \tau \to 0$$

Extension of Theories via Soft Limits

The CHY formula is given by:

$$\left(\text{Term} \quad \mathrm{Tr}\left(T^{a_1} T^{a_2} \cdots T^{a_n}\right)\right)$$

$$A_n(12 \dots n) = \int \prod_{a=1}^{n} d\sigma_a \, \delta\left(\sum_b \frac{s_{ab}}{\sigma_{ab}}\right) \frac{P_F \, \mathbb{A}_n}{\sigma_{12}\,\sigma_{23} \cdots \sigma_{n-1\,n}\,\sigma_{n1}}$$

$$[\mathbb{A}_n]_{ab} \equiv \begin{cases} \dfrac{s_{ab}}{\sigma_{ab}} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

Extension of Theories via Soft Limits

In the soft limit it is easy to write it as: $\left(k_n^{\mu} = \tau q^{\mu} \right)$

$$A_n(12\cdots n) = \tau \sum_{a=2}^{n-2} k_a \cdot q \int \prod_{c=1}^{n-1} d\sigma_c \, \delta\left(\sum_b \frac{s_{cb}}{\sigma_{cb}} \right) \frac{\mathrm{P}_F \, A^{n-1,a,1}_{n-1,a,1}}{(\sigma_{12}\cdots\sigma_{n-1\,1})(\sigma_{n-1\,a}\sigma_{a1}\sigma_{1\,n-1})}$$

(FC, P. Cha, S. Mizera 2016)

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Could these new objects be amplitudes of another theory?

(FC, P. Cha, S. Mizera 2016)

Extension of Theories via Soft Limits

Hints:

$$A_n(12\cdots n|i_1\cdots i_m)=\int \prod d\sigma\, \delta\left(\frac{\partial F}{\partial \sigma}\right)\frac{P_F\, A^{i_1\cdots i_m}_{i_1\cdots i_m}}{\underbrace{(\sigma_{12}\cdots\sigma_{n1})}_{U(N)}\underbrace{(\sigma_{i_1i_2}\cdots\sigma_{i_mi_1})}_{U(\tilde{N})}}$$

new!

If $m=n$ the numerator is 1

$\Rightarrow$ No derivative interactions

(FC, P. Cha, S. Mizera 2016)

Extension of Theories via Soft Limits

NLSM + Biadjoint scalar!

$$\mathcal{L} = \mathrm{tr}\,\partial_\mu U \partial^\mu U^\dagger + \partial_\mu \varphi^{a\tilde{b}} \partial^\mu \varphi_{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \varphi^{a\tilde{a}} \varphi^{b\tilde{b}} \varphi^{c\tilde{c}}$$
$$+ \mathcal{L}_{mix}(\pi^a, \varphi^{a\tilde{a}})$$

The biadjoint scalar naturally emerged from the NLSM. Note that a new flavor group also emerged! This is what we call the extension of the NLSM.

(FC, P. Cha, S. Mizera 2016)