Decay of a bound muon

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Outline

• Muonic atoms
• Muon electron coherent conversion
• Spectrum of the bound muons
  • Central region
  • Endpoint region
  • Radiative correction to the spectrum
Muonic atoms & Muon electron conversion
General characteristic

* One of the electrons is replaced by a muon

* Muon orbit is much smaller than the electron orbit
  \[ \frac{r_\mu}{r_e} \sim \frac{m_e}{m_\mu} \]
  * Much larger momentum

* Muons are more sensitive to the structure of the nucleus
  \[ \frac{1}{m_\mu} < r_N \]

* Muon can be captured by the nucleus or it can decay
Muon DIO

DIO — Decay In Orbit

- Muon DIO: standard muon decay into an electron and two neutrinos, with the muon and a nucleus forming a bound state
- For DIO momentum can be exchanged between the nucleus and both the muon and the electron

\[ \bar{\nu}_e, \nu_\mu \]
Muon electron coherent conversion

Neutrinos not produced

CLFV
Why conversion?

- Plethora of models gives large CLFV
- Muon g-2 discrepancy
  - $3.5\sigma$ Lattice calculation (Chakraborty, Davies, de Oliveira, Koponen, Lepage, 2016)
  - $3.3\sigma$ $e^+e^- + \tau$ (Jegerlehner, Szafron, 2011)
  - $3.3\sigma$ $e^+e^-$ (Hagiwara, Liao, Martin, Nomura, Teubner, 2011)
- Proton radius puzzle
- Today’s LHC statistical fluctuations
  - Signal is clean and the background is small
  - SM background can be well understood
Characteristic scales of muonic atom

- Nucleus mass: $M_{Al}$
- Muon mass: $m_\mu$
- Muon momentum: $Z\alpha m_\mu$
- Muon binding energy: $(Z\alpha)^2 m_\mu$
- Electron cloud: $\sim m_e$

$M_{Al} \gg m_\mu \gg m_\mu Z\alpha \gg m_\mu (Z\alpha)^2$
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Endpoint energy

\[ E_{\text{max}} = m_\mu + E_b + E_{\text{rec}} \]

**Binding energy**
\[ E_b \approx -m_\mu \frac{(Z \alpha)^2}{2} \]
(+ higher orders)

**Recoil energy**
\[ E_{\text{rec}} \approx -\frac{m_\mu^2}{2m_N} \]
(kinetic energy of the nucleus)

Both corrections decrease the endpoint energy
Conversion signal
(theoretical perspective)

...but energetic charged particles are accompanied by radiation...

Conversion signal

\[ E_e \text{ [MeV]} \]
Conversion spectrum

Emission of photons decreases the electron energy

Types of photons:
A. hard \( E_\gamma \sim m_\mu \) \[\rightarrow\] collinear \( p_\gamma p_e \sim m_e^2 \)
B. soft \( E_\gamma \sim m_\mu Z \alpha \)
C. ultrasoft \( E_\gamma \sim m_\mu (Z \alpha)^2 \)
Conversion spectrum

Emission of photons decreases the electron energy

Types of photons:
A. hard \( E_\gamma \sim m_\mu \)
B. soft \( E_\gamma \sim m_\mu Z\alpha \)
C. ultrasoft \( E_\gamma \sim m_\mu (Z\alpha)^2 \)

\[ E_\gamma \sim m_\mu, \quad p_e p_\gamma \sim m_e^2 \]

Requires NRQED
(see orthopositronium case)
Signal is reduced by the radiative corrections.
Signal is reduced by the radiative corrections.
# Corrections to the conversion signal

<table>
<thead>
<tr>
<th>Signal window [MeV]</th>
<th>0.1</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Universal part</td>
<td>0.861</td>
<td>0.923</td>
<td>0.930</td>
</tr>
<tr>
<td>Model I</td>
<td>0.861</td>
<td>0.923</td>
<td>0.930</td>
</tr>
<tr>
<td>Model II</td>
<td>0.861</td>
<td>0.924</td>
<td>0.930</td>
</tr>
<tr>
<td>Model III</td>
<td>0.858</td>
<td>0.921</td>
<td>0.927</td>
</tr>
</tbody>
</table>

Number of electrons that can reach the detector per one conversion

Details of the model are not important!
Bound muon spectrum
Why study bound muon spectrum?

- Background for a conversion process
- If not CLFV is found then at least we will have precise measurement of the DIO spectrum
- Underlying physics!
- Many similarities with the heavy quark decay where the perturbation theory breaks down at a scale $\sim \Lambda_{QCD}$
- For muons, pure theoretical calculation is possible without input from experiments
Bound muon spectrum

- Heavy quarks physics
- QED corrections to free muon
- pQCD methods
- Region expansion
- Endpoint expansion
- Schwinger background field method
- Quark fragmentation function
- SCET & QCD factorization theorems
- Leading logarithms
- Ultrasoft corrections
- NRQED pNRQED
- Numerical Dirac Equation solution

Bound muon spectrum

Techniques and methods:

- Schwinger background field method
- NRQED pNRQED
DIO spectrum regions

- Measured by the TWIST experiment in 2009
- Muon motion dominates

\[ E_e \sim \frac{m_\mu}{2} \]

- Background for the conversion experiments
- Will be measured in conversion experiments
Central region

- Free muon decay is the Leading Order effect
- Binding effects are only a correction
- Typical momentum transfer between nucleus and muon is of the order of $m_\mu Z\alpha$
- Binding effects need to be re-summed; wavefunction cannot be expanded

$$\psi(q) \sim \frac{1}{[q^2 + m_\mu^2 (Z\alpha)^2]^2}$$
Factorization

(shape function)

Following QCD approach a factorization theorem can be derived

\[
\frac{d\Gamma_{\text{DIO}}}{dE_e} = \frac{d\Gamma_{\text{free}}}{dE_e} \otimes S
\]

Free muon spectrum
It is associated with the hard scale \( m_\mu \)

QED Shape function
It is associated with the soft scale \( m_\mu Z\alpha \)

Separation of scales
\( m_\mu Z\alpha \ll m_\mu \)

QCD case:
Neubert 1993; Mannel, Neubert 1994; Bigi, Shifman, Uraltsev, Vainshtein, 1994
Shape function

For a point-like nucleus, the LO shape function can be calculated analytically

\[ S(\lambda) = \frac{8m_\mu^5 Z^5 \alpha^5}{3\pi \left[ \lambda^2 + m_\mu^2 Z^2 \alpha^2 \right]^3}. \]

Szafron, Czarnecki, 2015
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[Diagram of a bell curve with the peak at \( \lambda = 0 \) and \( \frac{1}{m_\mu Z \alpha} \) on the y-axis.]
Shape function

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\[ S'(\lambda) = \frac{8m_\mu^5 Z^5 \alpha^5}{3\pi \left[ \lambda^2 + m_\mu^2 Z^2 \alpha^2 \right]^3}. \]

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\[ S(\lambda) = \frac{8m_\mu^5 Z^5 \alpha^5}{3\pi \left[ \lambda^2 + m_\mu^2 Z^2 \alpha^2 \right]^3}. \]

Scaling $\lambda \sim m_\mu Z \alpha$
First moment is zero
\[ \int d\lambda \lambda S(\lambda) = 0 \]

\[ \Gamma_{\text{DIO}} = \Gamma_0 + \mathcal{O}(Z^2 \alpha^2) \]
Results for real atom and their relation to the TWIST data

Czarnecki, Dowling, Garcia i Tormo, Marciano, Szafron; 2014

Free muon

LO

NLO

Shape function

$E_e$ [MeV]

Events $\times 10^{-4}$/MeV
Leading Corrections

and their relation to the TWIST data

---

Data – theory

theory

data – theory

All orders in $Z \alpha$

but no RadCor

Factorization including RadCor

$E_e$ [MeV]
Endpoint Region
(conversion background)

\[ E_e \sim m_\mu \]

- Free muon spectrum is nonexistent in this region.
- Binding effects constitute the LO terms.
- Typical momentum transfer between the nucleus and the muon is large\((q^2 \sim m_\mu^2)\).
- Both wave functions and propagators can be expanded in powers of \(Z\alpha\).
Endpoint expansion

Near the endpoint, the dominant contribution comes from the exchange of hard virtual photons. Szafron, Czarnecki; 2015

\[
\frac{m_\mu}{\Gamma_{Free}} \frac{d\Gamma}{dE_e} \approx \frac{1024}{5\pi} (Z\alpha)^5 \left( \frac{\Delta}{m_\mu} \right)^5
\]

\[
\Delta = E_{max} - E_e
\]

\[q^2 = -m_{\mu}^2\]
Phase space suppression

\[ \int \frac{d^3 \nu}{\nu_0} \frac{d^3 \bar{\nu}_0}{\bar{\nu}_0} \delta (\Delta - \nu_0 - \bar{\nu}_0) \ldots \psi \ldots \bar{\psi} \sim \Delta^5 \]

Each neutrino gives 3 powers of $\Delta$

Can be used to constrain effective BSM operators!
Binding suppression

\[ |\mathcal{M}|^2 \sim |\psi(0)|^2 \times |V(m_\mu^2)|^2 \sim (Z \alpha)^3 \times (Z \alpha)^2 \]

\[ |\psi(0)|^2 \sim (Z \alpha)^3 \]

\[ V(k^2) \sim -\frac{Z \alpha}{k^2} \]

\[ \sim |\psi(0)|^2 \times |V(m_\mu^2)|^2 \]
Endpoint Radiative Correction

\[ \frac{1}{\Gamma_{Free}} \frac{d\Gamma}{dE_e} = \Delta^5 \frac{1024}{5\pi m^6_\mu} (Z\alpha)^5 \left( \frac{\Delta}{m_\mu} \right)^{\alpha/\pi} \delta_S (1 + \frac{\alpha}{\pi} \delta_{VP} + \frac{\alpha}{\pi} \delta_H) \]

\[ \Delta = E_{max} - E_e \]

Background suppression ~15%
Endpoint Radiative Correction

- Soft vacuum polarization correction to the muon wave-function at the origin (running to \( m_\mu Z\alpha \))

\[
\frac{1}{\Gamma_{\text{Free}}} \frac{d\Gamma}{dE_e} = \Delta^5 \frac{1024}{5\pi m_\mu^6} (Z\alpha)^5 \left( \frac{\Delta}{m_\mu} \right)^{\frac{\alpha}{\pi}} \delta_S \left( 1 + \frac{\alpha}{\pi} \delta_{VP} + \frac{\alpha}{\pi} \delta_H \right)
\]

\[
\Delta = E_{max} - E_e
\]

Background suppression \(~15\%\)
Endpoint Radiative Correction

- Soft vacuum polarization correction to the muon wave-function at the origin (running to $m_\mu Z \alpha$)

- Hard vacuum polarization

\[ \Delta = E_{max} - E_e \]

\[ \frac{1}{\Gamma_{Free}} \frac{d\Gamma}{dE_e} = \Delta^5 \frac{1024}{5\pi m_\mu^6} (Z\alpha)^5 \left( \frac{\Delta}{m_\mu} \right)^\frac{\alpha}{\pi} \delta_S \left( 1 + \frac{\alpha}{\pi} \delta_{VP} + \frac{\alpha}{\pi} \delta_H \right) \]
Endpoint Radiative Correction

- Soft vacuum polarization correction to the muon wave-function at the origin (running to $m_\mu Z\alpha$)
- Hard vacuum polarization
- Soft photon emission

\[
\Delta = E_{\text{max}} - E_e
\]

\[
\frac{1}{\Gamma_{\text{Free}}} \frac{d\Gamma}{dE_e} = \Delta^5 \frac{1024}{5\pi m_\mu^6} (Z\alpha)^5 \left( \frac{\Delta}{m_\mu} \right)^{\frac{\alpha}{\pi}} \delta_S \left( 1 + \frac{\alpha}{\pi} \delta_{VP} + \frac{\alpha}{\pi} \delta_H \right)
\]
Endpoint Radiative Correction

- Soft vacuum polarization correction to the muon wave-function at the origin (running to $m_\mu Z\alpha$)

- Hard vacuum polarization

- Soft photon emission

- Hard correction

$$\Delta = E_{max} - E_e$$

$$\frac{1}{\Gamma_{Free}} \frac{d\Gamma}{dE_e} = \Delta^5 \frac{1024}{5\pi m_\mu^6} (Z\alpha)^5 \left( \frac{\Delta}{m_\mu} \right)^\frac{\alpha}{\pi} \delta_S (1 + \frac{\alpha}{\pi} \delta_{VP} + \frac{\alpha}{\pi} \delta_H)$$
Interpolating between regions

- We also need to know the spectrum for intermediate electron energies
- We have identified the leading corrections and it is possible to calculate them!

1. Real radiation can be approximated by taking into account collinear photon emission
2. Vacuum polarization can be included when we solve the Dirac equation numerically
Vacuum polarization

\[ V(r) = -\frac{Z\alpha}{r} + Z\alpha \frac{\alpha}{\pi} V_U(r, m_e) \]

Electron loop generates long distance potential and this leads to large logarithmic corrections

\[ r \sim \frac{1}{m_e} \gg \frac{1}{m_\mu Z\alpha} \]

Correction range

Atom size

Correction range

Atom size

\[ \ln \frac{Z\alpha m_\mu}{m_e} \]

\[ r \sim 1 \]

\[ m_\mu \]

\[ m_\mu \]

\[ q^2 \sim m_\mu^2 \]

\[ e^- , \mu^- \]
Soft-Collinear Factorization

\[ d\Gamma_{LL} \left( \frac{dE_e}{dE_e} \right) = d\Gamma_{LO} \left( \frac{dE_e}{dE_e} \right) \otimes D_e \]

with the perturbative fragmentation function

\[ D_e(x) = \delta(1 - x) + \frac{\alpha}{2\pi} \ln \left( \frac{m^2_\mu}{m^2_e} \right) P_{ee}^{(0)}(x) + \ldots \]
Correction to the DIO spectrum

![Graph showing correction to the DIO spectrum with different lines for LL only, LL+VP, and VP only. The x-axis represents $E_e$ [MeV], and the y-axis represents correction.]
Endpoint region

Vacuum Polarization correction is very important!
Vacuum polarization correction

\[ E_b \to E_b + \frac{\alpha}{\pi} \delta E_b \]

Correction to the endpoint energy

\[ \psi(p) \to \psi(p) + \frac{\alpha}{\pi} \delta \psi(p) \]

Corrections to the wave-functions

\[
\begin{align*}
\text{Vacuum polarization correction} & \quad \text{Corrections to the wave-functions} \\
\quad \text{Correction to wave-function only} & \quad \text{Energy shift only} \\
\quad \text{Total VP} & \\
\end{align*}
\]
Exponentiation

Near the endpoint emission of soft photons is logarithmically enhanced

\[
\frac{\alpha}{\pi} \delta_S \ln \left( \frac{E_{\text{max}} - E_e}{E_{\text{max}}} \right) \rightarrow \left( \frac{E_{\text{max}} - E_e}{E_{\text{max}}} \right)^{\frac{\alpha}{\pi} \delta_S}
\]

Relative correction vs. \( E_e [\text{MeV}] \)
Exponentiation

<table>
<thead>
<tr>
<th>Bin Size</th>
<th>1 MeV</th>
<th>0.5 MeV</th>
<th>0.1 MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed order/Exponentiated</td>
<td>0.2%</td>
<td>0.4%</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

Graph showing the ratio of fixed order to exponentiated for different bin sizes with a blue line representing the trend.

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Summary

• Conversion spectrum is not sensitive to the BSM model

• We can correctly reproduce TWIST measurement

• Vacuum polarization gives large, nonfactorizable correction to the DIO spectrum

• Endpoint spectrum is very sensitive to the binding energy (Lamb shift)

• Large finite nucleus size effects
If you want to discover New Physics, first you have to understand the Standard Model.

DIO spectrum — a quantity that is changing by more than 16 orders is calculated including the leading corrections.
Backup
Free / Bound

Ratio
Free/Bound

\((Z\alpha)^0\)  \((Z\alpha)^5\)