Intense Beam Research for HEP and related items
J. Cary (U. Colorado, Tech-X)

- Motivation
- Optics
- Symplecticity may not be needed, but smoothness is
- Non-self-consistent equilibria
- Oscillations
- Self-consistent, linear-lattice equilibria
- Nonlinear lattices
- Tracking with space charge
- Equilibria with space charge(is there a Grad-Shafranov equation?)
- Mismatch, halo generation, nonlinear lattices
- Theoretical needs for IOTA

This talk will ignore non-Vlasov effects. I.e., no intrabeam scattering, no scattering off background particles

Why would we want integrable, nonlinear beams?

- Intensity frontier: develop beams with large space charge
- Linear integrability of Courant-Snyder does give a solution, but it is in practice not applicable (for circular systems)
  - Strong interactions with perturbations cause beam blow up when betatron tune is near (low) rational
  - Chromaticity causes the tune to vary, so that off-energy particles can have near-low rational tunes
  - Sextupoles control chromaticity (keeping betatron tune constant with energy), but also introduce nonlinear perturbations which lead to chaotic motion
- Much of these controllable at low intensity, but all bets are off at high intensity, where the beam self fields change the tune and cause perturbations
- Integrable nonlinear systems are known to have KAM stability to perturbations, so hope is that they can guard against instabilities caused by beam self fields

Linear lattices

- Dynamics
- Numerical testing
- Equilibria
- Oscillations
- Self-consistent equilibria
- Self-consistent oscillations

Courant-Snyder, strong focusing

- E. D. Courant and H. S. Snyder, Annals of Physics 3, 1 (1958)
- Follow charged particles in linear fields that vary along the beam $\frac{dx}{ds} = x' \quad \frac{dx'}{ds} = F(s)$
- But real lattices have errors, nonlinear terms, etc. How to test? Tracking – so important that each lab had to have at least one tracking code
  - Sixtrack (1994)
  - MAD (1998)
  - TEAPOT (1985)
  - COSY (1987)
  - …

Long time integrations need special techniques

- Hamiltonian motion preserves the Poincare invariants
  - Sum of surface areas in planes for each degree of freedom – (mostly useful for only uncouple DoFs)
  - …
  - Volume (Liousville's theorem)
- Symplectic integration preserves the same invariants, so…
  - KAM theorem: nearby maps have same dynamics and so same invariant tori – but not useful in higher dimensions where KAM tori are not confining
  - Volume preservation (Boris push) may be enough

For general integration, may be reduced to volume preserving

$$\dot{\mathbf{u}} = (e/m)\left[\mathbf{E} + (\mathbf{u}/\gamma)\times\mathbf{B}\right] \qquad \dot{\mathbf{x}} = \mathbf{u}/\gamma$$

- Symplectic integrator for general electromagnetic fields not known*
- "Boris push" splits integration into 4 steps, each of which is volume preserving
  - Half acceleration (translation in p)
  - Rotation around B
  - Half acceleration (translation in p)
  - Move (translation in x)

Claims [JCP 270 (2014): 570-576] of symplecticity have not held up [JCP 282 (2015): 43-46], [arXiv:1509.02863 (2015)].

Numerical experiments indicate volume preserving is sufficient

Efficiency of a Boris-like integration scheme with spatial stepping

P.H. Stoltz* and J.R. Cary

Tech-X Corporation, 5541 Central Avenue, Suite 135, Boulder, Colorado 80301

G. Penn and J. Wurtele

Department of Physics, University of California, Berkeley, California 94720

(Received 4 June 2002; published 6 September 2002)

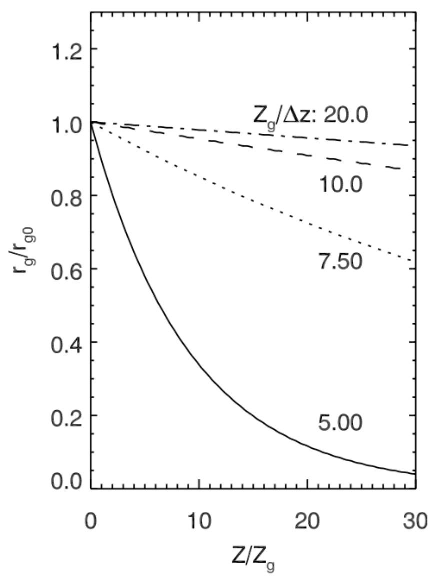


FIG. 1. The gyroradius (normalized to its initial value) as a function of distance (normalized to the gyroperiod) calculated using the fourth-order Runge-Kutta integration scheme. Plots are shown for step sizes of 5, 7.5, 10, and 20 steps per gyroperiod. The curves for perpendicular momentum as a function of distance are similar to these.

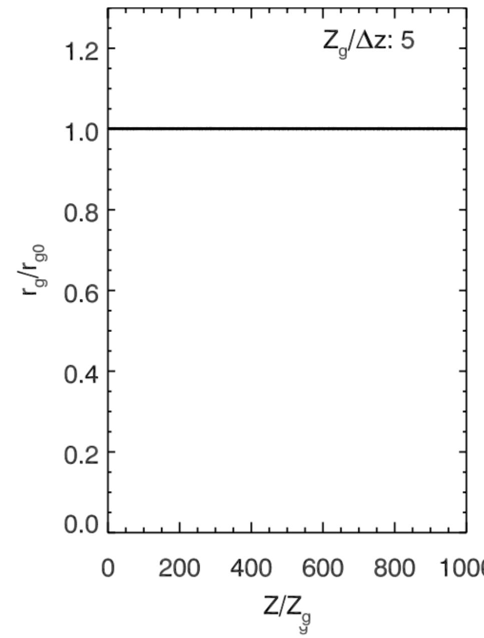


FIG. 3. The gyroradius and perpendicular momentum (normalized to their initial values) as a function of distance (normalized to the gyroperiod) calculated using the spatial-Boris scheme. These are plotted for a step size of five steps per gyroperiod. These quantities are perfectly conserved for any step size using the spatial-Boris scheme.

- Symplectic would not hurt, but what if you don't have it?

However, one must interpret the forces sufficiently smoothly

- Integration reduces a differential equation solve to the repeated application of a map
- Meiss, RMP 64 795-848 (1992) discusses the theorem (of Herman) that one needs 3+ε derivatives for KAM theorem to apply
- Hand waving explanation
  - Rationals are dense.
  - Resonant islands have width ~ $1/\sqrt{(\text{amplitude} = a)}$
  - There is a resonance at every rational surface with $a_{m,n} \sim (1/m)^p$, where p is the differentiability
  - For $\Sigma\, a_{m,n}$ < length, p rationals per m, need $(1-p/2) < -1$ or $p > 4$ (overestimate)

Linear lattice equilibria

- Standard theory: make phase independent distribution
- Twiss (α, β,) parameters give spread of beam in x, p, per emittance
- Why do we care?
  - Caveat: I am not a lattice designer
  - Beam width as a function of location – how small a pipe
  - Matching incoming beam means matching the Twiss parameters
  - Linear systems particularly easy because there is no need to worry about the action distribution

Oscillations in linear lattices are boring

- Action-angle variables $\mathbf{J}, \theta(periodic)$

$$\dot{\mathbf{J}} = 0, \dot{\theta} = \omega(\mathbf{J}) = \omega_0$$

$$\theta = \theta_0 + \omega_0 t$$

- Phase-space distribution $f(J, \theta_0) = f(J, \theta - \omega_0 t)$

- Nonlinear function (measurement) of f gives harmonics of betatron frequencies
- Persist forever for linear lattices

Linear, self-consistent equilibria

- Kapchinsky-Vladimirsky (Int. Conf. on High Energy Accelerators, 1959)
  - Linear lattices
  - Unphysical distribution function: delta function of invariant actions $f(J,\theta_0) = f_0 \delta(J_1 + J_2 - \bar{J})$

$$\frac{d^2X}{ds^2} = -k_x X + \frac{2Q}{X+Y} + \frac{\varepsilon_x^2}{X^3} \qquad \frac{d^2Y}{ds^2} = -k_y Y + \frac{2Q}{X+Y} + \frac{\varepsilon_y^2}{Y^3}$$

space charge defocusing   emittance defocusing

- No other general equilibria known
- Interesting work by Danilov, Self-consistent Space Charge Distributions: Theory and Applications, CASA Seminar 2004

Intense beams: space charge tune shift (tune depression) is large

Nonlinear, integrable beams

- Dynamics
- Numerical testing
- Equilibria
- Oscillations
- Self-consistent dynamics

Nonlinear beams presume a set of invariants (in involution)

- Hamiltonian
- Frequencies are action dependent
- Now have a range of frequencies – nonlinear tune shift
- Observable frequencies occupy a continuum
- Long known
- How to get there?

$$H(J_1, J_2)$$

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i}(J_1, J_2)$$

Nonlinear lattices: multiple approaches

- Island canceling (Chow-Cary, 94; Wan-Cary, 98)
- Perturbation approach to finding integrable magnetic fields (Sonnad-Cary, 2004)
- Circular beams
  - Integrable, nonlinear focusing magnets (Danilov, Nagaitsev)
  - Electron lenses with circular beams (Stancari)

Whatever approach is taken, testing is needed:

- Tracking needed for perturbative approaches as perturbation theory not a proof
- Tracking needed for exact solutions to understand effects of perturbations
- Sonnad-Cary:

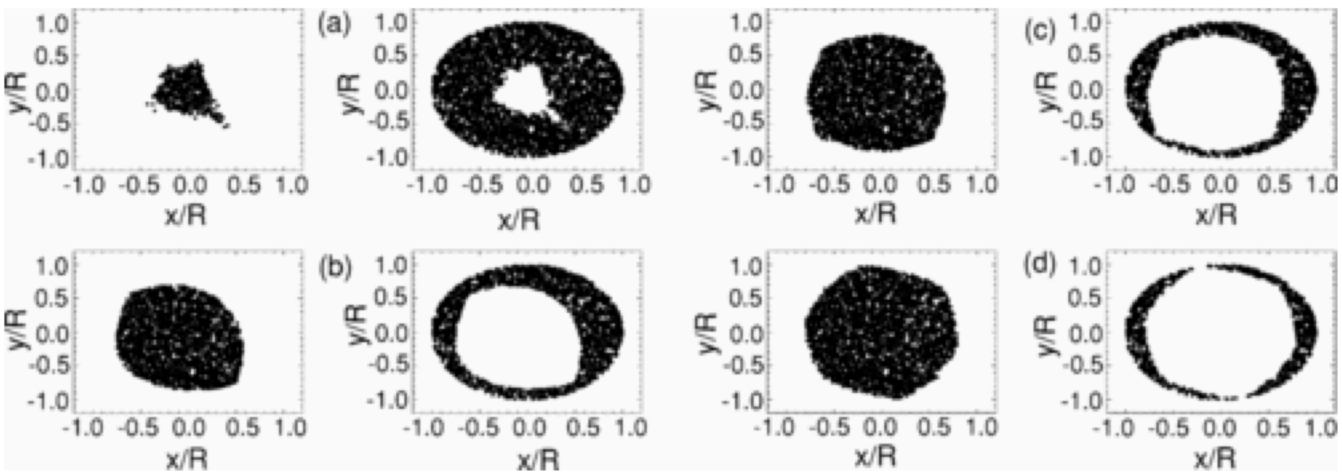


Figure 6

Initial distribution of confined and unconfined particles lying on the x–y plane for ψ = (a) $0°$, (b) $30°$,

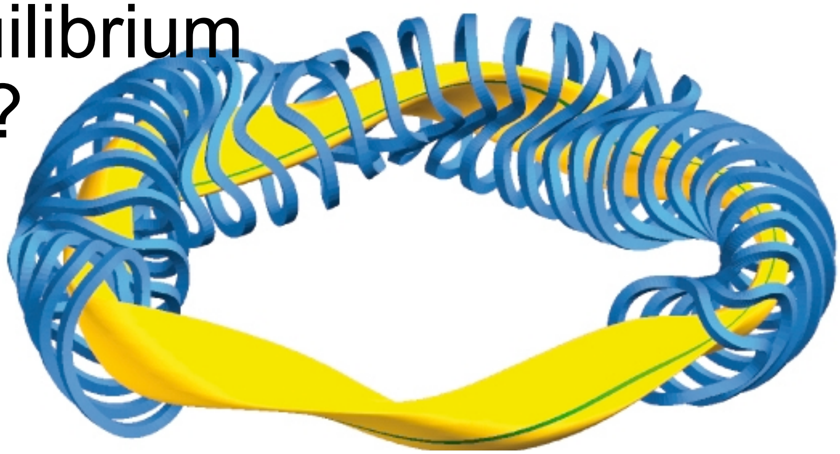
Can find equilibria: any beam with initial conditions in confined region

Tracking does not give a *self-consistent* equilibrium

- Tracking with space charge is no longer a single-particle problem
- Tracking can say what would happen to a distribution, but only one at a time, each requiring a large-scale simulation
- Tracking mixes equilibrium with non-equilibrium processes (oscillations)
- Would be much easier to match if one could simply calculate the equilibrium to know what has to be transported to the nonlinear lattice

How to compute equilibria: stellarator (fusion) analog

- Stellarator (Spitzer, 1950) analog
  - Plasma generates currents for confinement
  - Currents modify the magnetic field
- Grad-Shafranov equation (a differential equation) determines the equilibrium
- But does GS have a solution?
  - Unknown but
  - A solution is a minimum of an energy principle
- Implies an algorithm:
  - Define magnetic surfaces parametrically
  - Vary parameters to reduce the energy



A path forward indicated by Lund, Ryne, …

- Ryne, arXiv preprint acc-phys/9502001 (1995)
- Lund et al, PR ST/AB 9.064201 (2006)
- Lund et al, PR ST/AB 12.114801 (2009)
- Define equilibrium: a self-consistent beam that has the periodicity of the lattice
- Use the KV equations as “principal trajectories” that give the beam evolution.
- Look for fixed points – solutions that return to themselves after integration through the lattice.
- So far done for only linear-focusing lattices
- Above needs generalization for nonlinear lattices, possible through more principal trajectories and/or a moment hierarchy

But matching will never be perfect, so now want to estimate the effects of mismatch

- Gluckstern (94): particle core model indicated that beam mismatch could be a source of beam halo
- Physics for a linear lattice
  - Mismatch causes a particle oscillation at the betatron frequency
  - This causes self-consistent oscillations at twice the betatron frequency
  - This is exactly 2-1 resonant with the betatron motion of particles
  - This pumps particles to large amplitudes, with the "swing resonance"
- Scrape off the halo with collimators, but one has to do this at several phases

Our earlier work asked – what if one had a nonlinear lattice?

- Sonnad-Cary, PR ST/AB 8.064202
- Nonlinear – at large amplitudes the frequency would change
- The particles would fall out of resonance
- The large excursions would therefore be limited

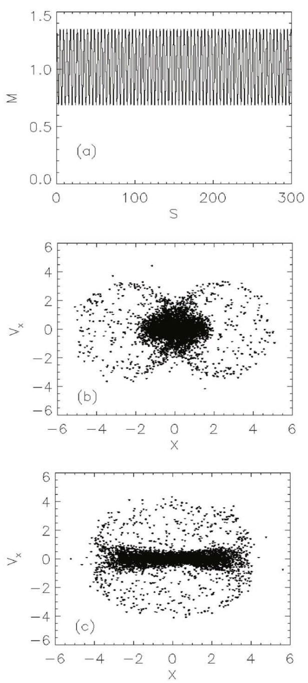


FIG. 1. Linear oscillations. (a) Oscillation of the rms width of the core with $\mu = 1.35$, (b) test particle distribution at last minimum rms width when $S \leq 300$, (c) test distribution at last maximum rms width when $S \leq 300$.

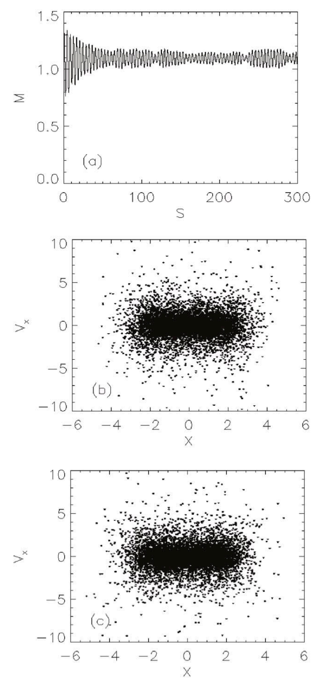


FIG. 2. Nonlinear oscillations. (a) Oscillation of the rms width of the core with $\mu = 1.35$, (b) particle distribution at last minimum rms width when $S \leq 300$, (c) distribution at last maximum rms width when $S \leq 300$.

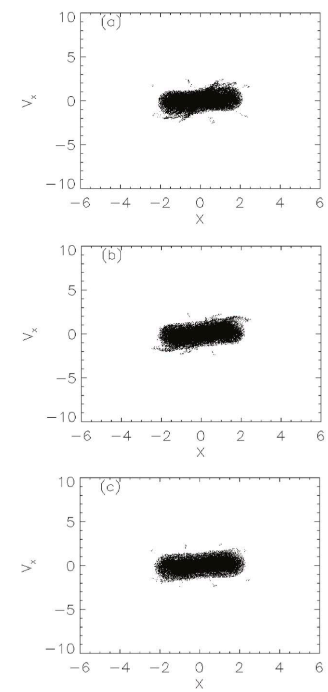


FIG. 10. Phase space distribution of particles just after collimation for a beam with nonlinear focusing for $\mu =$ (a) 1.5, (b) 1.35, (c) 1.2.

It is now important to extend these calculations to realistic lattices

- Significant space charge – so that the tune depression is O(1)
  - Self-charge defocusing is as large as the focusing, BUT it cannot be written as a map
  - Need direct integration methods
- Need to ensure that numerical methods do not lose essential physics
  - High-order smoothness
  - High-order solves?
  - High-order interpolations?
  - Volume preserving okay? Or full symplectic?

Summary of theory/computation needs for a successful IOTA program

- Cannot just "simulate with space charge"
- Methods for computing self-consistent beam equilibria for nonlinear lattices needed to know what matches
  - Otherwise – flying blind in the matching problem
  - Re-examine generalization of Danilov work
  - Extend principal trajectories of Lund
    - For nonlinear lattices
    - More principal trajectories
- Methods for tracking with space charge needed to understand how given mismatch will generate halo
  - Must work for space charge defocusing as large as lattice focusing – traditional map methods will not work
  - Must preserve invariants
    - no symplectic integrators, examine volume preserving
    - Must be very high order in smoothness (C^3)
    - May need similar order of accuracy
    - EM/ES to be decided

And before I go…

- JR Cary, DT Abell, GI Bell. BM Cowan, JR King, D Meiser, IV Pogorelov, GR Werner, “Select Advances in Computational Accelerator Physics”, IEEE Trans. Nucl Science (2016)
- On to curing cancer! (VSim now in use for photodynamic therapy.)