Factorization and Resummation for Jet Processes

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In collaboration with T. Becher, L. Rothen & Ding Yu Shao
(PRL 116 (2016) 192001 & arXiv:1605.02737)

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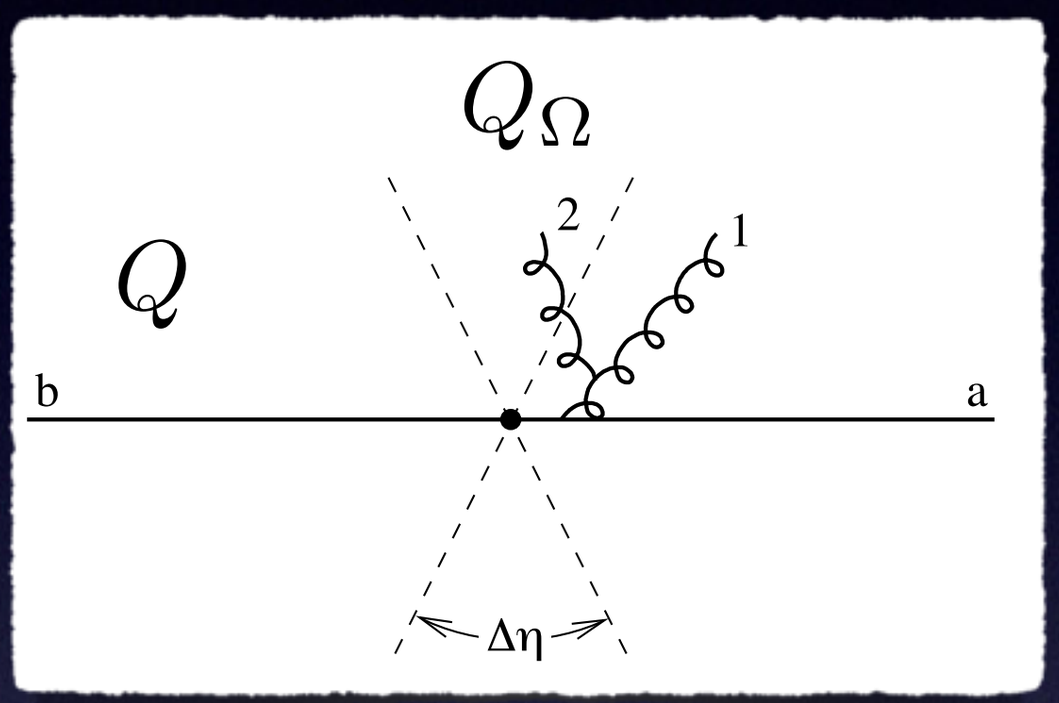
An Effective Field Theory Assault on the Zeptometer Scale:
Exploring the Origins of Flavor and Electroweak Symmetry Breaking

Non-global logarithms (NGLs)

(Dasgupta & Salam 2001,2002)

Observables which are insensitive to emissions into certain regions of phase space involve NGLs **not captured** by the usual resummation formula:

GLs : $$\exp\left[-4\,C_F\Delta\eta\int_{\alpha_s(Q_\Omega)}^{\alpha_s(Q)}\frac{d\alpha}{\beta(\alpha)}\,\frac{\alpha}{2\pi}\right] = 1+4\frac{\alpha_s}{2\pi}C_F\Delta\eta\ln\frac{Q_\Omega}{Q}$$
$$+\left(\frac{\alpha_s}{2\pi}\right)^2\left(8\,C_F^2\Delta\eta^2-\frac{22}{3}\,C_FC_A\Delta\eta+\frac{8}{3}C_FT_Fn_f\Delta\eta\right)\ln^2\frac{Q_\Omega}{Q}$$



NGLs :

$$\left(\frac{\alpha_s}{2\pi}\right)^2 C_FC_A\left[-\frac{2\pi^2}{3}+4\,\mathrm{Li}_2\left(e^{-2\Delta\eta}\right)\right]\ln^2\frac{Q_\Omega}{Q}$$

Leading-log resummation

Banfi, Marchesini & Smye 2002

- The leading logarithms arise from a configuration in which the emitted gluons are strongly ordered:

$$E_1 \gg E_2 \gg \cdots \gg E_m$$

- In the large-N_c limit, multi-gluon emission amplitudes become simple:

$$N_c^m g^{2m} \sum_{(1\cdots m)} \frac{p_a \cdot p_b}{(p_a \cdot p_1)(p_1 \cdot p_2)\cdots(p_m \cdot p_b)}$$

- Based on this structure, Banfi, Marchesini & Smye derived an integro-differential equation for resuming NG logarithms at LL level in the large-N_c limit:

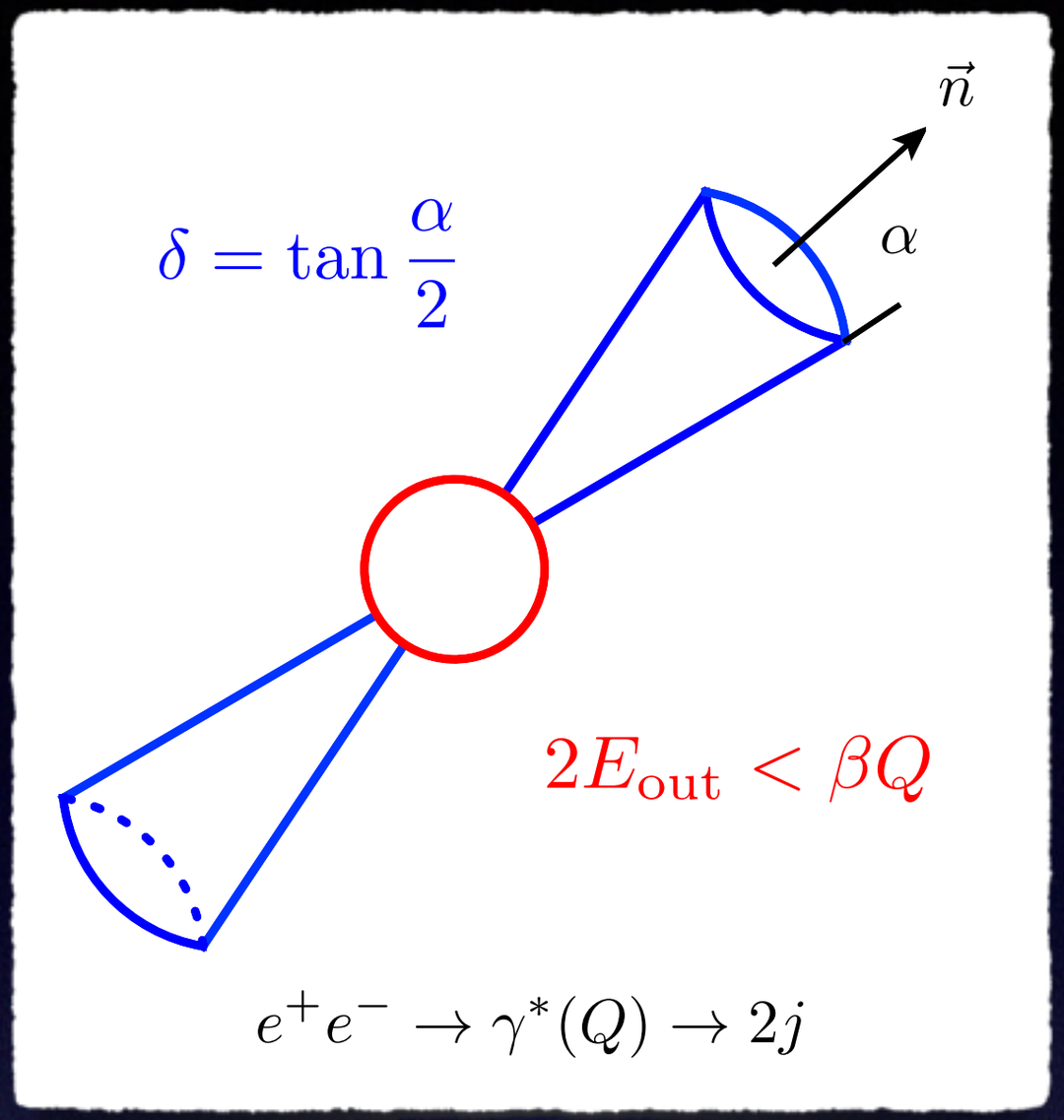
BMS equation: $$\partial_L G_{ab}(L) = \int \frac{d\Omega_j}{4\pi} W_{ab}^j \left[\Theta_{\rm in}^{n\bar{n}}(j) G_{aj}(L) G_{jb}(L) - G_{ab}(L)\right]$$

Some recent progress

- Resummation of LL NGLs beyond large N_c Hatta Ueda '13 + Hagiwara '15;
- Fixed-order results:
  - two-loop hemisphere soft function Kelley, Schwartz, Schabinger & Zhu '11; Horning, Lee, Stewart, Walsh & Zuberi '11
  - with jet-cone Kelley, Schwartz, Schabinger & Zhu '11; von Manteuffel, Schabinger & Zhu '13
  - LL NGLs (5-loop large N_c & 4-loop finite N_c) Schwartz, Zhu '14; Delenda, Khelifa-Kerfa '15
- Color density matrix (two-loop anomalous dimension) Caron-Huot '15
- Expansion in dressed gluons Larkoski, Moult & Neill '15; Neill '15; Laroski, Moult '15
- Avoid NGLs Dasgupta, Fregoso, Marzani & Powling '13; Dasgupta, Fregoso, Marzani & Salam '13; Larkoski, Marzani, Soyez & Thaler '14; Frye, Larkoski, Matthew & Yan '16; ...

Sterman-Weinberg dijets

(Sterman & Weinberg 1977)



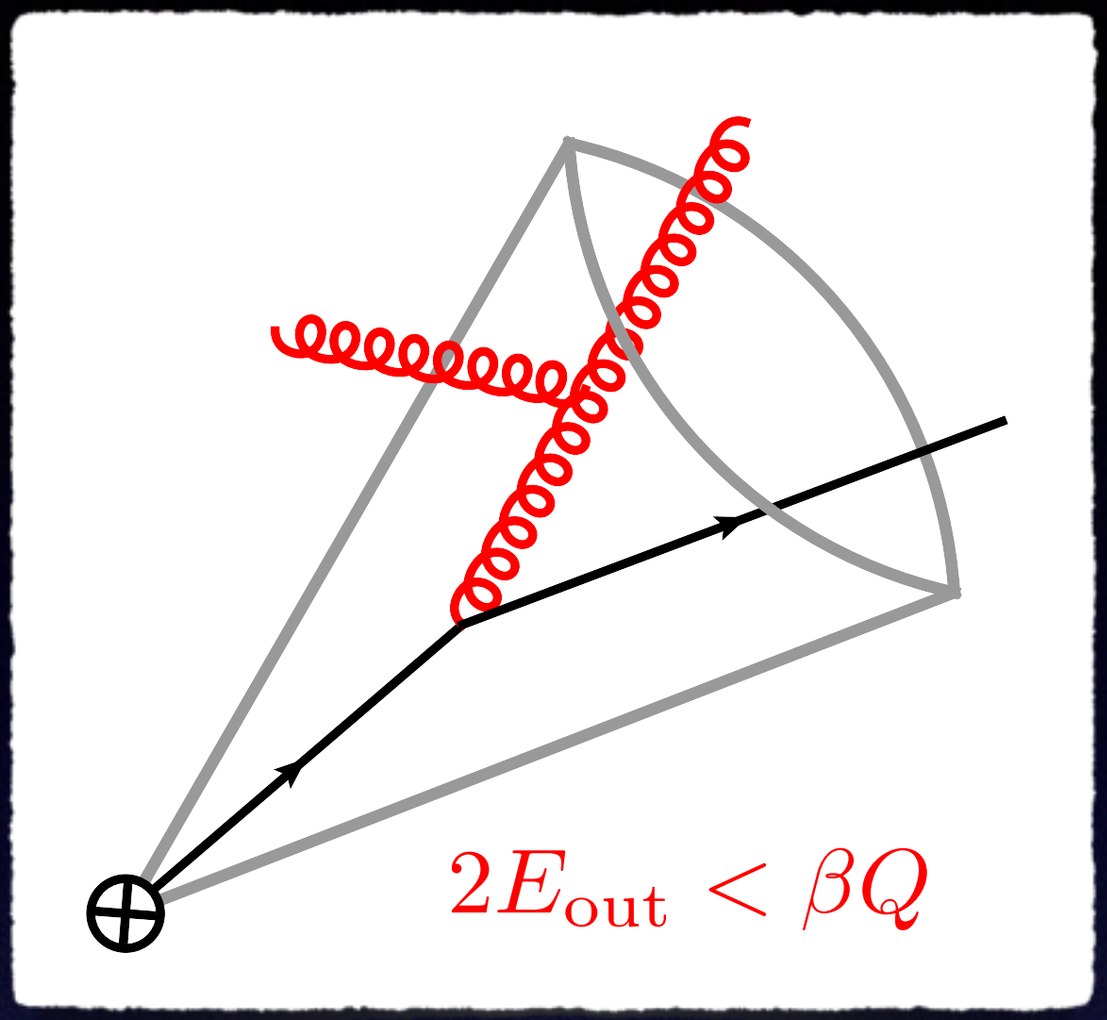
$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{3\pi}\left[-16\ln\delta\ln\beta - 12\ln\delta + 10 - \frac{4\pi^2}{3}\right]$$

IR finite, but problems for small β, δ

- Large logs can spoil perturbative expansion
- Scale choice?

$$\mu = Q,\ Q\beta,\ Q\delta,\ Q\beta\delta\ ?$$

NGLs in jet observables



Jet observables involve NGLs because they are insensitive to emissions inside the cone

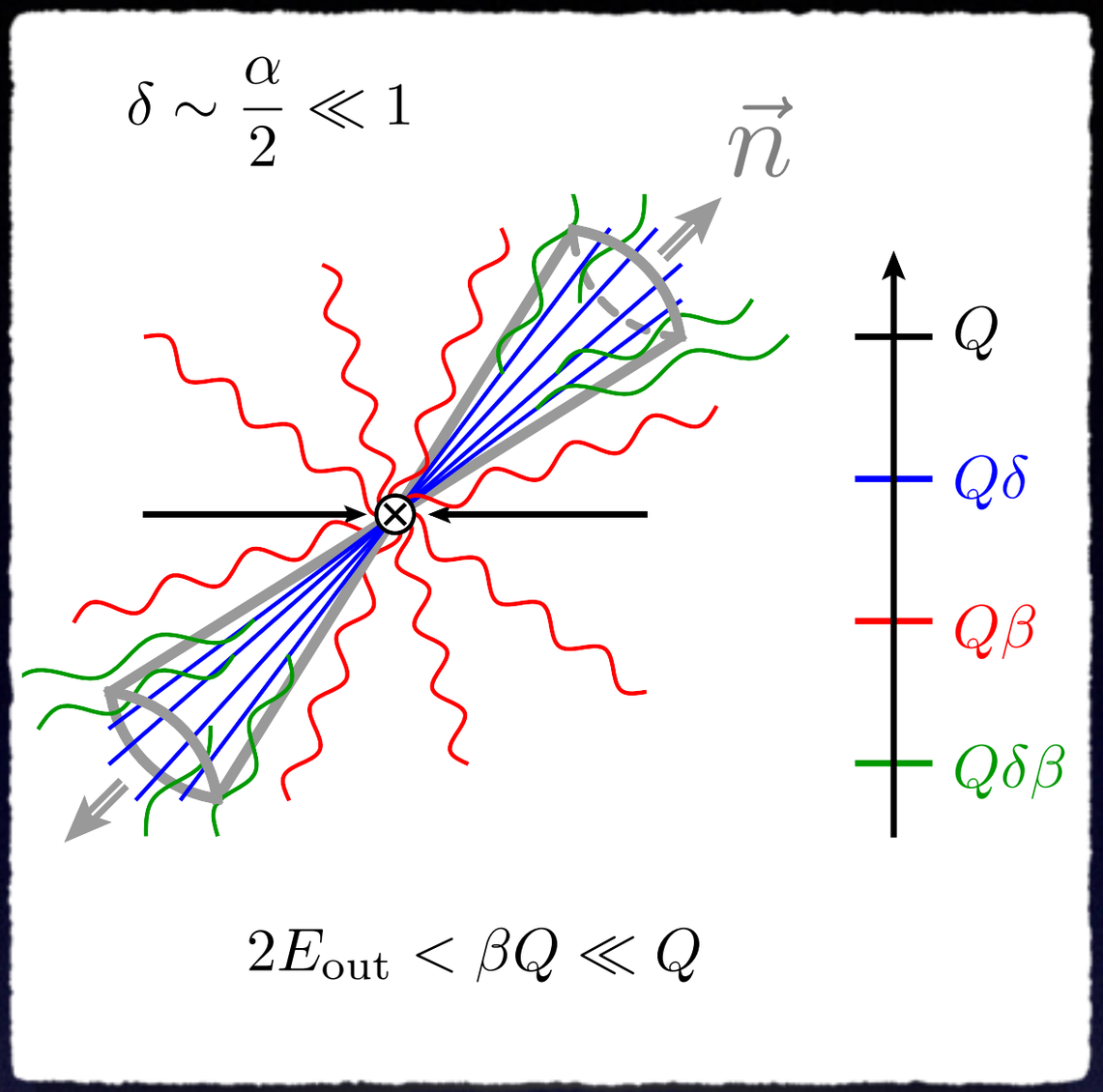
$$\left(\frac{\alpha_s}{2\pi}\right)^2 C_F C_A \left(-\frac{2\pi^2}{3}\right) \ln^2 \beta$$

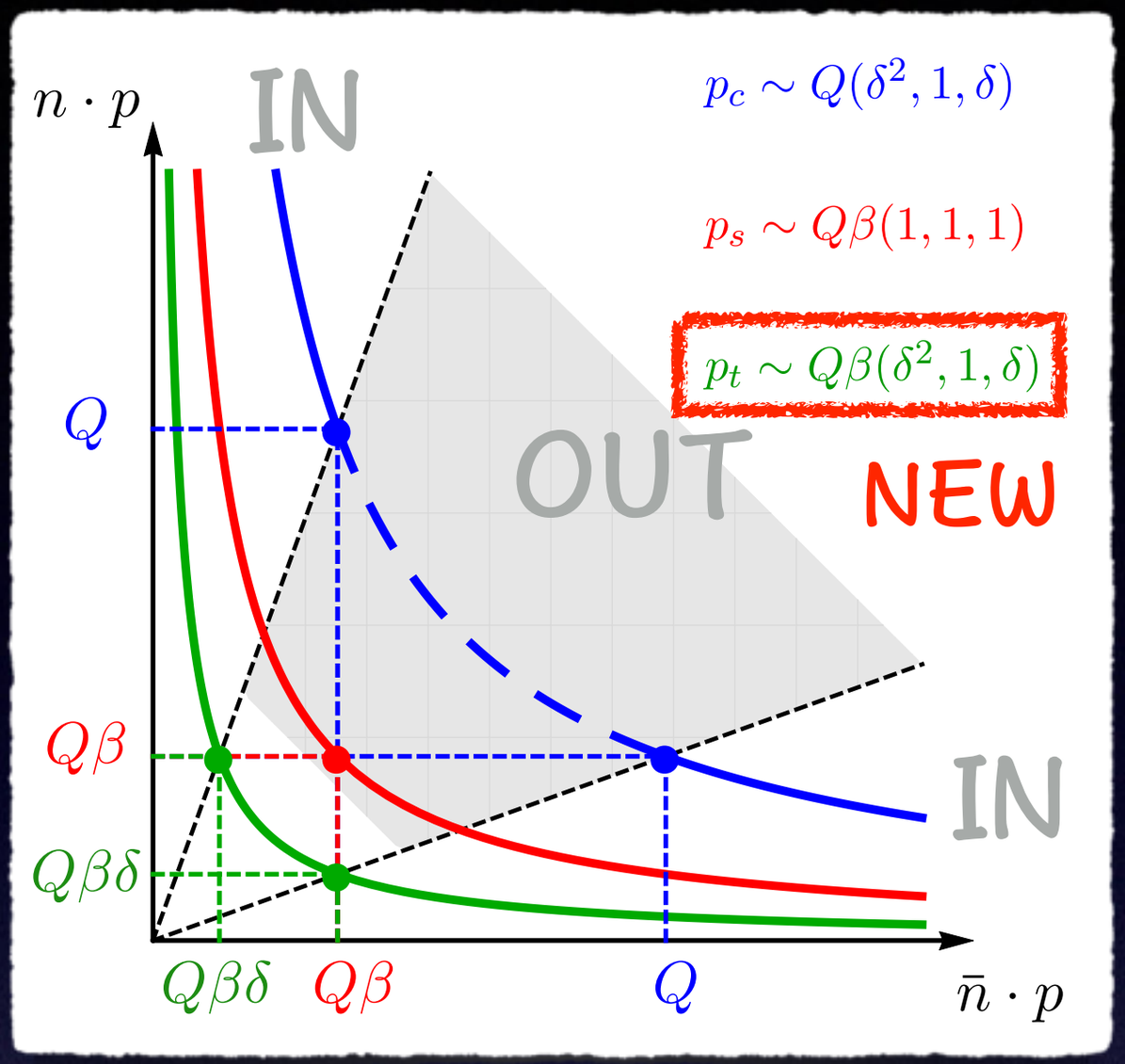
These types of logarithm do not exponentiate in the usual way

EFT for Sterman-Weinberg dijets

(Becher, MN, Rothen & Shao, PRL 116 (2016) 192001)

$$p \sim (n\cdot p, \bar{n}\cdot p, \vec{p}_\perp)$$





One-loop region analysis

Hard

$$\Delta\sigma_h = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}-\frac{6}{\epsilon}+\frac{7\pi^2}{3}-16\right)$$

Collinear

$$\Delta\sigma_{c+\bar{c}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}+\frac{6}{\epsilon}+c_0\right)$$

"Soft"

$$\Delta\sigma_s = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\beta}\right)^{2\epsilon}\left(\frac{8}{\epsilon}\ln\delta - 8\ln^2\delta - \frac{2\pi^2}{3}\right)$$

(Cheung, Luke, Zuberi 2009……)

$$\Delta\sigma^{\rm tot} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(-16\ln\delta\ln\beta - 12\ln\delta + c_0 + \frac{5\pi^2}{3} - 16\right)$$

Constant c_0 depends on the definition of jet axis:

$c_0 = -3\pi^2 + 26$ **(Sterman-Weinberg)**

$c_0 = -5\pi^2/3 + 14 + 12\ln 2$ **(thrust axis)**

One-loop region analysis

Hard	$\Delta\sigma_h = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}-\frac{6}{\epsilon}+\frac{7\pi^2}{3}-16\right)$
Collinear	$\Delta\sigma_{c+\bar{c}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}+\frac{6}{\epsilon}+c_0\right)$
Soft	$\Delta\sigma_s = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\beta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}-\pi^2\right)$
Coft	$\Delta\sigma_{t+\bar{t}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta\beta}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}+\frac{\pi^2}{3}\right)$

$$\Delta\sigma^{\rm tot} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(-16\ln\delta\ln\beta - 12\ln\delta + c_0 + \frac{5\pi^2}{3} - 16\right)$$

Constant c_0 depends on the definition of jet axis:

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$c_0 = -5\pi^2/3 + 14 + 12\ln 2$ (thrust axis)

Soft radiation

Large-angle soft radiation off a jet of collinear particles does not resolve individual energetic patrons:

$$\sum_i Q_i \frac{p_i \cdot \epsilon}{p_i \cdot k} \approx Q_{\rm tot} \frac{n \cdot \epsilon}{n \cdot k}$$

But this approximation breaks down for soft radiation collinear to the jet!

$$k^\mu = \omega n^\mu$$

Typically this small region of phase space does not give an $\mathcal{O}(1)$ contribution.

However, it does for non-global observables!

Factorization formula

$$\widetilde{\sigma}(\tau,\delta) = \int_0^\infty d\beta\, e^{-\beta/(\tau e^{\gamma_E})}\, \frac{d\sigma(\delta)}{d\beta}$$

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q)\, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m(Q\delta) \otimes \widetilde{\mathcal{U}}_m(Q\delta\tau) \right\rangle \right]^2$$

color trace

integration over angles

hard function

soft function

coft function with m Wilson lines

jet function with m partons at fixed direction $\mathcal{J}_m \sim \mathcal{O}(\alpha_s^{m-1})$

First all-order factorization theorem for a non-global observable, achieving full scale separation!

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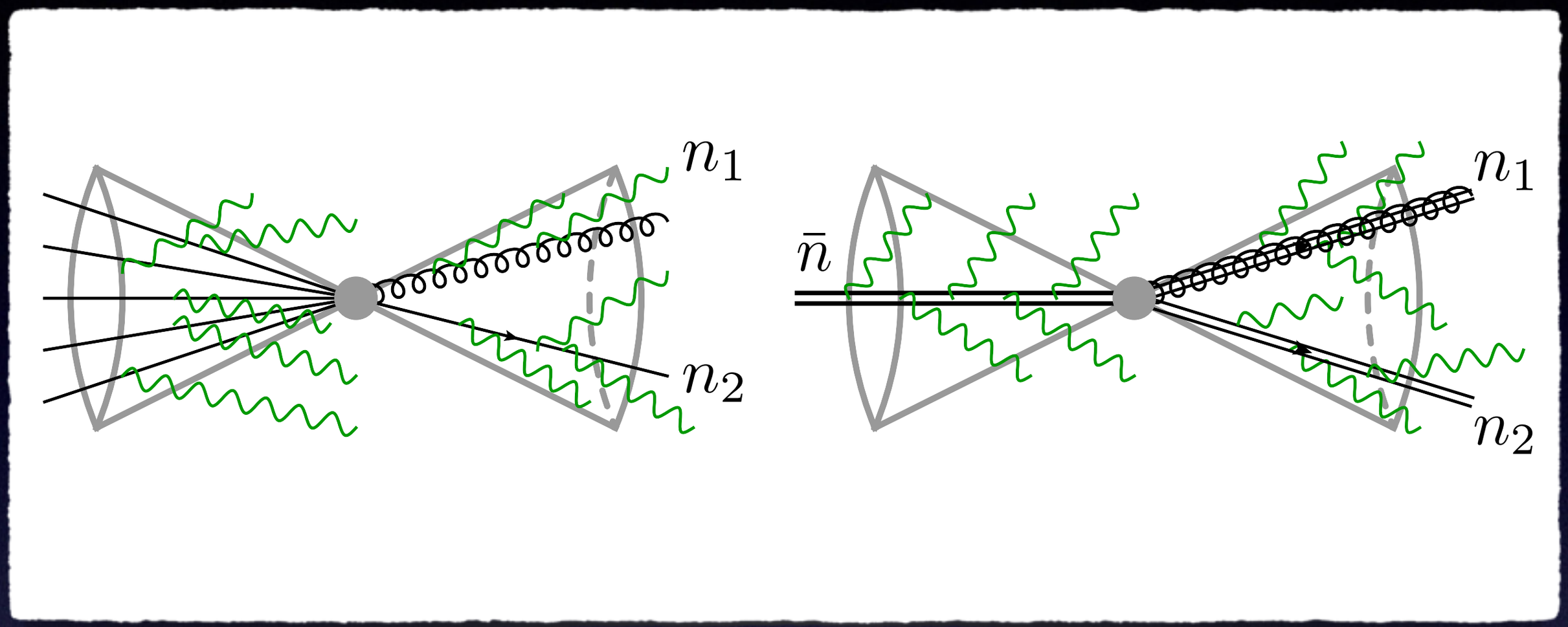
First all-order factorization theorem for a non-global observable, achieving full scale separation!

Note that the coft scale $\Lambda = Q\delta\tau$ can easily be 1 GeV, even if the collinear and soft scales are perturbative!

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon)\, \big\langle \mathcal{J}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\mathcal{U}}_1(\{n_1\},Q\delta\tau,\epsilon)$$
$$+ \underline{\mathcal{J}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\mathcal{U}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon)} + \mathcal{J}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \ldots \big\rangle^2$$



NNLO check

$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} A(\beta,\delta) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\beta,\delta) + \ldots$$

$$\begin{aligned}
B(\beta,\delta) = C_F^2 \Bigg[& \left(32\ln^2\beta + 48\ln\beta + 18 - \frac{16\pi^2}{3}\right)\ln^2\delta + \left(-2 + 10\zeta_3 - 12\ln^2 2 + 4\ln 2\right)\ln\beta \\
& + \left((8 - 48\ln 2)\ln\beta + \frac{9}{2} + 2\pi^2 - 24\zeta_3 - 36\ln 2\right)\ln\delta + c_2^F \Bigg] \\
& + C_F C_A \Bigg[\left(\frac{44\ln\beta}{3} + 11\right)\ln^2\delta - \frac{2\pi^2}{3}\ln^2\beta + \left(\frac{8}{3} - \frac{31\pi^2}{18} - 4\zeta_3 - 6\ln^2 2 - 4\ln 2\right)\ln\beta \\
& \quad + \left(\frac{44\ln^2\beta}{3} + \left(-\frac{268}{9} + \frac{4\pi^2}{3}\right)\ln\beta - \frac{57}{2} + 12\zeta_3 - 22\ln 2\right)\ln\delta + c_2^A \Bigg] \\
& + C_F T_F n_f \Bigg[\left(-\frac{16\ln\beta}{3} - 4\right)\ln^2\delta + \left(-\frac{16}{3}\ln^2\beta + \frac{80\ln\beta}{9} + 10 + 8\ln 2\right)\ln\delta \\
& \quad + \left(-\frac{4}{3} + \frac{4\pi^2}{9}\right)\ln\beta + c_2^f \Bigg] .
\end{aligned}$$

- Consistent with EVENT2

NNLO check

$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} A(\beta,\delta) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\beta,\delta) + \ldots$$

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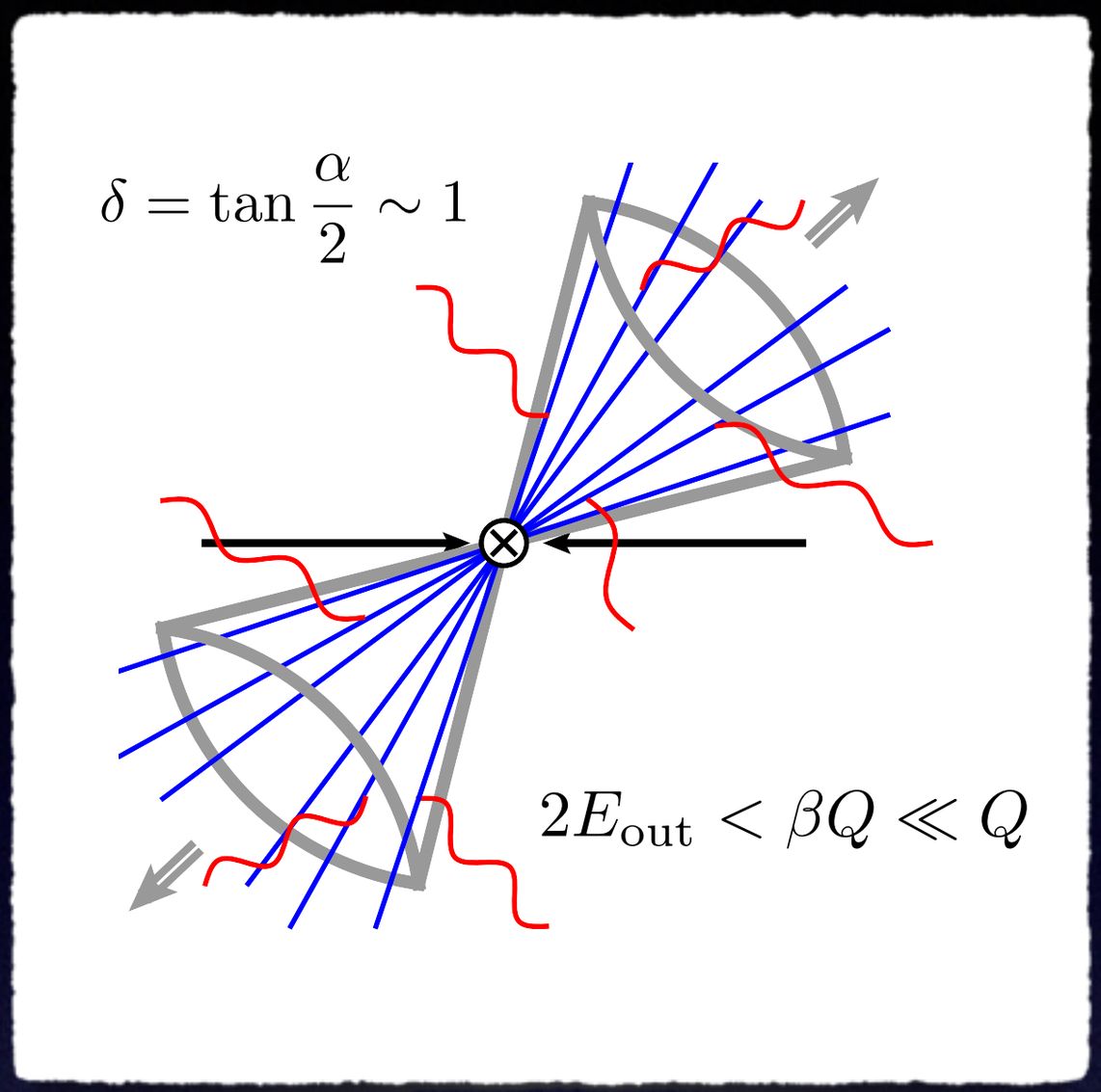
& \quad + \left(\frac{44\ln^2\beta}{3} + \left(-\frac{268}{9} + \frac{4\pi^2}{3}\right)\ln\beta - \frac{57}{2} + 12\zeta_3 - 22\ln 2\right)\ln\delta + c_2^A \Bigg] \\
& + C_F T_F n_f \Bigg[\left(-\frac{16\ln\beta}{3} - 4\right)\ln^2\delta + \left(-\frac{16}{3}\ln^2\beta + \frac{80\ln\beta}{9} + 10 + 8\ln 2\right)\ln\delta \\
& \quad + \left(-\frac{4}{3} + \frac{4\pi^2}{9}\right)\ln\beta + c_2^f \Bigg] .
\end{aligned}$$

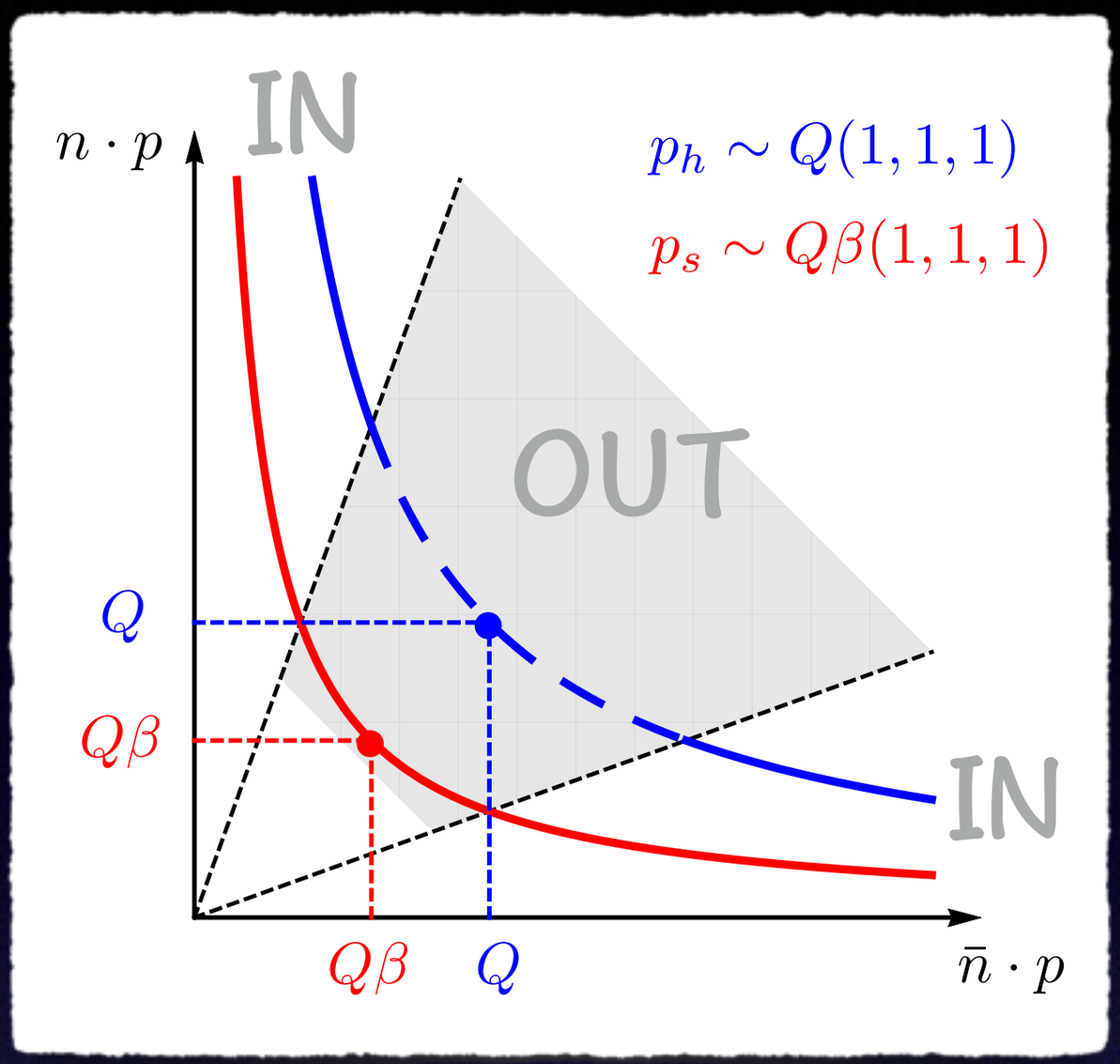
Leading NGL

- Consistent with EVENT2

EFT for interjet energy flow

(Becher, MN, Rothen & Shao 1605.02737)





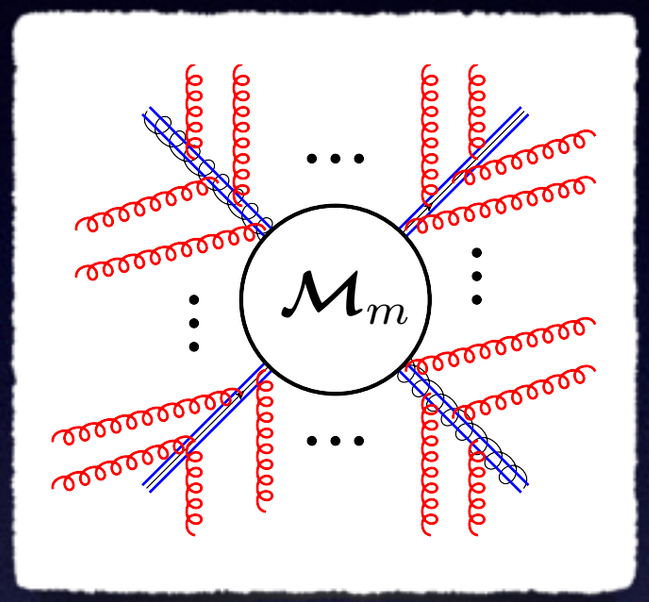
$$\Delta\eta = -2\ln\delta$$

Factorization

- Hard parton -> collinear fields $\Phi_i \in \{\chi_i, \bar{\chi}_i, \mathcal{A}^\mu_{i\perp}\}$ along $n_i^\mu = (1, \vec{n}_i)$
- Performing SCET decoupling transformation: $\Phi_i = \boldsymbol{S}_i(n_i)\,\Phi_i^{(0)}$

$$\boldsymbol{S}_i(n_i) = \mathbf{P}\exp\left(ig_s\int_0^\infty ds\, n_i\cdot A_s^a(sn_i)\,\boldsymbol{T}_i^a\right)$$

- The operator for the emission from an amplitude with m hard partons:



hard scattering amplitude with m particles (vector in color space)

$$S_1(n_1)\,S_2(n_2)\,\ldots\,S_m(n_m)|\mathcal{M}_m(\{\underline{p}\})\rangle$$

soft Wilson lines along the directions of the energetic particles (color matrices)

Factorization

- Then the cross section can be written in factorized form as:

$$\boxed{\sigma(\beta,\delta) = \sum_{m=2}^{\infty} \left\langle \mathcal{H}_m(\{\underline{n}\},Q,\delta) \otimes \mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) \right\rangle}$$

- We define the squared matrix element of the soft operator as:

$$\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) = \sum\!\!\!\!\!\!\int_X \langle 0|S_1^\dagger(n_1)\,\ldots\,S_m^\dagger(n_m)|X_s\rangle\langle X_s|S_1(n_1)\,\ldots\,S_m(n_m)|0\rangle\,\theta\left(Q\beta - 2E_{\rm out}\right)$$

- The hard functions are obtained by integrating over the energies of the hard particles, while keeping their direction fixed:

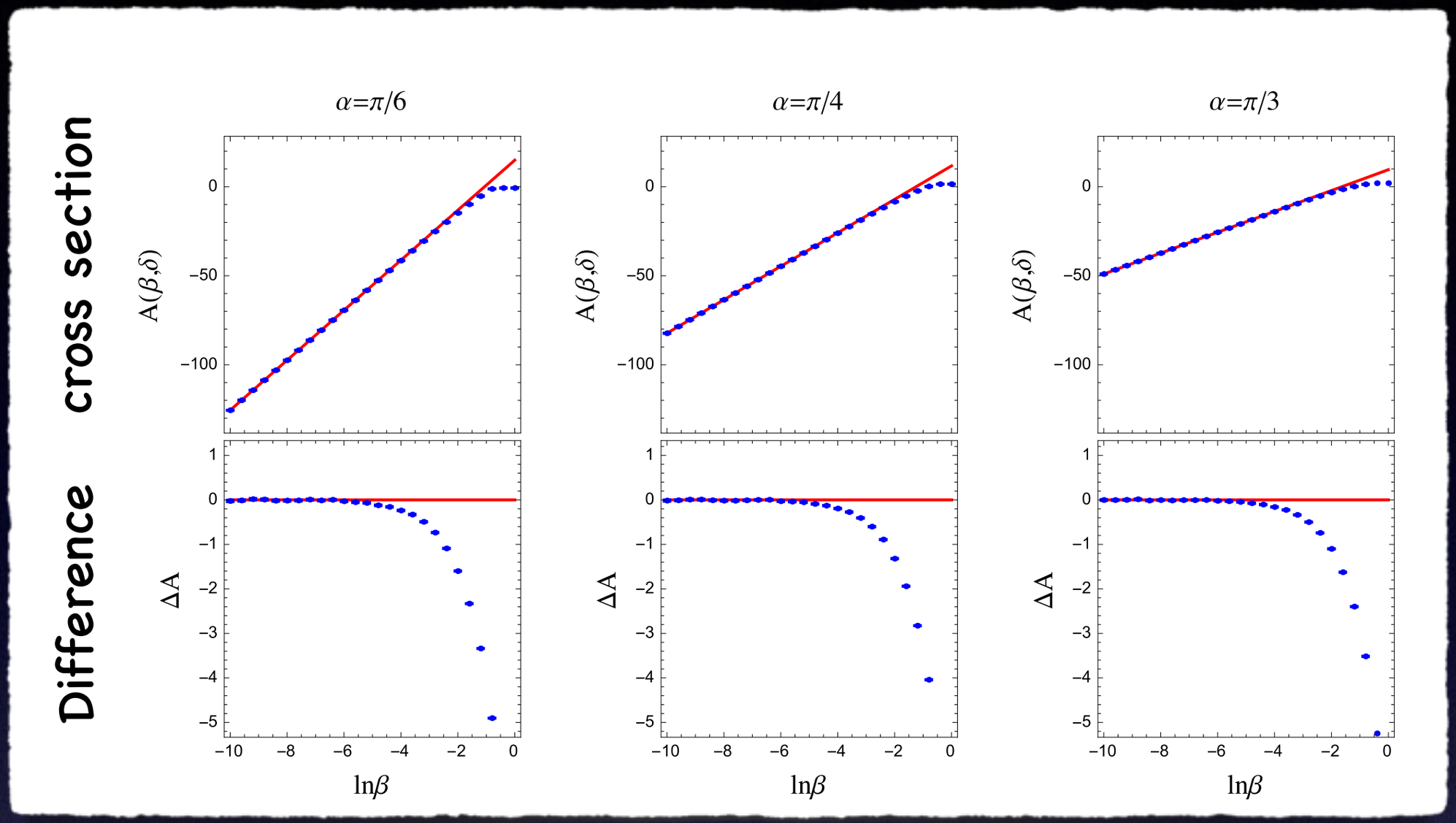
$$\mathcal{H}_m(\{\underline{n}\},Q,\delta) = \frac{1}{2Q^2}\sum_{\rm spins}\prod_{i=1}^{m}\int\frac{d\omega_i\,\omega_i^{d-3}}{(2\pi)^{d-2}}|\mathcal{M}_m\rangle\langle\mathcal{M}_m|\delta\Big(Q-\sum_{i=1}^{m}\omega_i\Big)\delta^{d-1}(\vec{p}_{\rm tot})\,\Theta_{\rm in}^{n\bar{n}}(\{\underline{p}\})$$

- $\bigotimes$ indicates integration over the direction of the energetic partons:

$$\mathcal{H}_m(\{\underline{n}\},Q,\delta)\otimes\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) = \prod_{i=1}^{m}\int\frac{d\Omega(n_i)}{4\pi}\,\mathcal{H}_m(\{\underline{n}\},Q,\delta)\,\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta)$$

One-loop coefficient vs. EVENT2

$$A(\beta,\delta) = C_F\left[-8\ln\delta\ln\beta - 1 + 6\ln 2 - 6\ln\delta - 6\,\delta^2 + \left(\frac{9}{2} - 6\ln 2\right)\delta^4 - 4\,\mathrm{Li}_2(-\delta^2) + 4\,\mathrm{Li}_2(\delta^2)\right]$$



Two-loop coefficient

$$B(\beta,\delta) = C_F^2 B_F + C_F C_A B_A + C_F T_F n_f B_f$$

$$\begin{aligned}
B_A = \Bigg[& \frac{44}{3}\ln\delta - \frac{2\pi^2}{3} + 4\,\mathrm{Li}_2(\delta^4)\Bigg]\ln^2\beta + \Bigg[\frac{4}{3\left(1-\delta^4\right)} - \frac{16\ln\delta}{3\left(1-\delta^4\right)} + \frac{16\ln\delta}{3\left(1-\delta^4\right)^2} \\
& -\frac{4}{3}\ln^3\left(1-\delta^2\right) - \frac{20}{3}\ln^3\left(1+\delta^2\right) + 32\ln\delta\ln^2\left(1-\delta^2\right) - 4\ln\left(1+\delta^2\right)\ln^2\left(1-\delta^2\right) \\
& -4\ln^2\left(1+\delta^2\right)\ln\left(1-\delta^2\right) + 64\ln\delta\ln^2\left(1+\delta^2\right) - 64\ln^2\delta\ln\left(1+\delta^2\right) \\
& +\frac{88}{3}\ln\delta\ln\left(1-\delta^2\right) - \frac{16}{3}\pi^2\ln\left(1-\delta^2\right) + 44\ln\delta\ln\left(1+\delta^2\right) + \frac{16}{3}\pi^2\ln\left(1+\delta^2\right) \\
& +\frac{44\ln^2\delta}{3} - \frac{16}{3}\pi^2\ln\delta - \frac{268\ln\delta}{9} + \frac{88\,\mathrm{Li}_2\left(\delta^4\right)}{3} - 4\,\mathrm{Li}_3\left(\delta^4\right) + 8\,\mathrm{Li}_3\left(-\frac{\delta^4}{1-\delta^4}\right) \\
& +8\ln 2\,\mathrm{Li}_2\left(\delta^4\right) - \frac{88\,\mathrm{Li}_2\left(\delta^2\right)}{3} - \frac{22}{3}\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) + \frac{22}{3}\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) + 32\,\mathrm{Li}_3\left(1-\delta^2\right) \\
& +32\,\mathrm{Li}_3\left(\frac{\delta^2}{1+\delta^2}\right) + 32\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^2\right) + 32\ln\delta\,\mathrm{Li}_2\left(\delta^2\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^2\right) \\
& +32\ln\delta\,\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\delta\,\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) \\
& +32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) - 8\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) + 8\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) - 24\,\zeta_3 \\
& -\frac{2}{3} - \frac{4}{3}\pi^2\ln 2 - M_A^{[1]}(\delta)\Bigg]\ln\beta + c_2^A(\delta)
\end{aligned}$$

Two-loop coefficient

Leading NGL $B(\beta,\delta) = C_F^2 B_F + C_F C_A B_A + C_F T_F n_f B_f$

$$
\begin{aligned}
B_A = \Bigg[& \frac{44}{3}\ln\delta \boxed{- \frac{2\pi^2}{3} + 4\,\mathrm{Li}_2(\delta^4)} \ln^2\beta + \Bigg[\frac{4}{3\left(1-\delta^4\right)} - \frac{16\ln\delta}{3\left(1-\delta^4\right)} + \frac{16\ln\delta}{3\left(1-\delta^4\right)^2} \\
& - \frac{4}{3}\ln^3\left(1-\delta^2\right) - \frac{20}{3}\ln^3\left(1+\delta^2\right) + 32\ln\delta\ln^2\left(1-\delta^2\right) - 4\ln\left(1+\delta^2\right)\ln^2\left(1-\delta^2\right) \\
& -4\ln^2\left(1+\delta^2\right)\ln\left(1-\delta^2\right) + 64\ln\delta\ln^2\left(1+\delta^2\right) - 64\ln^2\delta\ln\left(1+\delta^2\right) \\
& + \frac{88}{3}\ln\delta\ln\left(1-\delta^2\right) - \frac{16}{3}\pi^2\ln\left(1-\delta^2\right) + 44\ln\delta\ln\left(1+\delta^2\right) + \frac{16}{3}\pi^2\ln\left(1+\delta^2\right) \\
& + \frac{44\ln^2\delta}{3} - \frac{16}{3}\pi^2\ln\delta - \frac{268\ln\delta}{9} + \frac{88\,\mathrm{Li}_2\left(\delta^4\right)}{3} - 4\,\mathrm{Li}_3\left(\delta^4\right) + 8\,\mathrm{Li}_3\left(-\frac{\delta^4}{1-\delta^4}\right) \\
& +8\ln 2\,\mathrm{Li}_2\left(\delta^4\right) - \frac{88\,\mathrm{Li}_2\left(\delta^2\right)}{3} - \frac{22}{3}\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) + \frac{22}{3}\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) + 32\,\mathrm{Li}_3\left(1-\delta^2\right) \\
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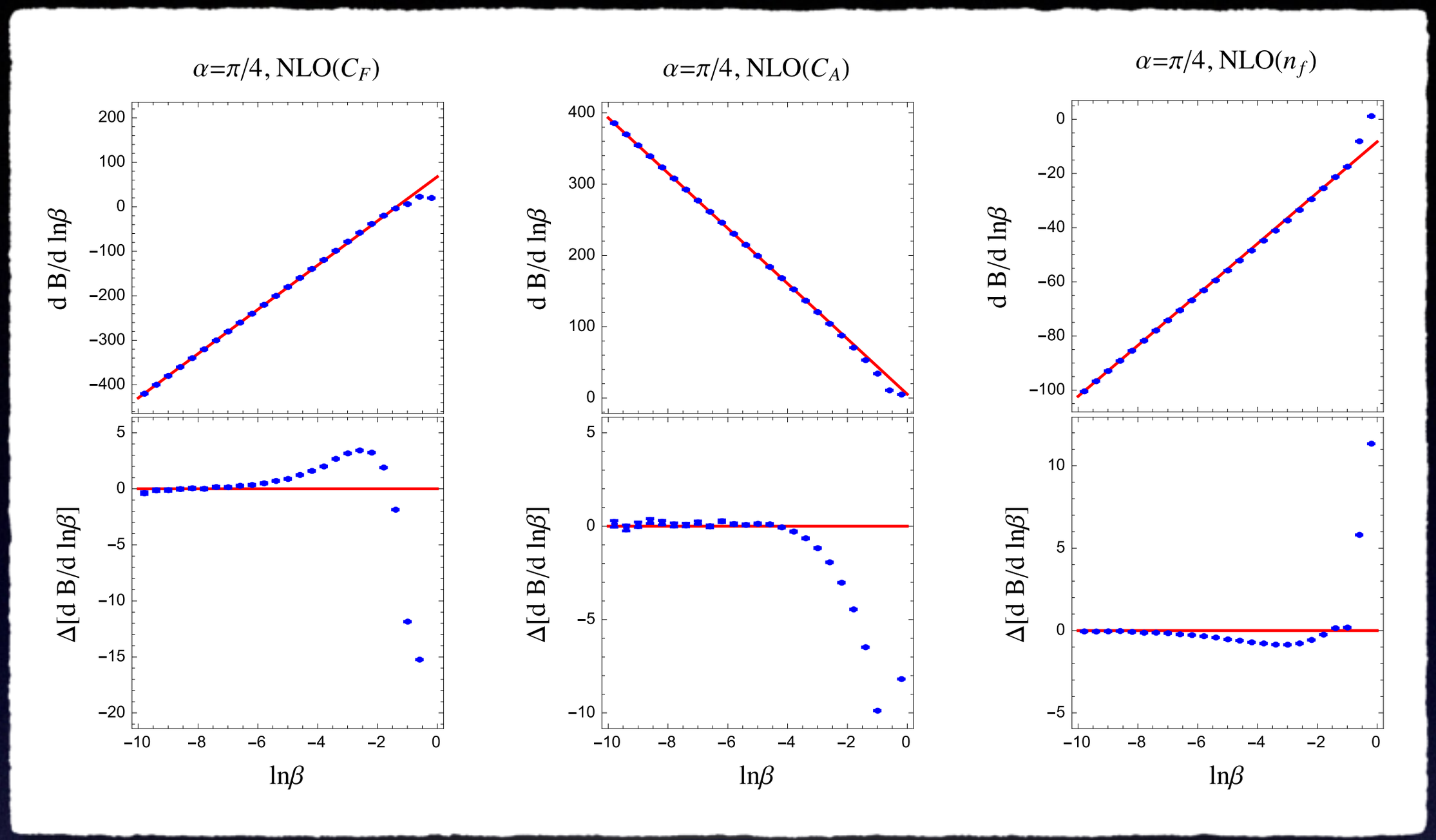
& +32\ln\delta\,\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\delta\,\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) \\
& +32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) - 8\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) + 8\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) - 24\,\zeta_3 \\
& - \frac{2}{3} - \frac{4}{3}\pi^2\ln 2 - M_A^{[1]}(\delta)\Bigg]\ln\beta + c_2^A(\delta)
\end{aligned}
$$

Two-loop coefficient vs. EVENT2



Renormalization

- We renormalize the bare hard function as usual:

$$\boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q, \delta, \epsilon) = \sum_{l=2}^{m} \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu)\, \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)$$

$$\boxed{\mathcal{H}_m \sim \mathcal{O}(\alpha_s^{m-2})}$$

e.g. $\mathcal{H}_2(\epsilon) = \mathcal{H}_2(\mu)\boldsymbol{Z}^H_{22}(\epsilon, \mu)$

$$\mathcal{H}_3(\epsilon) = \mathcal{H}_2(\mu)\boldsymbol{Z}^H_{23}(\epsilon, \mu) + \mathcal{H}_3(\mu)\boldsymbol{Z}^H_{33}(\epsilon, \mu)$$

- Z-factor has the structure:

$$\boldsymbol{Z}^H(\{\underline{n}\}, Q, \delta, \epsilon, \mu) = \begin{pmatrix} Z_{22} & Z_{23} & Z_{24} & Z_{25} & \dots \\ Z_{32} & Z_{33} & Z_{34} & Z_{35} & \dots \\ Z_{42} & Z_{43} & Z_{44} & Z_{45} & \dots \\ Z_{52} & Z_{53} & Z_{54} & Z_{55} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \sim \begin{pmatrix} 1 & \alpha_s & \alpha_s^2 & \alpha_s^3 & \dots \\ 0 & 1 & \alpha_s & \alpha_s^2 & \dots \\ 0 & 0 & 1 & \alpha_s & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Renormalization

- By consistency, matrix Z^H must render the soft function finite:

$$\boldsymbol{\mathcal{S}}_l(\{\underline{n}\}, Q\beta, \delta, \mu) = \sum_{m=l}^{\infty} \boldsymbol{Z}^H_{lm}(\{\underline{n'}\}, Q, \delta, \epsilon, \mu) \,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n'}\}, Q\beta, \delta, \epsilon)$$

- Have verified that Z^H renormalizes the two-loop soft function:

$$\mathcal{S}_2(\mu) = Z^H_{22}\,\mathcal{S}_2(\epsilon) + Z^H_{23} \,\hat{\otimes}\, \mathcal{S}_3(\epsilon) + Z^H_{24} \,\hat{\otimes}\, 1 + \mathcal{O}(\alpha_s^3)$$

and the general one-loop soft function:

$$\frac{\alpha_s}{4\pi}\, \boldsymbol{z}^{(1)}_{m,m}(\{\underline{n}\}, Q, \delta, \epsilon, \mu) + \frac{\alpha_s}{4\pi} \int \frac{d\Omega(n_{m+1})}{4\pi}\, \boldsymbol{z}^{(1)}_{m,m+1}(\{\underline{n}, n_{m+1}\}, Q, \delta, \epsilon, \mu)$$
$$+ \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \epsilon) = \text{finite}$$

Resummation

Therefore the resumed cross section reads:

$$\sigma(\beta,\delta) = \sum_{l=2}^{\infty} \Big\langle \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu_h) \otimes \sum_{m\geq l} \boldsymbol{U}^S_{lm}(\{\underline{n'}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n'}\}, Q\beta, \delta, \mu_s) \Big\rangle$$

with the (formal) evolution matrix:

$$\boldsymbol{U}^S(\{\underline{n}\}, \delta, \mu_s, \mu_h) = \mathbf{P}\exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu}\, \boldsymbol{\Gamma}^H(\{\underline{n}\}, \delta, \mu)\right]$$

The hard and soft matching scales are $\mu_h \sim Q$ and $\mu_s \sim Q\beta$; at these scales the hard and soft functions are free of large logs!

Leading-log resummation

At LL level:

$$\boldsymbol{\mathcal{S}}^T = (1, 1, \cdots, 1), \quad \mathcal{H} = (\sigma_0, 0, \cdots, 0), \quad \boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

V_m: divergences of one-loop virtual m-leg amplitudes

R_m: divergences from additional real radiation

$$\sigma_{\rm LL}(\delta, \beta) = \sigma_0 \left\langle \boldsymbol{\mathcal{S}}_2(\{n, \bar{n}\}, Q\beta, \delta, \mu_h) \right\rangle = \sigma_0 \sum_{m=2}^{\infty} \left\langle \boldsymbol{U}^S_{2m}(\{\underline{n}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \mathbf{1} \right\rangle$$

The symbol $\hat{\otimes}$ indicates that one has to integrate over the additional directions present in the higher-multiplicity anomalous dimensions R_m and V_m

Leading-log expansion

Expand RG equation order by order:

$$W_{ij}^k = \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_j \cdot n_k}$$

$$\mathcal{S}_2^{(1)} = -\,(4N_c) \int_\Omega \mathbf{3}_{\rm Out}\, W_{12}^3\,,$$

$$\mathcal{S}_2^{(2)} = \frac{1}{2!}\,(4N_c)^2 \int_\Omega \Big[-\,\mathbf{3}_{\rm In}\,\mathbf{4}_{\rm Out}\,\big(P_{12}^{34} - W_{12}^3\,W_{12}^4\big) + \mathbf{3}_{\rm Out}\,\mathbf{4}_{\rm Out}\,W_{12}^3\,W_{12}^4\Big]\,,$$

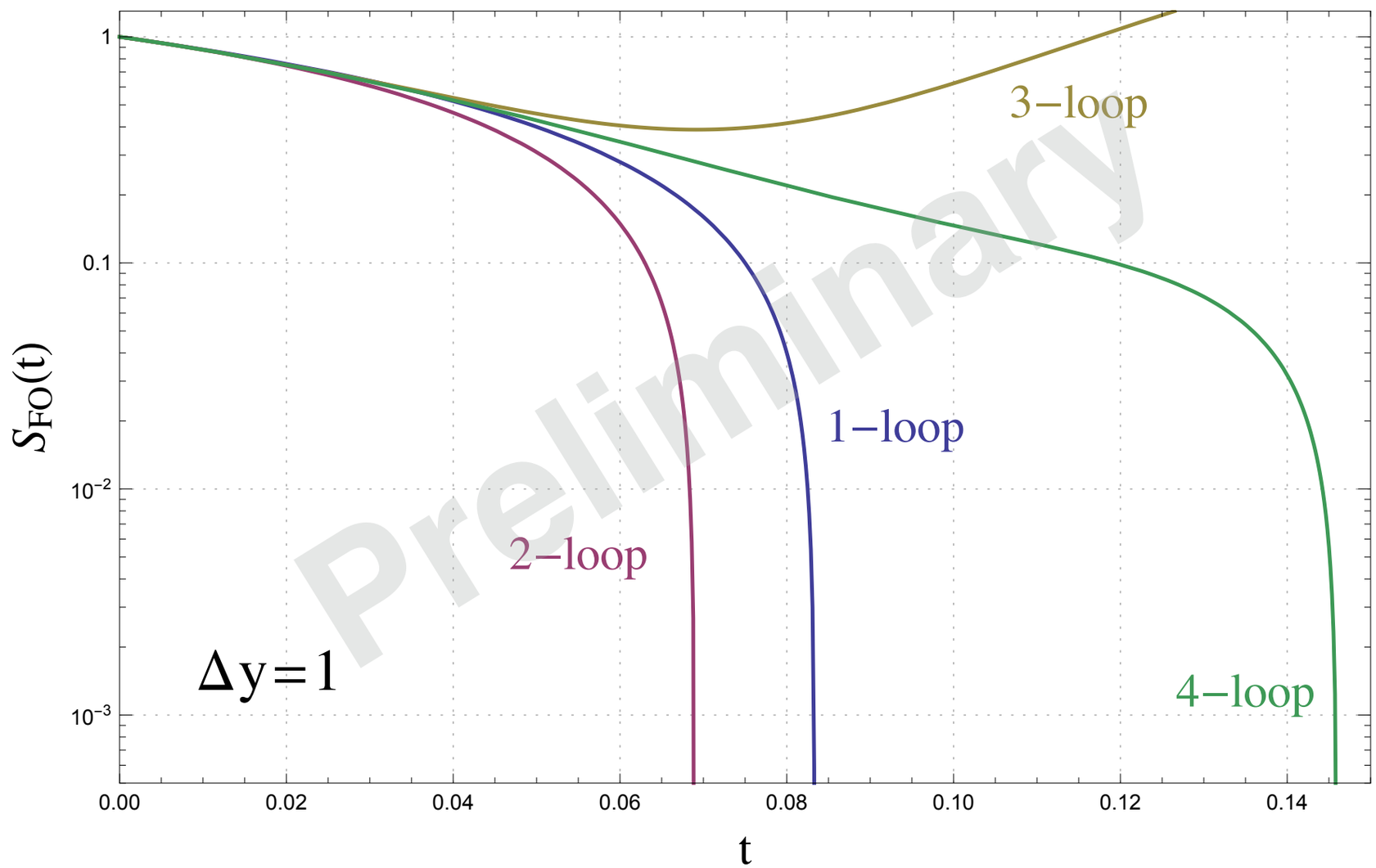
$$\begin{aligned}\mathcal{S}_2^{(3)} = \frac{1}{3!}\,(4N_c)^3 \int_\Omega \Big[& \mathbf{3}_{\rm In}\,\mathbf{4}_{\rm Out}\,\mathbf{5}_{\rm Out}\,\big[P_{12}^{34}\,(W_{13}^5 + W_{32}^5 + W_{12}^5) - 2W_{12}^3\,W_{12}^4\,W_{12}^5\big] \\ & -\,\mathbf{3}_{\rm In}\,\mathbf{4}_{\rm In}\,\mathbf{5}_{\rm Out}\,W_{12}^3\,\big[(P_{13}^{45} - W_{13}^4\,W_{13}^5) + (P_{32}^{45} - W_{32}^4\,W_{32}^5) - (P_{12}^{45} - W_{12}^4\,W_{12}^5)\big] \\ & -\,\mathbf{3}_{\rm Out}\,\mathbf{4}_{\rm Out}\,\mathbf{5}_{\rm Out}\,W_{12}^3\,W_{12}^4\,W_{12}^5\Big]\end{aligned}$$

Agrees with order-by-order expansion of BMS equation:

$$\partial_L G_{12}(L) = \int \frac{d\Omega_j}{4\pi} W_{12}^j \left[\Theta_{\rm in}^{n\bar{n}}(j) G_{1j}(L) G_{j2}(L) - G_{12}(L)\right]$$

Schwartz, Zhu '14

Leading-log expansion



Leading-log resummation

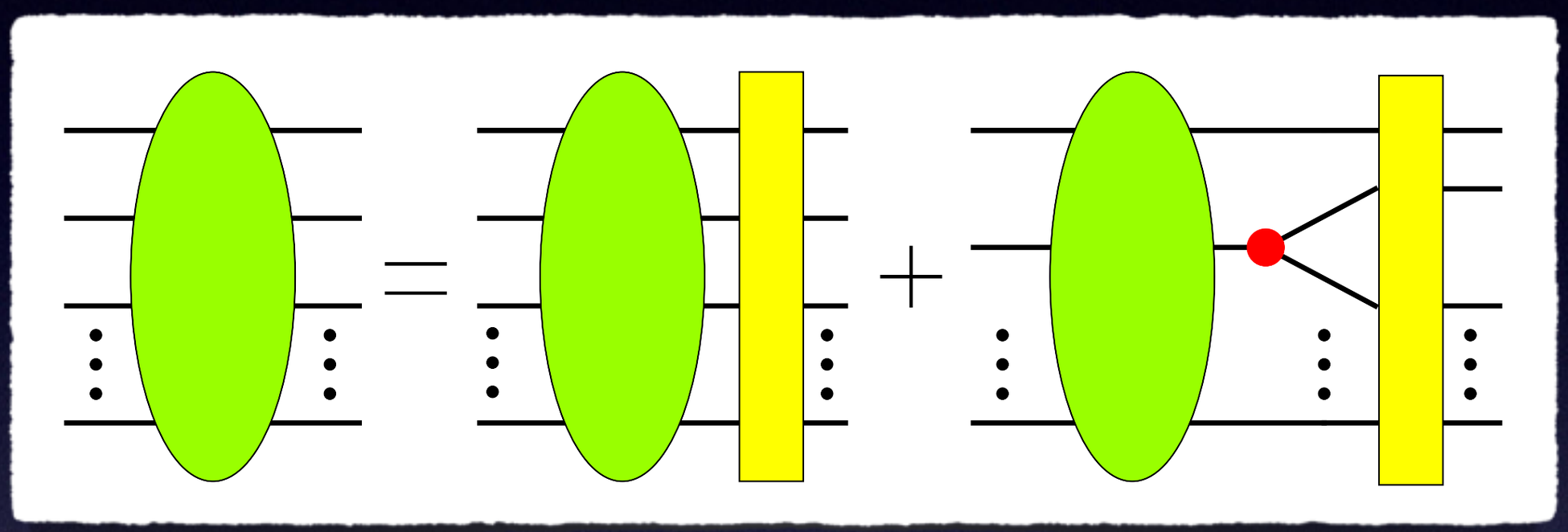
LL evolution equation: $\frac{d}{dt}\mathcal{H}_n(t) = \mathcal{H}_n(t)V_n + \mathcal{H}_{n-1}(t)R_{n-1}$

$$t = \int_{\alpha(\mu_h)}^{\alpha(\mu_s)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha}{4\pi}$$

Solution:

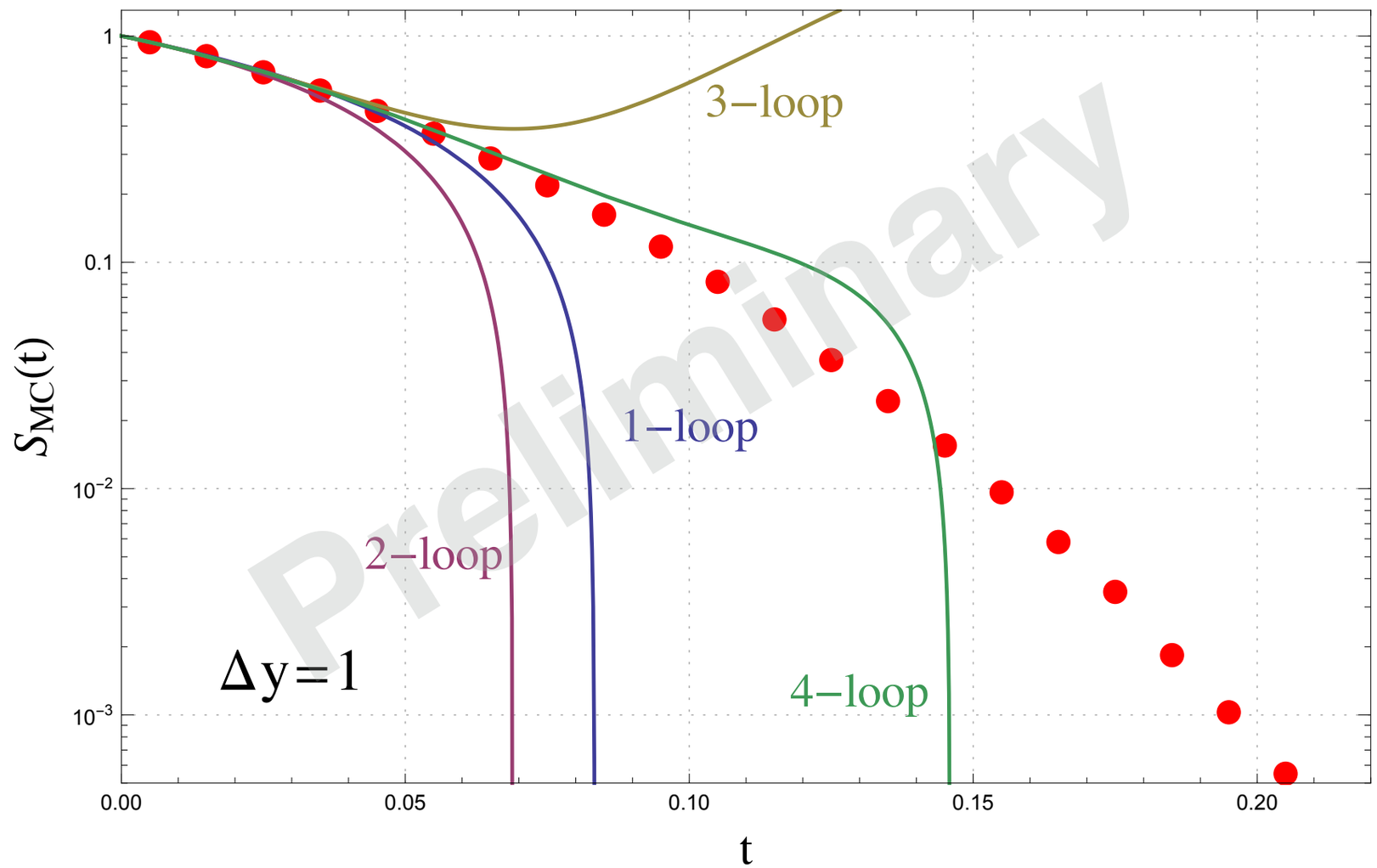
$$\mathcal{H}_n(t) = \mathcal{H}_n(t_1)e^{(t-t_1)V_n} + \int_{t_1}^{t} d\,t' \mathcal{H}_{n-1}(t')R_{n-1}e^{(t-t')V_n}$$

This form is exactly what is implemented in a standard parton shower MC!



MC numerical results

(Becher & Shao, in preparation)



Conclusion

- We have derived the first factorization formulae for NG observables: Sterman-Weinberge dijet cross section and interjet energy flow

$$\widetilde{\sigma} = \sigma_0\, H\, \widetilde{S} \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m \otimes \widetilde{\mathcal{U}}_m \right\rangle \right]^2$$

$$\sigma = \sum_m \langle \mathcal{H}_m \otimes \mathcal{S}_m \rangle$$

- In both cases we have checked the factorization up to NNLO and reproduced the full QCD results
- All scales are separated -> RG evolution can be used to resum all large logarithms, including the NGLs
- We have applied MC methods to solve the associated RG equations at LL level (next step: NLL)
- Numerous possible applications: jet cross sections, jet substructure, jet veto, ...

Thank you!

Backup slides

Comparison to BMS

Consider real and virtual together, all collinear divergences drop out. Leading soft divergence obtained by the soft approximation for the emitted (real or virtual) gluon:

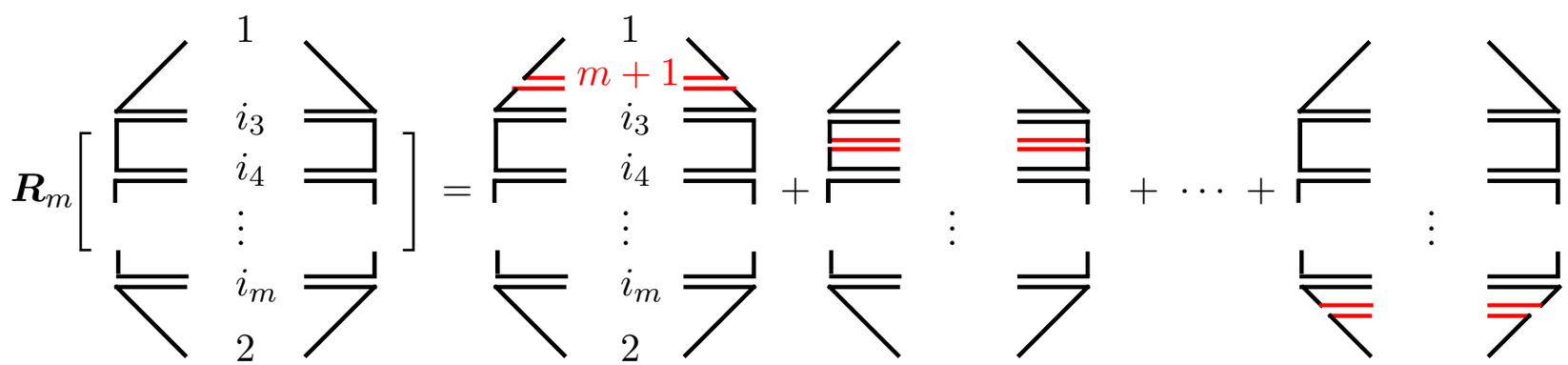
$$\boldsymbol{V}_m = \boldsymbol{\Gamma}^{(1)}_{m,m} = -4 \sum_{(ij)} \frac{1}{2} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \int \frac{d\Omega(n_k)}{4\pi} W^k_{ij} \left[\Theta^{n\bar{n}}_{\text{in}}(k) + \Theta^{n\bar{n}}_{\text{out}}(k) \right]$$

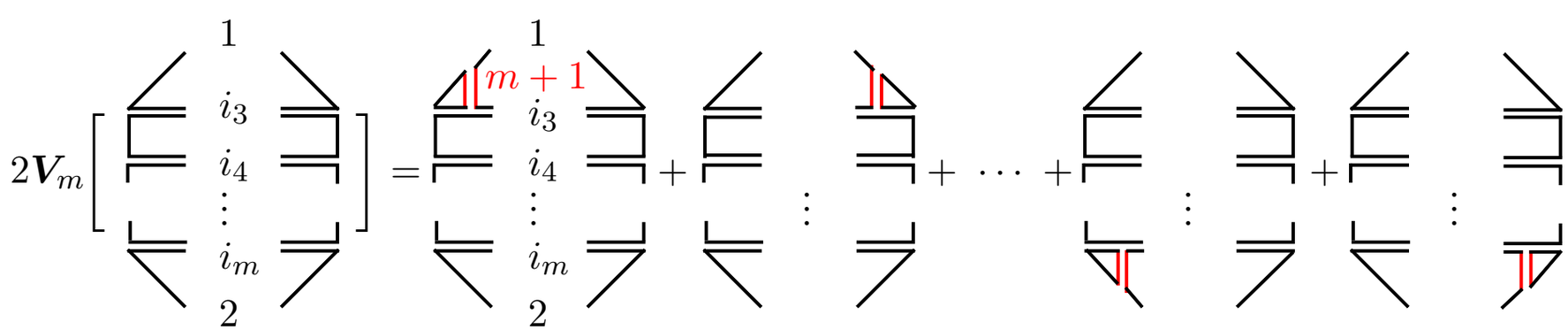
$$\boldsymbol{R}_m = \boldsymbol{\Gamma}^{(1)}_{m,m+1} = 4 \sum_{(ij)} \boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,R} \, W^k_{ij} \, \Theta^{n\bar{n}}_{\text{in}}(k)$$

Virtual has the same form as the real-emission contribution, because the principal-value part of the propagator of the emission does not contribute.

Leading-log resummation

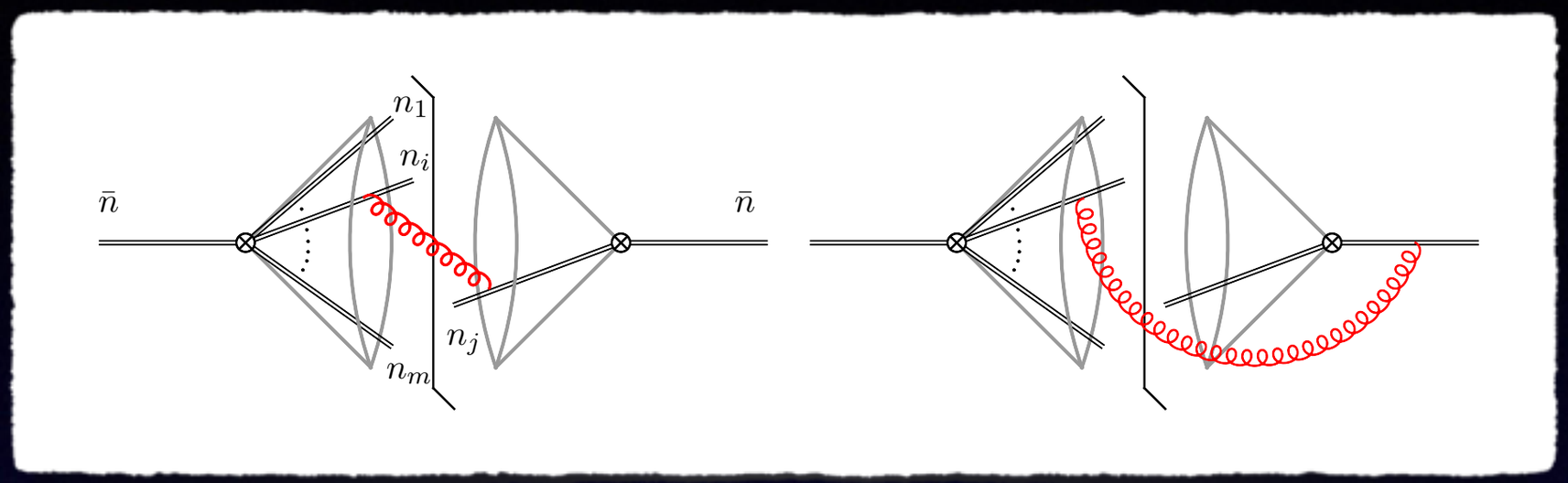
In the large-N_C limit, the color structure becomes trivial:





One-loop renormalization for the narrow-angle jet process

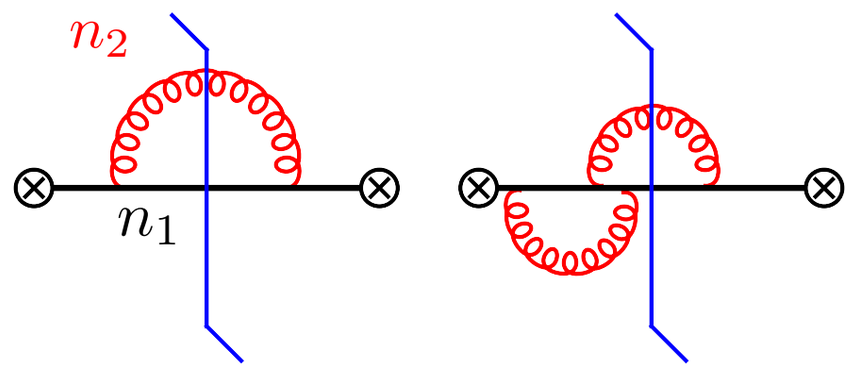
$$\frac{1}{2}\mathcal{H}^{(1)}\cdot\mathbf{1}+\frac{1}{2}\widetilde{\mathcal{S}}^{(1)}\cdot\mathbf{1}+\boldsymbol{z}^{(1)}_{m,m}+\boldsymbol{z}^{(1)}_{m,m+1}+\widetilde{\boldsymbol{\mathcal{U}}}^{(1)}_{m}=\text{fin.}$$



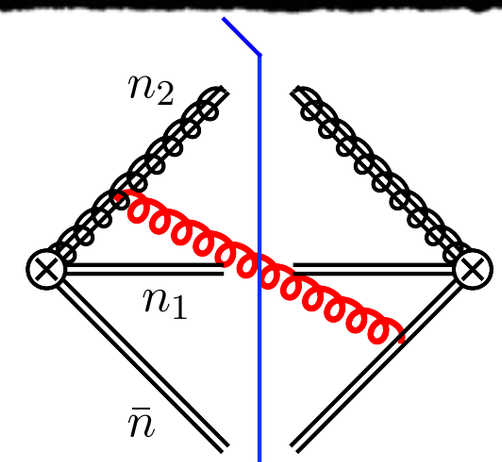
$$\widetilde{\mathcal{U}}^{(1)}_m(\{\underline{n}\},\epsilon)=-\frac{1}{\epsilon}\sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left[\ln\left(1-\hat{\theta}_i^2\right)+\ln\left(1-\hat{\theta}_j^2\right)-\ln\left(1-2\cos\phi_j\hat{\theta}_i\hat{\theta}_j+\hat{\theta}_i^2\hat{\theta}_j^2\right)\right]$$

$$-\frac{2}{\epsilon}\sum_{i=1}^{l}\boldsymbol{T}_0\cdot\boldsymbol{T}_i\ln\left(1-\hat{\theta}_i^2\right)+\boldsymbol{T}_0\cdot\boldsymbol{T}_0\left(-\frac{2}{\epsilon^2}+\frac{4\,L_{Q\tau\delta}}{\epsilon}\right)$$

NNLO check



Jet function: $\mathcal{J}_2$

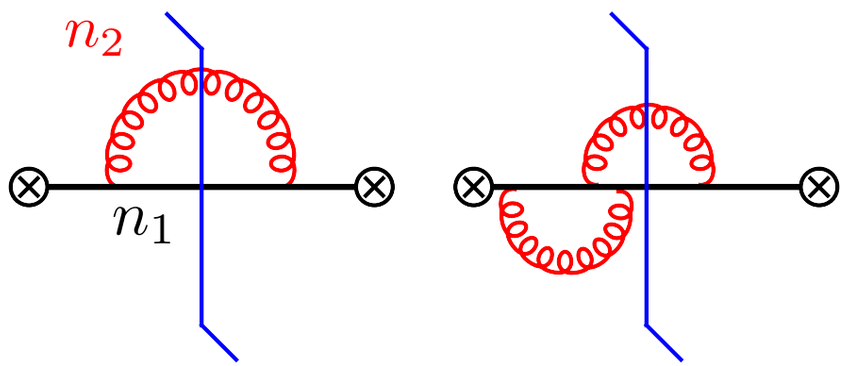


Coft function: $\mathcal{U}_2$

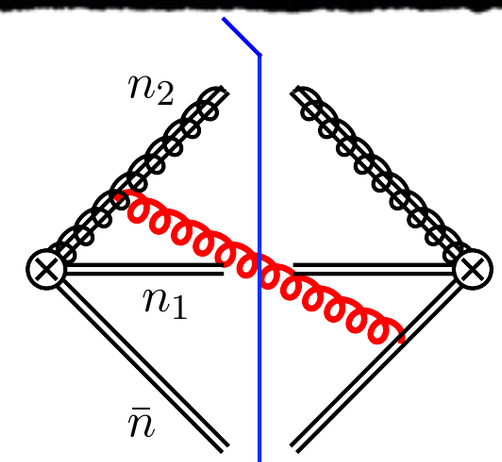
$$\mathcal{J}_2^{(1)}(\hat\theta_1,\hat\theta_2,\phi_2,Q\delta,\epsilon) = C_F\,\delta(\phi_2-\pi)\,e^{-2\epsilon L_c}$$
$$\times\Bigg\{\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+7-\frac{5\pi^2}{6}+6\ln 2\right)\delta(\hat\theta_1)\,\delta(\hat\theta_2)-\frac{4}{\epsilon}\,\delta(\hat\theta_1)\left[\frac{1}{\hat\theta_2}\right]_+ + 8\,\delta(\hat\theta_1)\left[\frac{\ln\hat\theta_2}{\hat\theta_2}\right]_+$$
$$+4\,\frac{dy}{d\hat\theta_2}\left[\frac{1}{\hat\theta_1}\right]_+\frac{1+2y+2y^2}{(1+y)^3}\,\theta(\hat\theta_1-\hat\theta_2)$$
$$+4\,\frac{dy}{d\hat\theta_1}\left[\frac{1}{\hat\theta_2}\right]_+\left(2\left[\frac{1}{y}\right]_+-\frac{4+5y+2y^2}{(1+y)^3}\right)\theta(\hat\theta_2-\hat\theta_1)+\mathcal{O}(\epsilon)\Bigg\}\,\mathbf{1}$$

$$\widetilde{\mathcal{U}}_2(\hat\theta_1,\hat\theta_2,\phi_2,Q\tau\delta,\epsilon) = \mathbf{1}+\frac{\alpha_0}{4\pi}\,e^{-2\,\epsilon L_t}\Big[C_F\,u_F(\hat\theta_1)+C_A\,u_A(\hat\theta_1,\hat\theta_2,\phi_2)\Big]\,\mathbf{1}$$

NNLO check



Jet function: $\mathcal{J}_2$



Coft function: $\mathcal{U}_2$

$$\langle \mathcal{J}_2^{(1)} \otimes \widetilde{\mathcal{U}}_2^{(1)} \rangle = e^{-2\epsilon(L_c+L_t)} \left(C_F^2 M_F + C_F C_A M_A\right)$$

$$\begin{aligned} M_F = &-\frac{4}{\epsilon^4} - \frac{6}{\epsilon^3} + \frac{1}{\epsilon^2}\left(-14 + \frac{2\pi^2}{3} - 12\ln 2\right) + \frac{1}{\epsilon}\left(-26 - \pi^2 + 10\,\zeta_3 - 32\ln 2\right) \\ &- 52 - \frac{10\pi^2}{3} - 27\zeta_3 + \frac{11\pi^4}{30} - \frac{4}{3}\ln^4 2 - 8\ln^3 2 - 4\ln^2 2 + \frac{4\,\pi^2}{3}\ln^2 2 \\ &- 52\ln 2 + 4\pi^2\ln 2 - 28\zeta_3\ln 2 - 32\,\mathrm{Li}_4\left(\frac{1}{2}\right), \end{aligned}$$

$$\begin{aligned} M_A = &\frac{2\pi^2}{3\epsilon^2} + \frac{1}{\epsilon}\left(-2 + \frac{\pi^2}{2} + 12\,\zeta_3 + 6\ln^2 2 + 4\ln 2\right) - 4 + \frac{7\pi^2}{6} - 24\zeta_3 - \frac{\pi^4}{6} + \frac{8}{3}\ln^4 2 \\ &- 4\ln^3 2 + 6\ln^2 2 - \frac{8\pi^2}{3}\ln^2 2 - 4\ln 2 + 9\pi^2\ln 2 + 56\zeta_3\ln 2 + 64\,\mathrm{Li}_4\left(\frac{1}{2}\right) \end{aligned}$$