TRANSVERSE MODE COUPLING INSTABILITY WITH SPACE CHARGE

V. Balbekov, APC seminar 04/14/2016

TMCI with space charge is considered in frameworks of the boxcar model.

Eigenfunctions of the bunch *without wake* are used as the basis

to investigate in depth the problem of instability *with the wake*.

Full text: V. Balbekov, FERMILAB-PUB-16-079-APC, arXiv:1603.03744

TMCI without space charge

TMCI has been observed first in PETRA (1980)

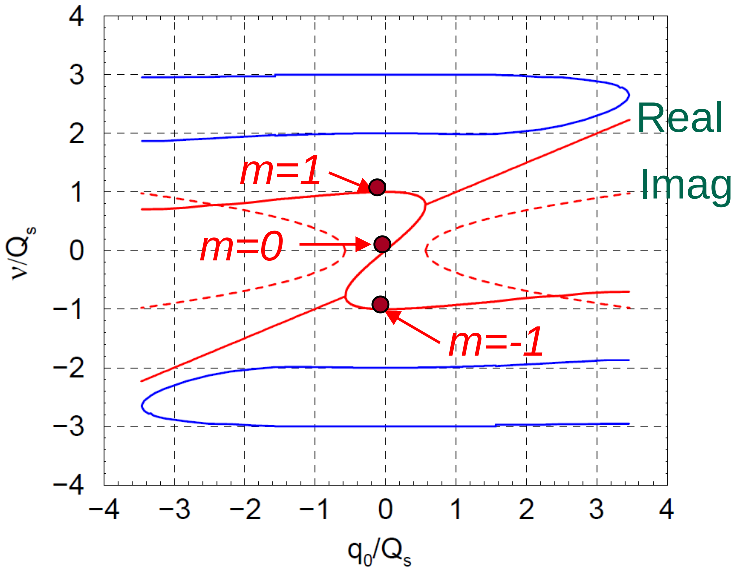
First explanation has been given in frameworks of two-particle model where the oscillating particles propel each other by a wake field (R.Kohaupt, DESY M-80/19, 1980).

The theory was evolved further in many works on the base of Vlasov equation.

At present, there is a complete understanding of the phenomenon *without space charge*:

The TMCI occurs when frequencies of two neighboring head-tail modes of the bunch (oscillators!) approach each other coalescing eventually due to the bunch wake field.

Example: tunes vs constant wake field



Lowest TMCI mode appears as a coalescence of multipoles *m=0* and *m=±1,* dependent on sign of the wake *q* (solid/dashed lines present real/imag parts).

The wake strength *q* is normalized to get the tune shift $\nu_0 = q$ at $q << Q_s$ for the lowest *(m=0)* head-tail mode.

In such a case, the TMCI threshold is $q_{thres} \approx 0.57\, Q_s$ almost independently on the bunch size am shape.

TMCI of higher modes is possible as well.

However, the understanding is not so well when space charge is essential.

TMCI with space charge by M.Blaskiewicz, PRSTAB 1, 044201 (1998).

The solution has been obtained in this paper in form of restricted series of multipoles ~exp($im\varphi$) where φ is synchrotron phase.

The TMCI growth rate is plotted in the graph against the space charge tune shift at constant negative wake.

The growth rate goes down in the beginning, and the stabilization occurs at

$\Delta Q/Q_s = 0.5\text{-}0.9$

This result is treated as proof of the state-ment that *space charge suppresses TMCI*

However, the instability is resumed at

$\Delta Q/Q_s > 2.2$

if the high order multipoles are used.

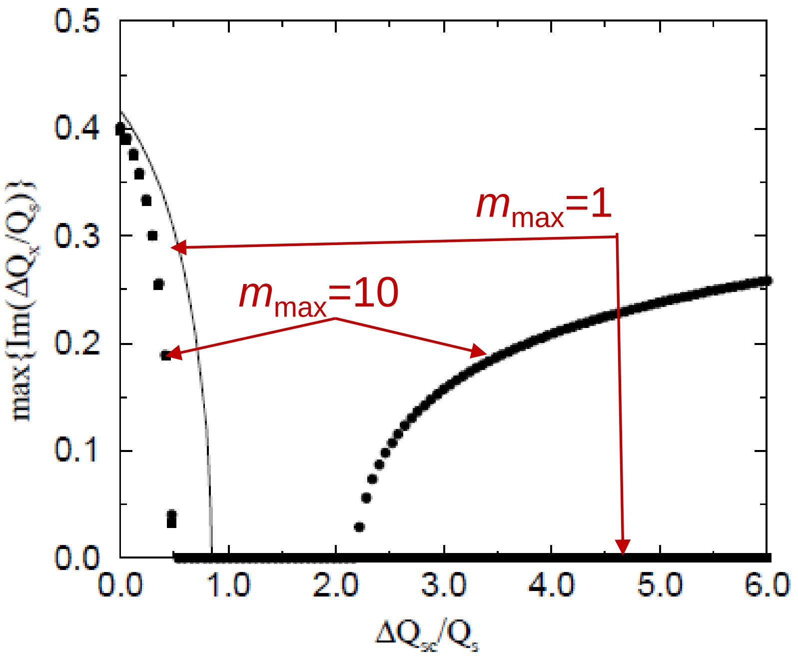


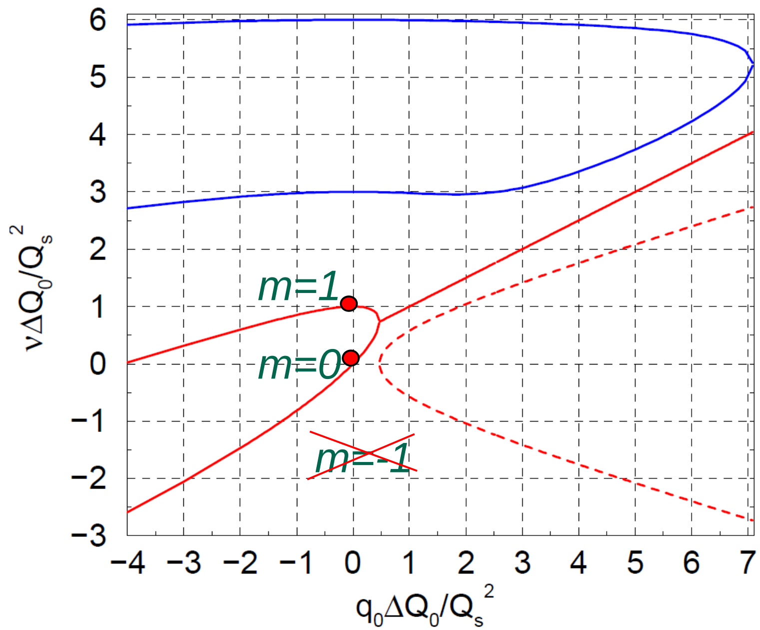
FIG. 1. Largest value of $\mathrm{Im}(\Delta Q_x/Q_s)$ as a function of $\Delta Q_{sc}/Q_s$ and $m_{\max}$ using matrix element (12). The value of W is twice the size needed to produce instability with $\Delta Q_{sc} = 0$: $m_{\max} = 1$, solid line; $m_{\max} = 5$, squares; $m_{\max} = 10$, circles.

The bad convergence does not provide a way to answer the question:

Is the high space charge a stabilizing or destabilizing factor?

“Vanishing TMCI”

1. A. Burov, PRSTAB 12, 044202 (2009),
2. V. Balbekov, PRSTAB 14, 094401 (2011).



The limiting case $\Delta Q >> Q_s$ has been considered in these papers.

It was shown that space charge enforces the instability *with positive wake* so that the TMCI threshold goes down

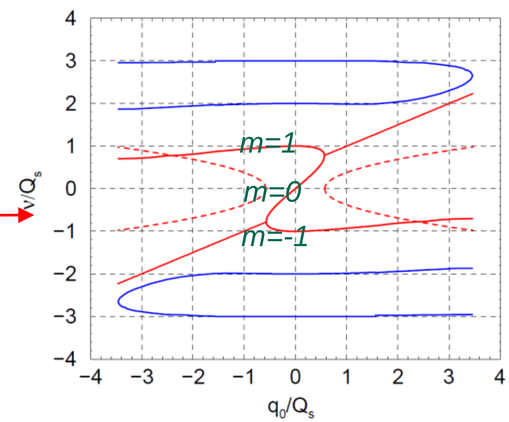
$$q_{thresh} \approx 0.5 Q_s^2 / \Delta Q \to 0$$

but it suppresses the instability excited by *negative wake*:

$$q_{thresh} \to -\infty \quad \text{at} \quad \Delta Q \to \infty$$

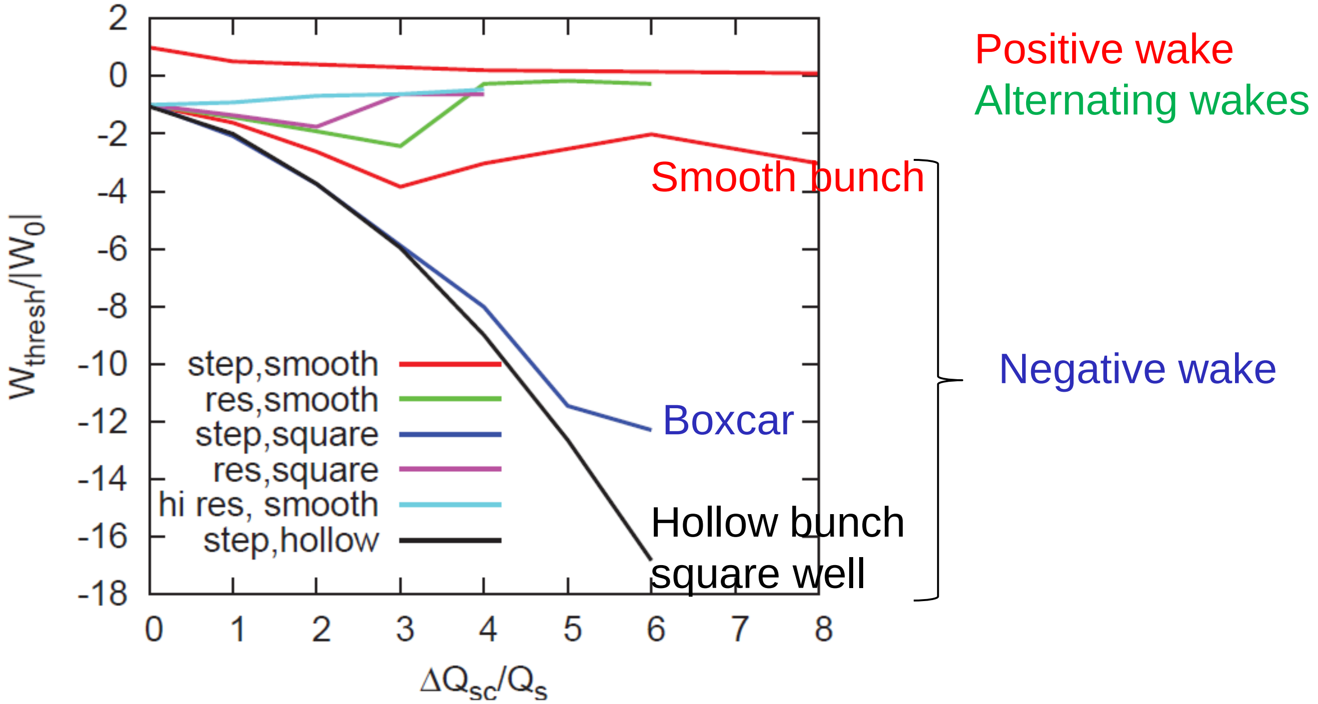
However, the last statement is questionable because *all negative multipoles were excluded from consideration* in the quoted papers due to used approximations.

Meanwhile, just the coalescence of the multipoles *m = 0* and *m = -1* is responsible for the TMCE with negative wake without space charge.



Numerical simulation by M. Blaskiewicz, IPAC 2012, p. 3165 (2012).

Code TRANFT has been used for the simulation



The results are in agreement with the suggestions that space charge enforces effect of positive wake and reduces it if the wake is negative.

The last statement is confirmed at $\Delta Q/Q_s<3$ and maybe at $\Delta Q/Q_s<6$.

However, there are no data at higher tune shifts.

Boxcar model

Square bunch (boxcar) model provides a unique possibility to study the problem in depth because its eigenfunctions without wake are known exactly at any tune shift (F. Sacherer, CERN-SI-BR-72-5, 1975).

They can be used as the basis set for solution of the problem with wake providing easy solvable and fast converging series of equations.

The boxcar eigenfunctions

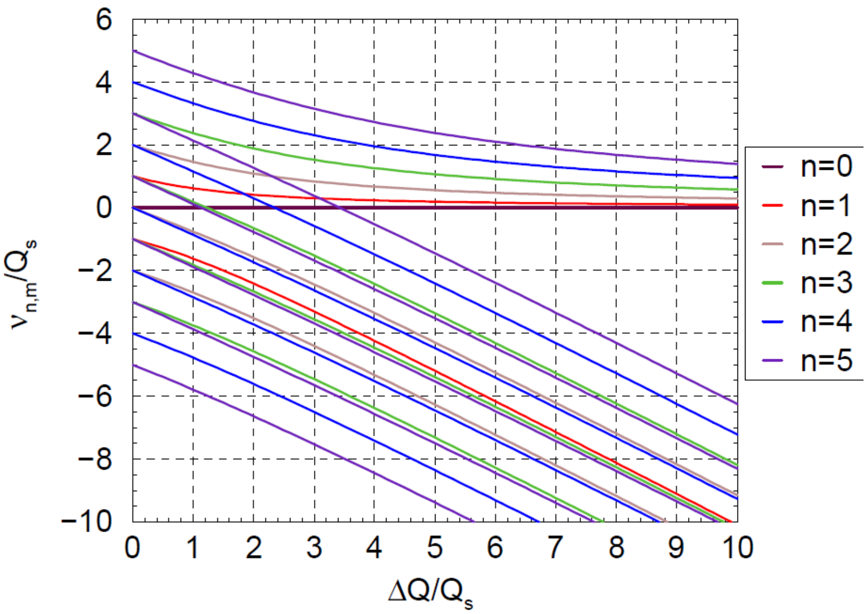
Coherent transverse displacement of the bunch, as a function of time and longitudinal coordinate in the rest frame, is (w/o chromaticity)

$$X \propto \mathrm{Re}\left\{P_n(\vartheta)e^{i(Q_0+\nu_{nm})\Omega_0 t}\right\}$$

where $P_n(\theta)$ are Legendre polynomials.

At any $n = 0, 1, \ldots$, there are $n+1$ eigenfunctions X_{nm} in the longitudinal phase space $(\theta, p-p_0)$~(A,φ). They have to be determined from the equation:

$$(\nu_{nm} + \Delta Q)X_{nm} + iQ_s \frac{\partial X_{nm}}{\partial \phi} = \Delta Q S_{nm} P_n(\theta)$$



The functions X_{nm} are polynomials of power n in the phase space, coefficients S_{nm} are needed to assure normalization of the functions.

The eigentunes ν_{nm} are plotted in the graph. At $\Delta Q = 0$, they start from the points $\nu_{nm} = mQ_s$ that is m is the multipole number whereas the index n is associated with number of radial mode, in traditional presentation.

The highest tunes ν_{nn} are positive at any ΔQ, and $\nu_{nn} \to 0$ at $\Delta Q \to \infty$

The expression $\nu_{nm} \approx mQ_s - \Delta Q$ is acceptable for other tunes being weakly dependent on n

Equations of the boxcar bunch with wake

Representing solution with wake in the form

$$X = \sum_{nm} C_{nm} X_{nm}$$

using properties of the Legendre polynomials

$$\int_{-1}^{1} P_N(x)dx \int_{x}^{1} P_n(y)dy = \frac{2\delta_{N-1,n}}{(2n-1)(2n+1)} + \frac{2\delta_{N+1,n}}{(2n+1)(2n+3)}$$

one can get dispersion equation for the bunch eigentunes as the infinite continued fraction

$$\nu - q = -\cfrac{(q/3)^2 W_1}{1 + \cfrac{(q/15)^2 W_1 W_2}{1 + \cfrac{(q/35)^2 W_2 W_3}{1 + \ldots\ldots\ldots\ldots}}}$$

$$W_n(\nu) = \sum_m \frac{|S_{nm}|^2}{\nu - \nu_{nm}}$$

Truncating the fraction by the assumption $W_n = 0$ at $n > n_{max}$ results in:

dispersion equation

$$T_{n_{\max}}(\nu) \approx 0$$

with the recursive relation

$$T_n = T_{n-1} + T_{n-2} \frac{q^2 W_{n-1} W_n}{(4n^2 - 1)^2}$$

and initial conditions

$$T_0 = \nu - q$$

$$T_1 = T_0 + \left(\frac{q}{3}\right)^2 \frac{3(\nu + \Delta Q)}{\nu(\nu + \Delta Q) - Q_s^2}$$

The problem can be explored step by step up to desirable n_{max}.

With any n, it is the algebraic equation of power $(n+1)(n+2)/2$.

Three-mode approximation

The lowest approximation $T_0(\nu) = 0$ is trivial in essence reflecting the dependence of minimal bunch eigentune on the wake strength with accepted normalized $\nu_{0,0} = q$

The first approximation which allows to see TMCI is $T_1(\nu) = 0$ that is in the expanded form

$$(\nu - q)\left(\nu - \frac{Q_s^2}{\nu + \Delta Q}\right) = -\frac{q^2}{3}$$

This equation is the partial case of my three-mode model (V.Balbekov,JINST10 P10032) although the last has been built using other basis and considerations, including additional factors like chromaticity, different wake form, etc.

However, the mentioned analysis has been restricted by the case of modest space charge:

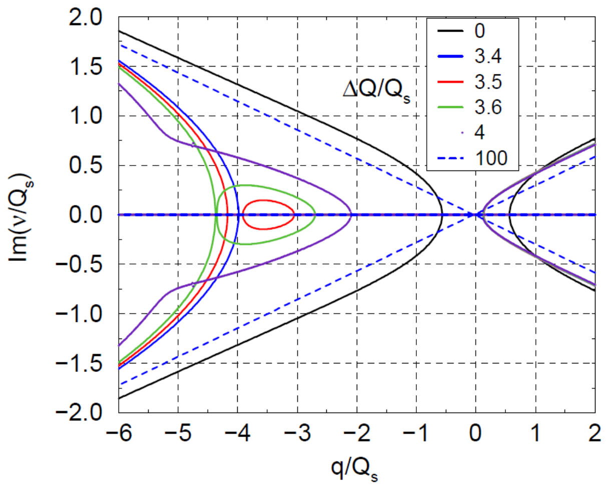
$$\Delta Q/Q_s < 3.5$$

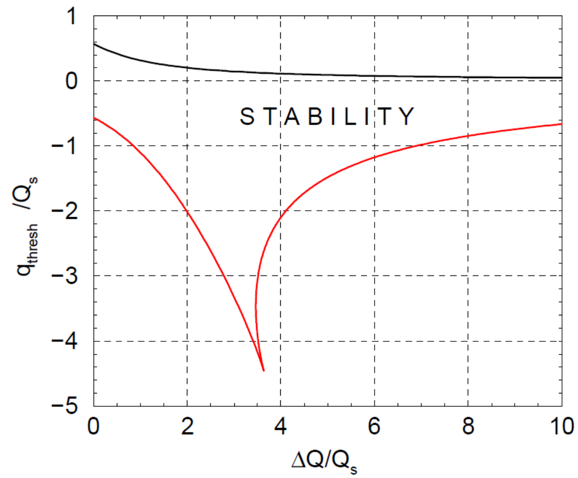
Now the restriction has to be removed because just the case of high space charge is a subject of special interest.

(In advance: it will be shown that this approximation provides qualitatively correct result)

Three-mode approximation: region of stability

$$(\nu - q)\left(\nu - \frac{Q_s^2}{\nu + \Delta Q}\right) = -\frac{q^2}{3}$$





Stability region in (q-ΔQ) plane

Imaginary part of the solutions is shown against the wake strength at different tune shifts.

The instability threshold $q_{th}/Q_s \approx \pm 0.57$ at $\Delta Q = 0$.

Then threshold of positive wake goes down when the tune shift increases.

Threshold of negative wake goes up in absolute value if $\Delta Q/Q_s$ increases from 0 to 3.465.

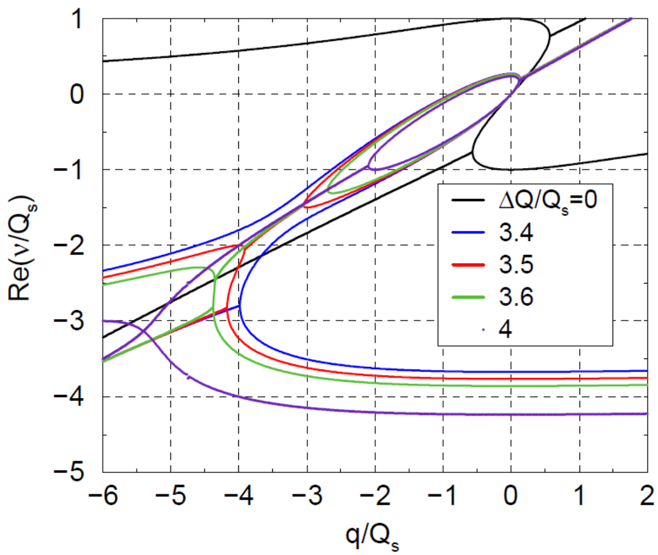
An additional region of instability appears after that which quickly extends (from red to green)

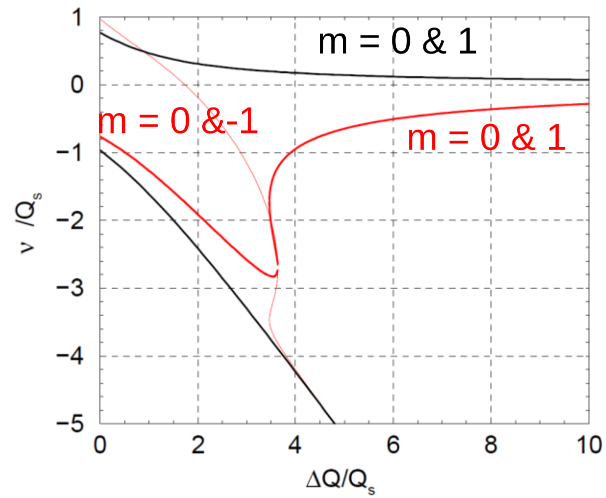
It coalesces with the main instability region at $\Delta Q/Q_s = 3.6$.

The instability threshold goes down in absolute value after that.

Three-mode approximation: eigentunes

$$(\nu - q)\left(\nu - \frac{Q_s^2}{\nu + \Delta Q}\right) = -\frac{q^2}{3}$$





Tunes on the boundary of the stab.reg. Black $q>0$, red $q<0$. The merged tunes are shown by bold lines.

Left: Real part of the solutions against the wake strength at different tune shifts.

Right: Tunes on the boundary of the stability region.

Coalescence of the modes $m=0$ and $m=1$ is responsible for the TMCI with positive wake.

With negative wake: the modes $m=0$ and $m=-1$ are coalesced at $\Delta Q/Q_s < 3.46$, and $m=0$ and $m=1$ -- at $\Delta Q/Q_s > 3.6$. All the tunes about coincide in the transition region

So the alternation of coalescing multipoles causes the dog-leg of the threshold curve.

Higher approximation: equations

Higher approximations are needed to validate the result and to make sure that the saturation is achieved.

Provided recursive formulae allow to investigate the process step by step.

Generally, it leads to an algebraic equation of power *$(n_{max}+1)(n_{max}+2)/2$* where n_{max} is max power of the Legendre polynomial in using.

Its real roots can be found numerically. The system is unstable if amount of the real roots is less then the equation power.

Example: $n_{max} = 2$

$$(\nu - q)\left(\nu - \frac{Q_s^2}{\nu + \Delta Q}\right) = -\frac{q_s^2}{3} \quad \text{1}^{\text{st}} \text{ apx}$$

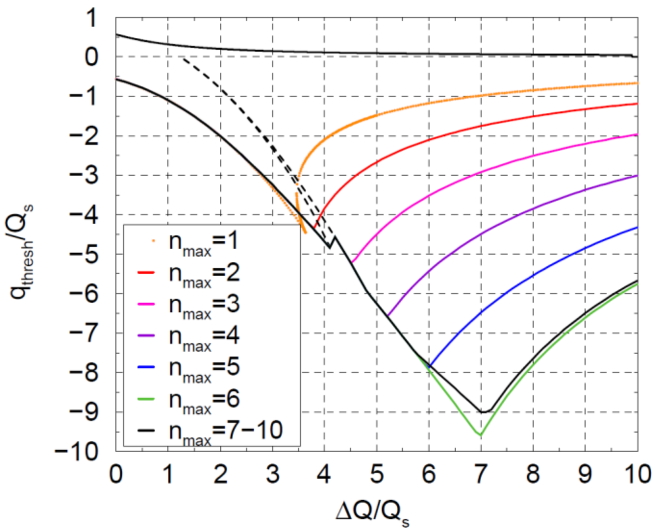
$$-\frac{q^2(\nu - q)}{75}\left(\frac{S_{2,-2}^2}{\nu - \nu_{2,-2}} + \frac{S_{2,0}^2}{\nu - \nu_{2,0}} + \frac{S_{2,2}^2}{\nu - \nu_{2,2}}\right) \quad \text{addition}$$

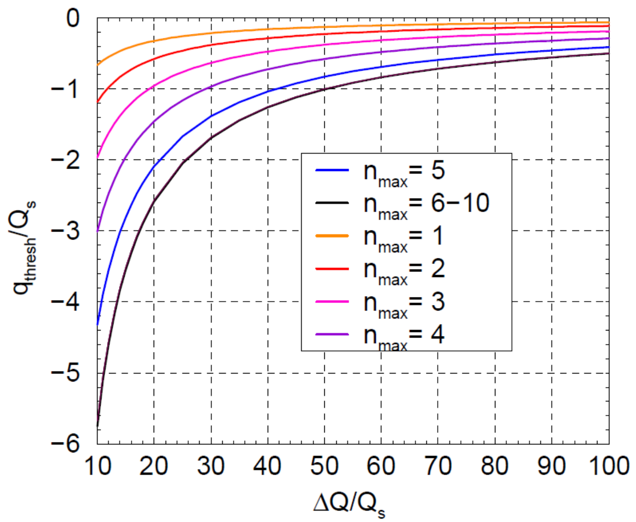
$$\nu_{n.m} = \hat{\nu}_{n,m} Q_s + \Delta Q$$

$$\hat{\nu}_{2,m}(\hat{\nu}_{2,m}^2 - 4) = \frac{\Delta Q}{Q_s}(\hat{\nu}_{2,m}^2 - 1)$$

$$S_{2,m}^2 = \frac{5(\hat{\nu}_{2,m}^2 - 1)^2}{\hat{\nu}_{2,m}^4 + \hat{\nu}_{2,m}^2 + 4}$$

Higher approximations: stability threshold





TMCI threshold q_{thresh}/Q_s is presented against $\Delta Q/Q_s$ in different approximations.

Black curve in the top of left graph is the threshold of positive wake at any approximation, other curves refer to the negative wake.

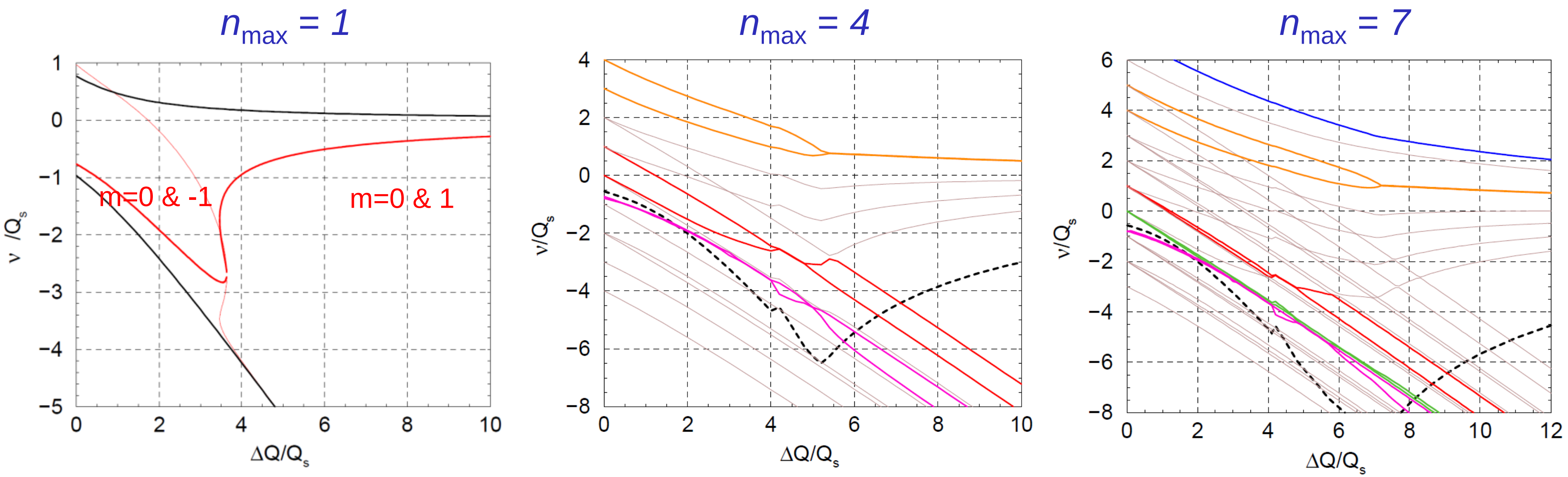
The dog-leg of the negative threshold curve appears in any approximation.

Due to the good convergence, the approximation $n_{max} \geq 6$ can be used with any ΔQ.

However, lower approximations are also acceptable if $\Delta Q/Q_s$ is rather small
In particular, three-mode approximation is really applicable at $\Delta Q/Q_s < 3.5$.

The strip bounded by dashed lines is very narrow region of instability inside the wide stable region. It firmly appears at $n_{max} \geq 3$.

Higher approximations: Threshold tunes



The bunch tunes are shown against the tune shift on the boundary of the stability region.

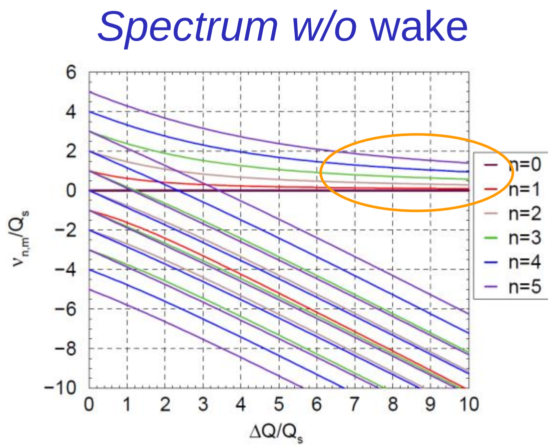
The left graph is being demonstrated to remind that the a sudden veer of the threshold happens when the coalescence is switched from the modes *m = 0 &-1* to *m = 0 &+1.*

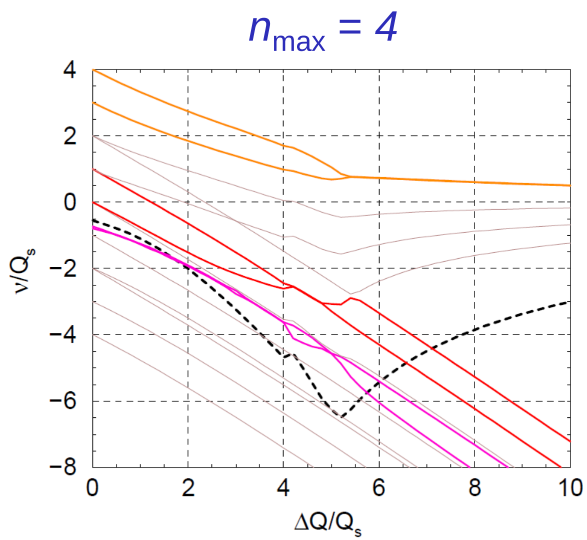
About the same happens at higher approximations:

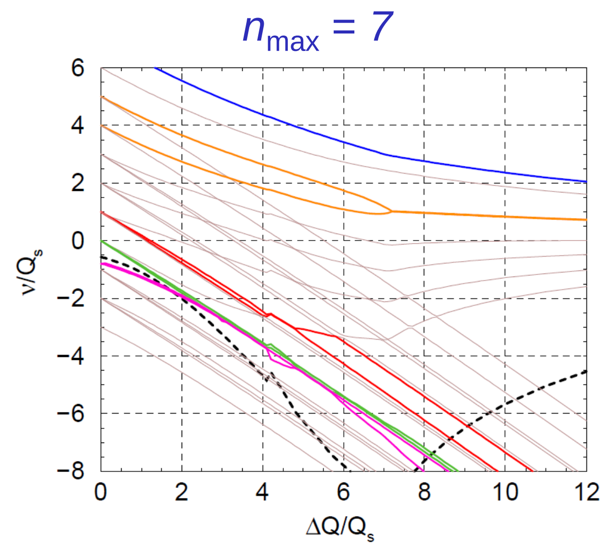
At modest $\Delta Q/Q_s$, the TMCI appears by coalescence of the modes *m = 0 &-1* but further it goes up to the combinations *m = 3 & 4* at $n_{max} = 4$, and *m = 4 & 5* at $n_{max} = 7$.

There are several transitional steps which can cause small strips or spots of instability in the wide stable region. One of them has been demonstrated before, but generally these mini-regions are elusive because of small size and the instability rate.

Why does the switching happen?







Coalescence of the multipoles *m = 0 &-1* causes the instability at $\Delta Q/Q_s < 4$ because tune of the mode *m=0* ($\nu_0 \approx q$) is pushed down by negative wake striving to the mode *m=-1*:, $\nu_{-1} \approx -Q_s$. This stage is presented by magenta lines in the figures.

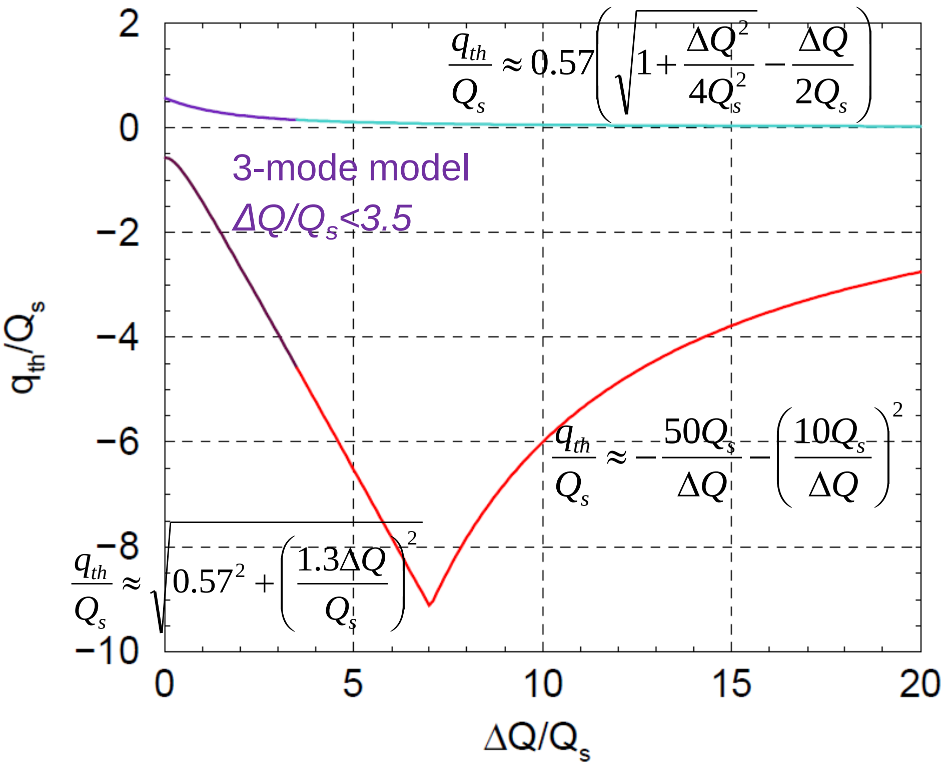
However, at growing ΔQ , the coalescence strive up to higher tunes where the bunch spectrum is are more tight (orange oval). The final positions are show by orange lines.

At $n_{max} = 4$, the final position is located in the edge of the spectrum being restricted by the used basis functions. It means that the equilibrium (saturation) is not reached yet.

But it is located inside the spectrum in the equilibrium (saturated) position as at $n_{max} = 7$. The saturation is achieved at $n_{max} = 6$, and extra basis vectors no longer affect the coupling.

Only lowest radial modes can be coupled eventually but the higher modes can take a part at the transition.

Summary: complete outline of the stability region

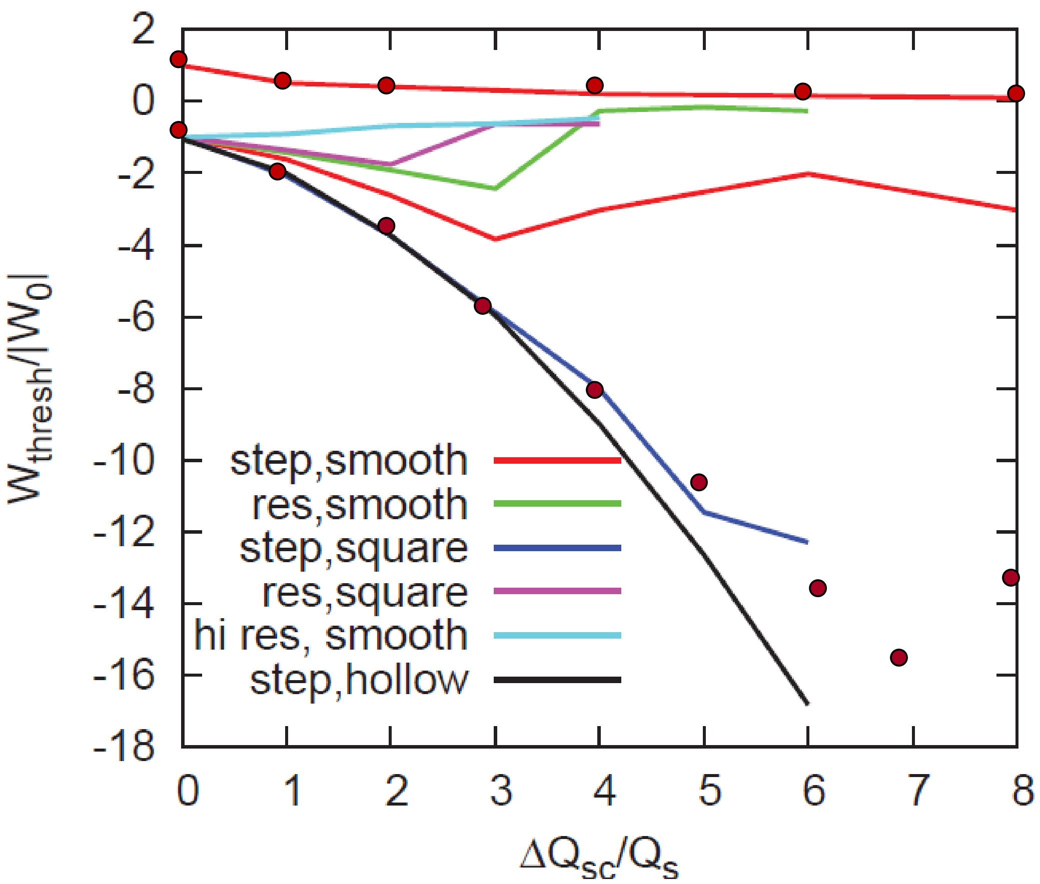


Different boundaries of the region are described by different equations.

Presented simplified formulae provide the accuracy 15% or better.

The simplest three-mode model resolves itself to the algebraic equation of 3rd order which is acceptable at $\Delta Q/Q_s<3.5$ and can include chromaticity, and different bunch and wake forms.

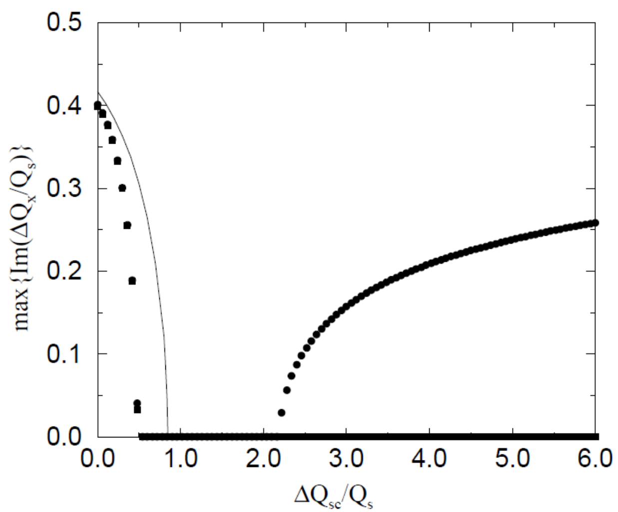
Comparison with numerical simulation

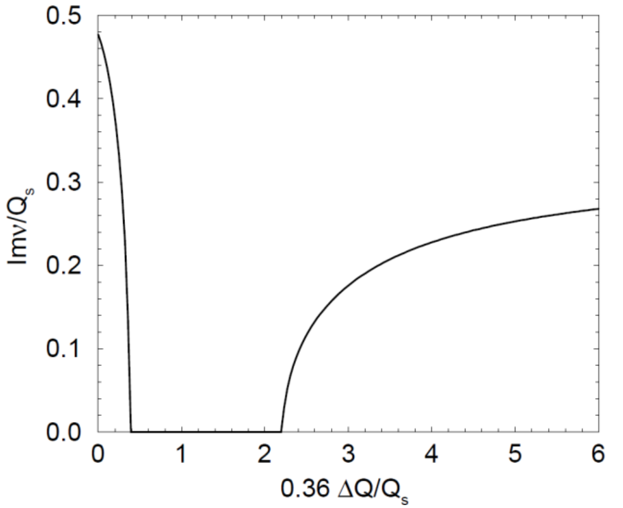


My results are inserted into the plot of M. Blaskiewicz (IPAC2012). There is a perfect agreement with the simulation results for the boxcar bunch with constant wakes of any signs.

(The middle curves relate to other bunch/wake models)

Comparison with analytical calculation





The TMCI growth rate is presented in the graphs against the space charge tune shift at negative constant wake (q=-1.13).

Top graph: M. Blaskiewicz, PRSTAB 1, 044201 (1998). Expansion technique has been applied in the paper but the basis differs from the boxcar eigenfunctions.

Bottom plot is obtained by my 3-mode approximation.

It could tell about an amazing similarity of the pictures if not the difference of the horizontal scales. Space between 2 regions of instability is $\Delta Q/Q_s \approx 1.5$ on the top plot, and 3 times more by the bottom one.

Another contradiction is that the second region of instability appears in the top plot only at $m_{max}= 10$. In my solution, it exists at any n_{max} but moves to right and reaches the equilibrium $\Delta Q/Q_s \sim 45$ at $n_{max} \geq 6$.

Thus one can tell about a qualitative similarity of the results but numerical consent is present firmly only at $\Delta Q/Q_s < 2$.

High space charge limit (vanishing TMCI)

It has been declared and supported that negative wake can't cause TMCI if $\Delta Q \to \infty$ (A. Burov, 2009: V. Balbekov, 2011).

Opposite conclusion follows from this work: TMCI threshold $\to 0$ at $\Delta Q \to \infty$

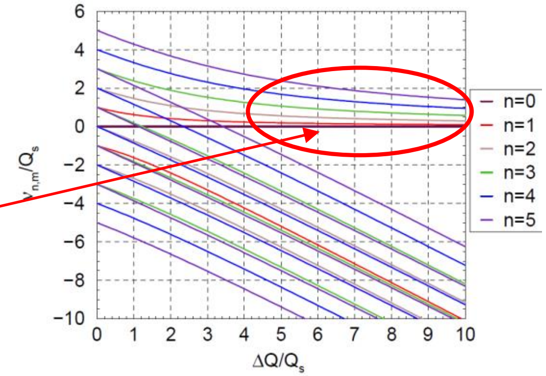
The idea that ignoring of negative multipoles could be a reason of the "vanishing" has little force now (at least for the boxcar bunch) because it has been established that just positive multipoles have main part in the instability at $\Delta Q > 7$.

The explanation can be reached if to note that, for the boxcar model, central equation of the mentioned papers is a partial case of the present paper, with additional assumptions concerning the basis functions (eigenmodes without wake):

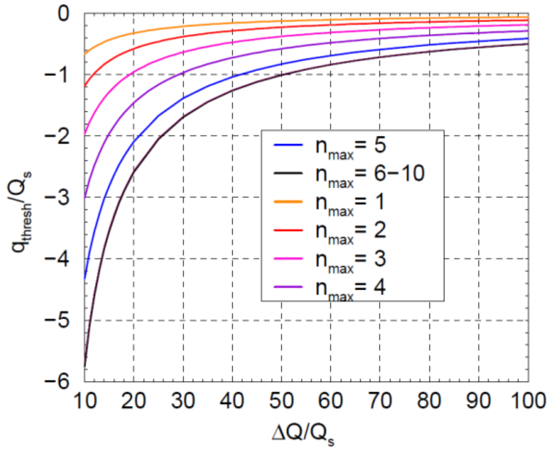
$$S_{nm}^2 = (2n+1)\delta_{nm}, \qquad \nu_{nn} = \frac{Q_s^2 n(n+1)}{2\Delta Q} \qquad \text{that is} \qquad W_n = \sum_m \frac{|S_{nm}|^2}{\nu - \nu_{nm}} \to \frac{|S_{nn}|^2}{\nu - \nu_{nn}}$$

This truncated basis includes only lowest radial modes of positive multipoles, and asymptotical values of remaining tunes are used

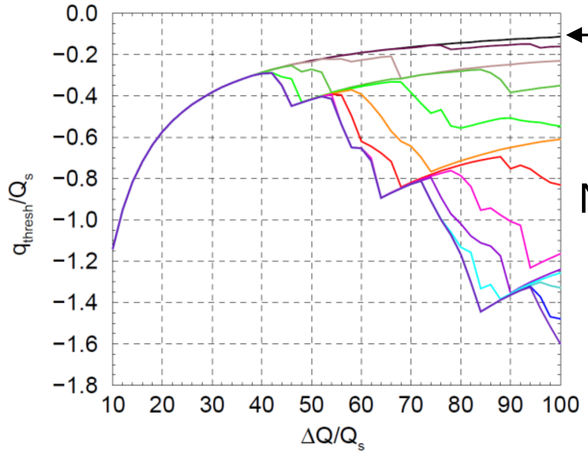
Only this part of the spectrum has been taken for the consideration



High space charge limit – threshold calculation



Full set of harmonics



Top: $n_{max} = 2\text{-}4$

Next: $n_{max} = 5\text{-}16$, $\Delta n = 1$

Truncated set

The truncation drastically affects the result.

The convergence is poor in the right-hand plot but the trend is rather clear – absolute value of the threshold increases at higher space charge.

I guess that the statement “vanishing TMCI” should not be applied, at least to boxcar bunch though its applicability to other distributions and especially to other wake forms is the open question.

Conclusion

Transverse Mode Coupling Instability of square bunch with space charge and constant wake is considered. Borders of the stability region are drawn around.

It is shown that the space charge lowers the instability threshold of positive wake

Threshold of negative wakes goes up at $\Delta Q/Q_s < 7$, and goes down to 0 at larger space charge.