Finite Integrals and Progress on the Four-Loop Form Factors in Massless QCD

Robert M. Schabinger

with Andreas von Manteuffel and Erik Panzer (Phys. Rev. **D93** (2016) no. 12, 125014)
and work in progress

Trinity College Dublin

Outline

What We're Doing

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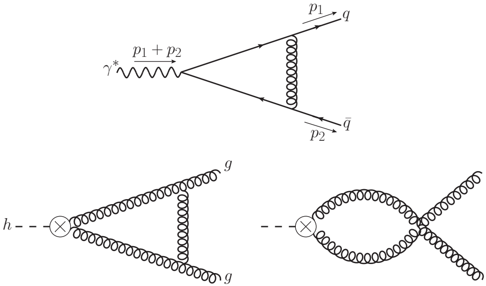

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At L loops, Γ_L characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

$\Longrightarrow$ Γ_4 is the last unknown ingredient needed for $\mathrm{N}^3\mathrm{LL}$ resummation!

A Dipole Formula For Gauge Theory IR Divergences?

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, JHEP **0906** (2009) 081; E. Gardi and L. Magnea, JHEP **0903** (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c)$ $\mathcal{N}=4$ super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1,\dots,p_n) = \exp\left\{-\frac{1}{2}\sum_{L=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^L \mu_\epsilon^{2L\epsilon}\int_0^{\mu_\epsilon^2} \mathrm{d}\mu^2\left(\mu^2\right)^{-1-L\epsilon}\right.$$
$$\left.\sum_{\substack{\mathrm{i,j}=1\\ \mathrm{i}<\mathrm{j}}}^{n}\left(\Gamma_{1;L}^{\mathcal{N}=4}\ln\left(\frac{\mu^2}{-s_{ij}}\right)+\mathcal{G}_{1;L}^{\mathcal{N}=4}\right)\frac{\mathbf{T}_\mathrm{i}\cdot\mathbf{T}_\mathrm{j}}{N_c}\right\}\sum_{L=0}^{\infty}\mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon;p_1,\dots,p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, JHEP **0706** (2007) 064). In a nutshell, the dipole conjecture is the suggestion that, with minor modifications, the above structure could hold for more general gauge theories like QCD.

When Something Sounds Too Good To Be True...

Although some three-loop evidence was collected by Dixon (Phys. Rev. **D79** (2009) 091501) for the n_f terms, it is now clear that the dipole conjecture fails for QCD due to three-loop calculations which probe the structure of the soft anomalous dimension matrix.

S. Caron-Huot, JHEP **1505** (2015) 093; Ø. Almelid *et. al.*, arXiv:1507.00047;

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In fact, Casimir scaling for the light-like cusp anomalous dimension

$$\Gamma_L^g \stackrel{?}{=} C_A/C_F \Gamma_L^q$$

is still very much an open problem at four loops.

R. Boels *et. al.*, JHEP **1302** (2013) 063; Nucl. Phys. **B902** (2016) 387;

A. Grozin *et. al.*, JHEP **1601** (2016) 140; J. Henn *et. al.*, JHEP **1605** (2016) 066

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$\Longrightarrow$ new approaches to multi-loop calculations are required!

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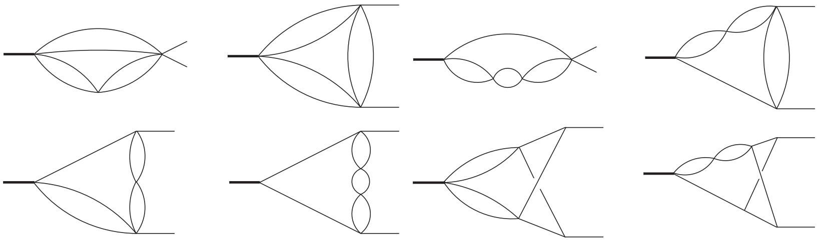
- Construct an alternative basis of finite integrals and rewrite everything in terms of it using auxiliary reductions.
- Evaluate all finite master integrals either analytically using `HyperInt` (Erik's program) or numerically using `FIESTA 4` (this talk).
  A. Smirnov, Comput. Phys. Commun. **204** (2016) 189;

  T. Hahn, Comput. Phys. Commun. **168** (2005) 78

The Master Integrals For The N_f^3 Contributions To The Four-Loop Gluon Form Factor

- From the reductions, it seemed initally that 10 master integrals would appear in the $C_F N_f^3$ and $C_A N_f^3$ color structures.
- Actually, two factorizable topologies drop out of the final results.
- All master integrals can be evaluated to all orders in ϵ.

R. J. Gonsalves, Phys. Rev. **D28** (1983) 1542; Gehrmann *et. al.*, Phys. Lett. **B640** (2006) 252



N_f^3 Part Of The Bare Four-Loop Gluon Form Factor

In the $\overline{\text{MS}}$ scheme, we find

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_F N_f^3} = -\frac{2}{3\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{32\zeta_3}{3} - \frac{145}{9}\right) + \frac{1}{\epsilon}\left(\frac{352\zeta_2^2}{45} + \frac{1040\zeta_3}{9} + \frac{68\zeta_2}{9} - \frac{10003}{54}\right) + \frac{4288\zeta_5}{27} - 64\zeta_3\zeta_2 + \frac{2288\zeta_2^2}{27} + \frac{24812\zeta_3}{27} + \frac{3074\zeta_2}{27} - \frac{508069}{324} + \mathcal{O}(\epsilon)$$

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_A N_f^3} = \frac{1}{27\epsilon^5} + \frac{5}{27\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{14\zeta_2}{27} - \frac{55}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{586\zeta_3}{81} - \frac{70\zeta_2}{27} - \frac{24167}{1458}\right) + \frac{1}{\epsilon}\left(-\frac{802\zeta_2^2}{135} - \frac{5450\zeta_3}{81} - \frac{262\zeta_2}{81} - \frac{465631}{2916}\right) - \frac{14474\zeta_5}{135} + \frac{4556}{81}\zeta_3\zeta_2 - \frac{1418}{27}\zeta_2^2 - \frac{99890\zeta_3}{243} + \frac{38489\zeta_2}{729} - \frac{20832641}{17496} + \mathcal{O}(\epsilon)$$

in both general R_ξ gauge and $\xi = 1$ background field gauge.

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- The N_f^3 gluon cusp anomalous dimension agrees with the prediction of the Casimir scaling principle!

$$\Gamma_4^g\Big|_{C_F N_f^3} = 0$$

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- Numerical checks on the expansion coefficients of all masters to part per mille precision using `FIESTA 4`.

From Conventional To Finite Integral Bases

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  E. Panzer, JHEP **1403** (2014) 071

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- For $x = \Delta d/2$ (the dimension shift divided by two), $y = \nu - N$ (the number of "extra" powers of the propagators or "dots"), and all fixed non-negative integers $n = x + y$, this test is carried out in practice by considering the integrals which correspond to all possible non-negative integer solutions $\{x, y\}$, beginning with the $n = 0$ case corresponding to the basic scalar integral in $d = 4 - 2\epsilon$.

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- Rotate from the old basis to the new basis using auxiliary IBPs.
- The computationally expensive part at this stage is to perform a Tarasov shift (Phys. Rev. **D54** (1996) 6479) on the old basis and then IBP reduce the resulting linear combination of integrals in $d + 2$ with a number of additional dots equal to the loop order. This connects the "conventional" integral bases in d and $d + 2$; it can be used iteratively if multiple dimension shifts are required.

What About The Auxiliary Reductions Needed For The Basis Rotation?

In his classic paper on dimension shifts, Tarasov also points out that one can, for any integral topology, eliminate all irreducible numerators in favor of higher-multiplicity propagators. A single irreducible numerator is eliminated at the cost of adding L additional dots and going from a single integral to a linear combination of integrals.

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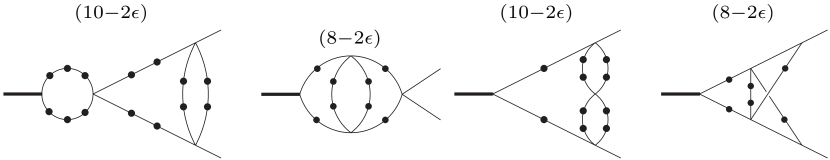

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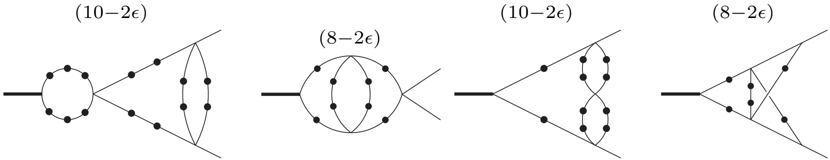
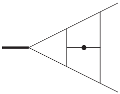
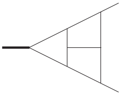
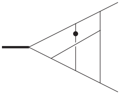
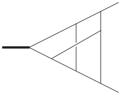
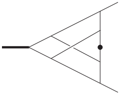
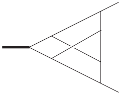


$\Longrightarrow$ Auxiliary reductions not a problem if using Feynman diagrams!

For Fixed Program Settings, Finite Integral Bases Offer Spectacular Performance Enhancements

Let's continue with our three-loop form factor example

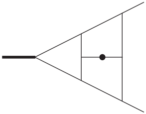
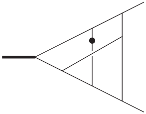
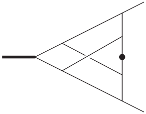
diagram	run time	relative accuracy	diagram	run time	relative accuracy
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	$(4-2\epsilon)$ 	39094 s	9.91×10^{-4}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	$(4-2\epsilon)$ 	19025 s	9.38×10^{-5}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	$(4-2\epsilon)$ 	19586 s	1.07×10^{-4}

up to and including contributions of weight six.

How About Three-Loop Form Factors @ Weight 8?

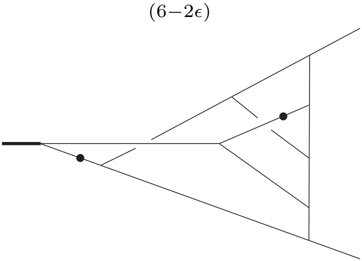
Important for our recent paper and as an answer to

K. G. Chetyrkin *et. al.*, Nucl. Phys. **B742** (2006) 208

diagram	run time	relative accuracy	run time	relative accuracy
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	491 s	2.22×10^{-5}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	761 s	5.84×10^{-6}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	485 s	8.45×10^{-6}

Finite Parts Of The Four-Loop Form Factors?

In about a day on an ancient desktop with `FIESTA 4`, we find



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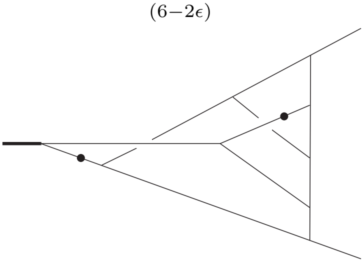

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This is the result through to weight 8 with 4 digit absolute accuracy!

Outlook

Overall, our IBP algorithm seems strong enough to significantly ameliorate one of the biggest performance bottlenecks to calculating the four-loop cusp anomalous dimensions in QCD and we have seen that `FIESTA 4` can deliver acceptably precise results for finite non-planar twelve-line four-loop integrals if any relevant masters are inaccessible to us using `HyperInt`-like methods. In addition, several other ideas for further research come to mind:

- Using Tarasov's ideas for the IBPs when the payoff is huge.
- Implementing a multivariate version of `FinRed`.
- Further N^2LO calculations using finite integrals numerically, in the spirit of what was done recently for double Higgs production.

  S. Borowka *et. al.*, Phys. Rev. Lett. **117** (2016) no. 1, 012001
- The mixed EW-QCD virtual corrections to Drell-Yan are already extremely challenging to calculate analytically even though the integrals are known to be expressible in terms of MPLs.

  Di Vita's talk this year, my talk last year $\Longrightarrow$ Can check using `SecDec 3`!