Moonshine, Supersymmetry and Gravity

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Moonshine refers to unexpected connections between the representation theory of sporadic groups and modular functions.

It is an area of mathematics, but one where conformal field theory and string theory have played a crucial role.

Recent developments in Moonshine exhibit surprising new connections to supersymmetry and possibly to three-dimensional gravity.

Examples of Moonshine

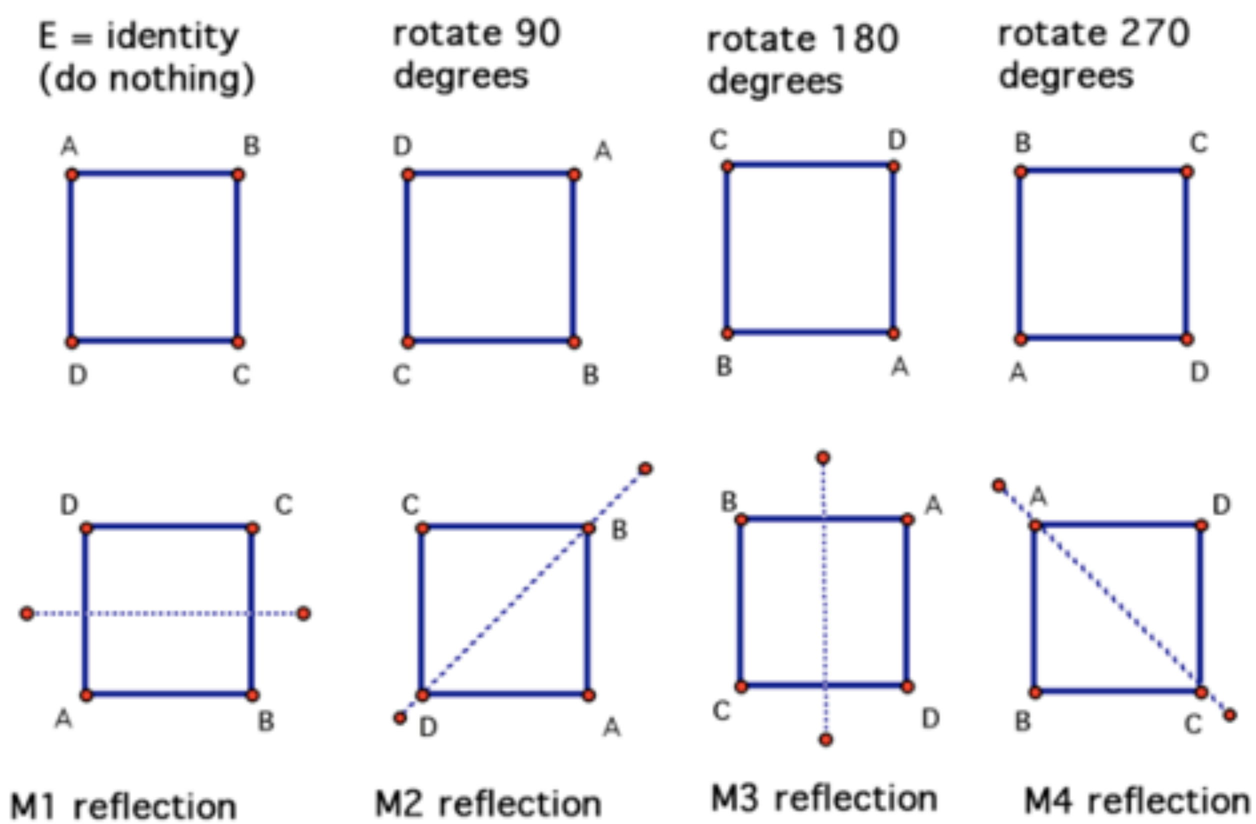
Monstrous Moonshine and Conway Moonshine (1979-1992)
McKay, Thompson, Conway, Norton, Frenkel, Lepowsky, Meurman, Borcherds

Mathieu Moonshine (2010) Eguchi, Ooguri, Tachikawa

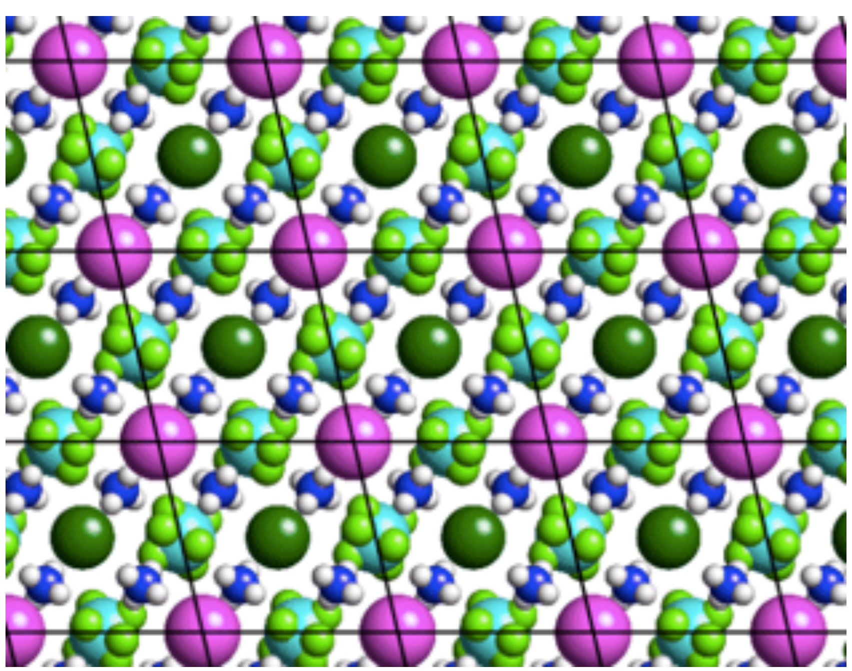
Umbral Moonshine (2012) Miranda Cheng, John Duncan, JH

Moonshine for the Thompson Group (2015) Brandon Rayhaun and JH

Finite Groups



The symmetry transformations of a square form a finite group



Symmetries of lattices of atoms lead to symmetries of crystals

In a way analogous to writing integers as products of primes, finite groups can be decomposed into simple groups.

A subgroup H of a group G is normal if for all $g \in G$

$$gH = Hg$$

In this case G/H is also a group.

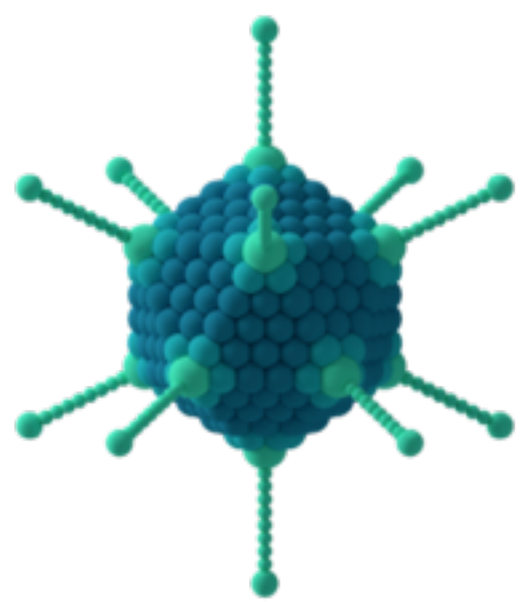
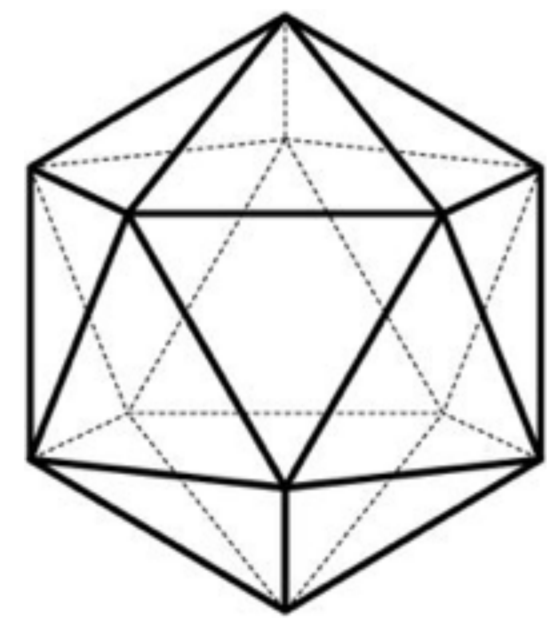
Example: Translations form a normal subgroup in the Euclidean group of translations, rotations and reflections since

(Rotate)(Translate)(Inverse Rotate)=Translate

The quotient Euclid/Translate=(Rotate, Reflect)

Simple groups are groups with no normal subgroups, they can't be broken down into smaller groups.

Examples: The group $\mathbb{Z}/3$ consisting of {0,1,2} with addition mod 3 is a simple group as is the symmetry group of the Icosahedron.

Remarkably, mathematicians have been able to classify all finite, simple groups. It is one of the major results of 20th century mathematics.

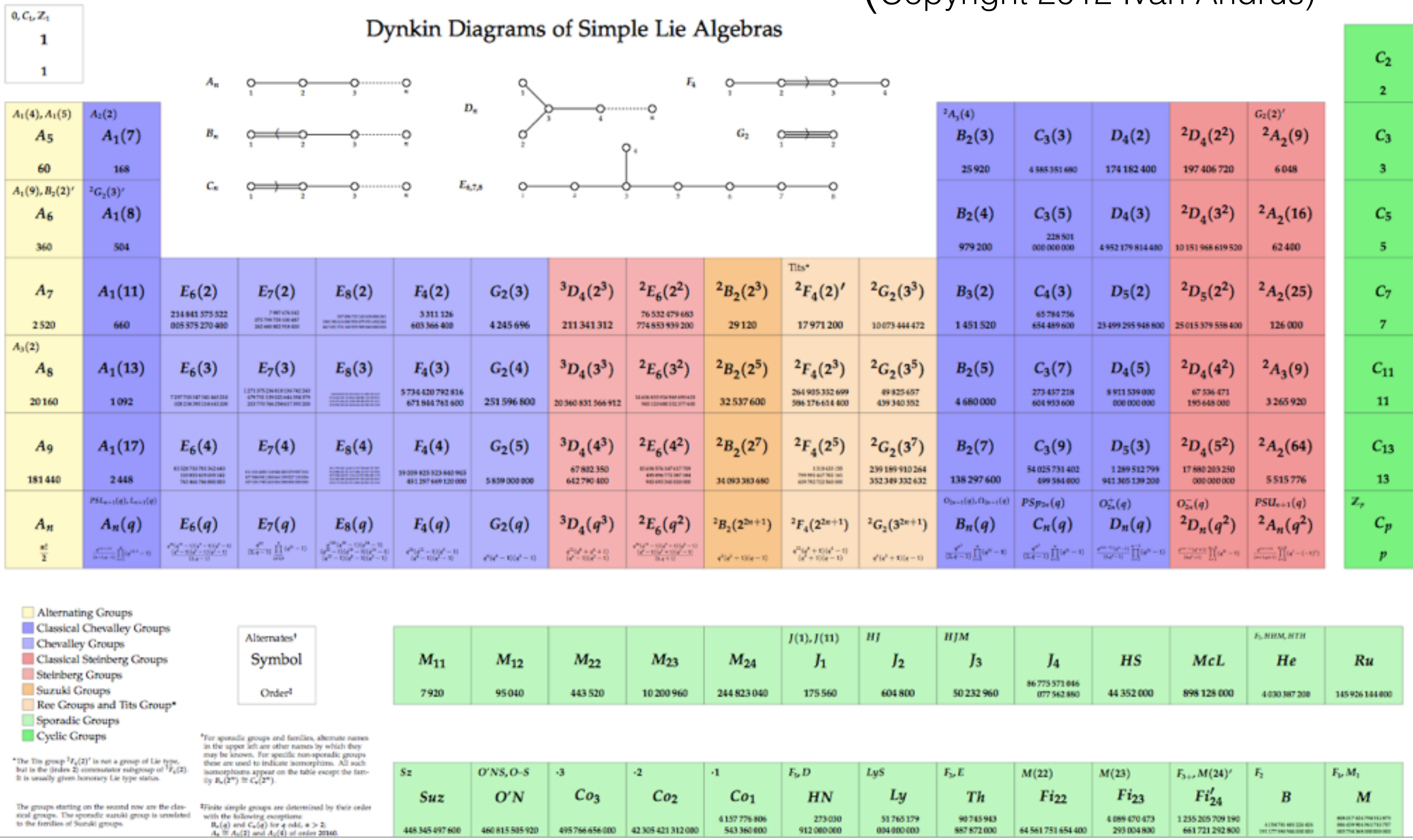
It is also an amazing example of the emergence of complexity from simple rules, not in a physical setting but in a mathematical setting. The simple rules are the axioms of group theory and the definition of a simple group.

A group is a set G and a composition law $\bullet$ obeying

Closure	$a, b \in G \to a \bullet b \in G$
Associativity	$(a \bullet b) \bullet c = a \bullet (b \bullet c)$
Identity	$a \bullet e = e \bullet a = a$
Inverse	$a \bullet a^{-1} = a^{-1} \bullet a = e$

The Periodic Table Of Finite Simple Groups

(Copyright 2012 Ivan Andrus)



There are several infinite families, the easiest to understand are the cyclic groups $\mathbb{Z}/p$ {0,1,…p-1} with addition mod p for p prime. There are other infinite families constructed out of continuous Lie groups. Then there are 26 (or 27) sporadic groups which don't fit into any families and are symmetries of large, bizarre, exceptional things.

Of the 26 sporadics, 20 can be found inside the largest of the sporadics, the MONSTER. So it seems fair to say that if we really understood why the MONSTER exists we would be a fair way to understanding why the sporadic groups exist.

The Monster group has ~ 10^{54} elements.

A representation of a finite group of dimension n is an action of the group on a n-dimensional space ~ a set of n by n matrices obeying the group multiplication rules.

Symmetries of a square: two-dimensional representation given by action on (x,y) coordinates of vertices.

Rotations in QM: For spin 1/2 we have a two-dimensional representation acting on $\begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}$

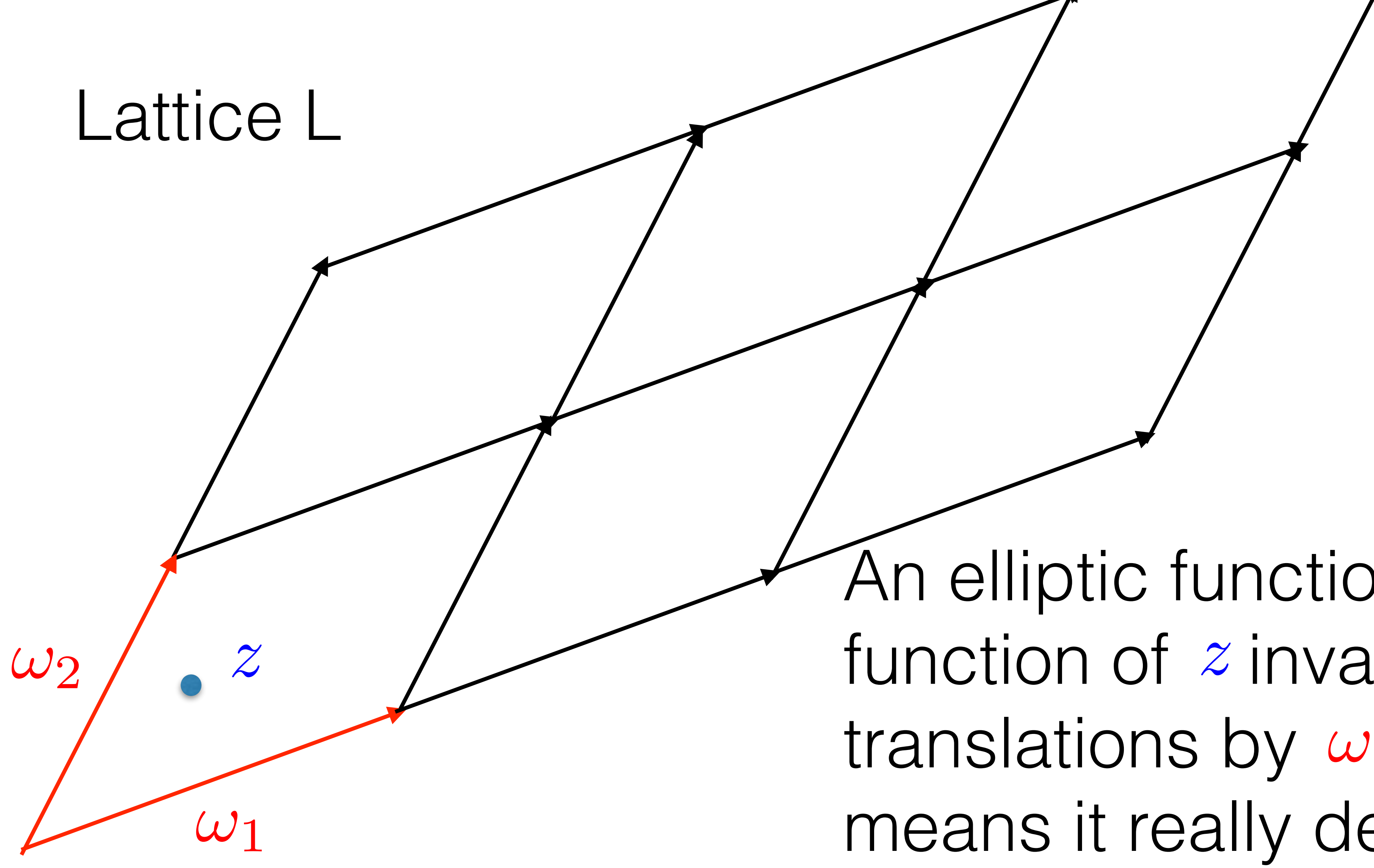
Monster: Other than the trivial one-dimensional representation, the smallest (irreducible) representation has dimension 196883 !!!

Although this barely scratches the surface, I will leave finite groups and go on to discuss modular functions. Modular functions in particular play a central role in many developments in theoretical physics and mathematics:

Phase transitions in one-space dimensional systems are described by 2d Conformal Field Theories and their thermal partition functions are naturally a type of modular form.

Chiral edge states in Quantum Hall systems and Topological Insulators are described by CFT and have properties constrained by modular invariance of their partition functions.

The recent proof of Fermat's last theorem involved a proof of the modularity theorem involving modular forms.



An elliptic function is a function of z invariant under translations by ω_1 , ω_2. That means it really depends on both z and ω_1, ω_2 .

However if we don't care about the overall scale, and if we fix the orientation, then the dependence on L is equivalent to a dependence on τ with $Im(\tau) > 0$.

$$(\omega_1, \omega_2) = \omega_2(\omega_1/\omega_2, 1) = \cancel{\omega_2}(\tau, 1).$$

Now changing the basis for the lattice L doesn't change the lattice, so we would expect such functions to be invariant under

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \to \begin{pmatrix} a\omega_1 + b\omega_2 \\ c\omega_1 + d\omega_2 \end{pmatrix}$$

or $\tau \to \dfrac{a\tau + b}{c\tau + d}$ with $a, b, c, d \in \mathbb{Z},\ ad - bc = 1$

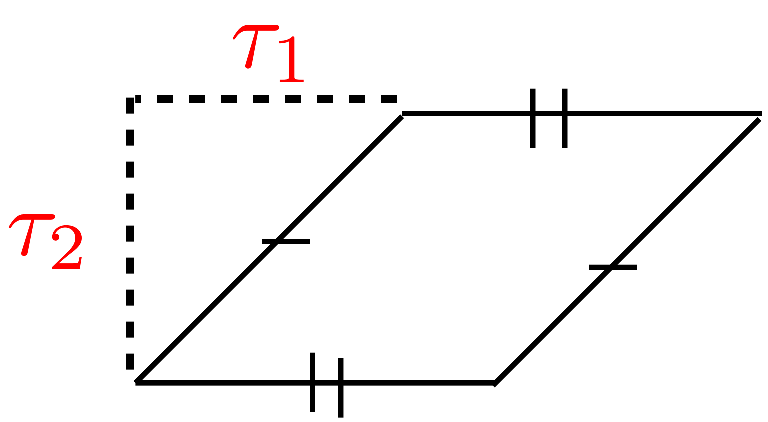
This is called the group of modular transformations, and functions which transform nicely under these transformations are modular forms. Mathematically they are to $SL(2,\mathbb{Z})\backslash\mathbb{H}$ as trig functions are to $S^1 = \mathbb{Z}\backslash\mathbb{R}$

where $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}\tau > 0\}$

They arise in 2d CFT because you can think of the partition function in the continuum limit with periodic boundary conditions as being defined on a circle, and the partition function is computed as a path integral on a circle in Euclidean time:

$$Z(\tau) = \mathrm{Tr} e^{-\tau_2 H - i\tau_1 P}$$

$$\int \mathcal{D}\Phi e^{-S_E(\Phi)}$$



Scale invariance and invariance under change of basis implies invariance under modular transformations. Cardy famously used this to restrict the operator content of CFTs

Now we come to something truly remarkable. In the CFT framework the partition function is generally a function of both $\tau, \bar{\tau}$ but in certain situations it factorizes as $|f(\tau)|^2$ (when the central charge is a multiple of 8 this can happen).

In math they like functions just of τ (holo or meromorphic)

All modular invariant functions of just τ are rational functions of a single function called the j function.

$$q = e^{2\pi i \tau}$$

$$j(\tau) = q^{-1} + 196884q + 21493760q^2 + \cdots \quad = \sum_n c(n) q^n$$

$$j(-1/\tau) = j(\tau)$$

$$j(\tau) = j(\tau + 1)$$

Monstrous Moonshine

John McKay: 196884=196883+1 (1978)

Coefficient of $j(\tau)$

Dimension of space on which the Monster group acts.

It was soon realized that similar relations held for the other coefficients, e.g.

Thompson, Conway&Norton

21493760=21296876+196883+1

Moonshine: a crazy or foolish idea. It also suggests something slightly illicit or illegal.

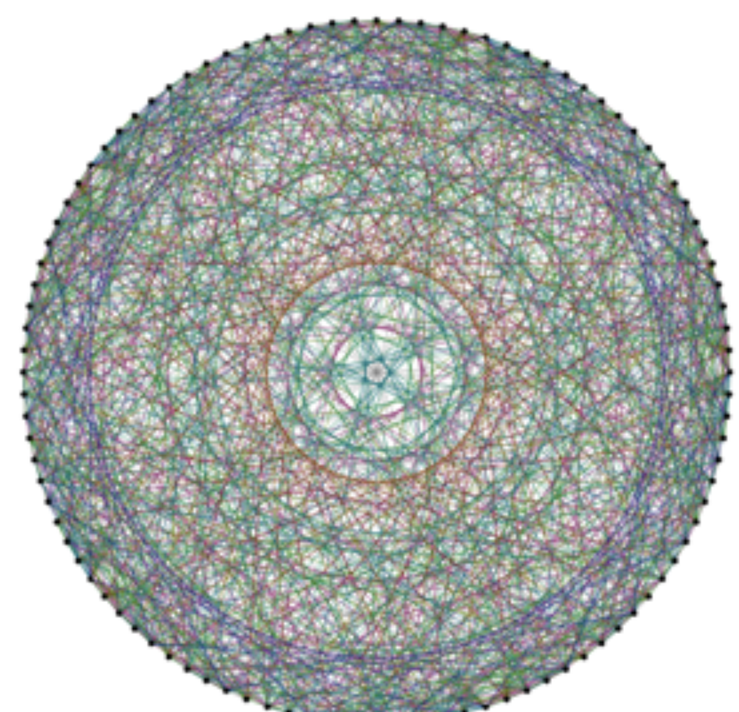
Monstrous moonshine is best understood through the study of a CFT defined via a 24-dimensional space

$$(\mathbb{R}^{24}/\Lambda_{Leech})/(\mathbb{Z}/2\mathbb{Z})$$

The Leech lattice is a remarkable object closely connected to several sporadic groups and is intimately connected to the solution to a common problem in an uncommon setting: what is the best way to pack spheres?

Three dimensions:

The densest lattice packing in 24 dimensions is given by the Leech lattice. Each sphere kisses 196560 of its neighbors! Recently proved to be densest of all packings.



Many aspects of the Monster group are now understood through construction of a c=24 CFT due to Frenkel-Lepowsky-Meurman with partition function $j(\tau)$ and with the Monster acting as a symmetry group, thus explaining at least part of the moonshine. However, there are many, many more remarkable features of this construction.

One example: $24 = 23\frac{1}{2} + \frac{1}{2}$

central charge of Ising model

A CFT with the Baby Monster as symmetry group

An order 2 element of the Monster preserved by the BM is simply $\psi \to -\psi$ of the free fermion of the Ising model.

Of course it is no coincidence that 24 is also the number of physical, transverse excitations of the bosonic string in 26 spacetime dimensions. And in fact many elements of string theory appear in the construction:

FLM: Used the Vertex Operators of string theory/CFT, and constructed the first asymmetric orbifold CFT.

Borcherds: Used the “no-ghost” theorem in his proof of certain genus-zero conjectures of Conway&Norton.

Moonshine and Supersymmetry

World-sheet supersymmetry:

The Monster CFT actually has a hidden superconformal structure (Dixon, Ginsparg, JH) because twisting 24 bosons by $X^I \to -X^I$ gives operators of dimension 24/16=3/2 and one of these fields has the correct OPE with the stress tensor to give a N=1 superconformal algebra.

Just as the Monster CFT arises from a compactification of the bosonic string to two-dimensions, there is also moonshine involving compactification of the superstring to two dimensions.

Conway Moonshine

Compactify (holomorphic part) of the superstring on the E8 lattice and do a Z/2 orbifold on the bosons and fermions, $X^a \to X^a, \psi^a \to \psi^a$. The resulting partition function (in the NS sector) is

$$Z_{NS} = q^{-1/2}(1 + 276q + 2048q^{3/2} + \cdots)$$

Dimensions of representations of Conway sporadic group

This theory was constructed by J. Duncan and Duncan and Mack-Crane proved it shares many of the same remarkable properties as monstrous moonshine. There is actually a SO(24) symmetry, broken to the Conway group by demanding invariance of the N=1 SCA.

Does spacetime supersymmetry also enter into the moonshine game?

With $N \geq 2$ spacetime susy there are BPS states in small representations of the susy algebra. The simplest non-trivial Calabi-Yau space is K3, and string theory on $K3 \times S^1$ has BPS states counted by the elliptic genus of K3.

Eguchi, Ooguri, Tachikawa (2010):

The elliptic genus of K3 can be decomposed into characters of the N=4 superconformal algebra, the multiplicities of massive representations are counted by the function

$$H^2(\tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + +2277q^4 + \cdots)$$

45,231,770,2277 are dimensions of M24 irreps

This is the start of a vast extension and generalization of moonshine which is still poorly understood but involves

Modular Forms → Mock Modular Forms

Monster → Niemeier lattice groups

CFT → Super CFT

and undoubtedly new ingredients as well, but all indications are that it is tied up with superstrings, their compactifications, and the structure of black holes in string theory.

A modular form of weight k is a holomorphic function $f(\tau)$ obeying

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$h(\tau)$ is a mock modular form of weight k if there is a pair $(h(\tau), g(\tau))$ where $g(\tau)$ is a holomorphic modular form of weight 2-k, known as the shadow of h, such that the non-holomorphic function

$$\hat{h}(\tau) = h(\tau) + \text{const} \int_{-\bar{\tau}}^{\infty} \overline{g(-\bar{z})}(z+\tau)^{-k} dz$$

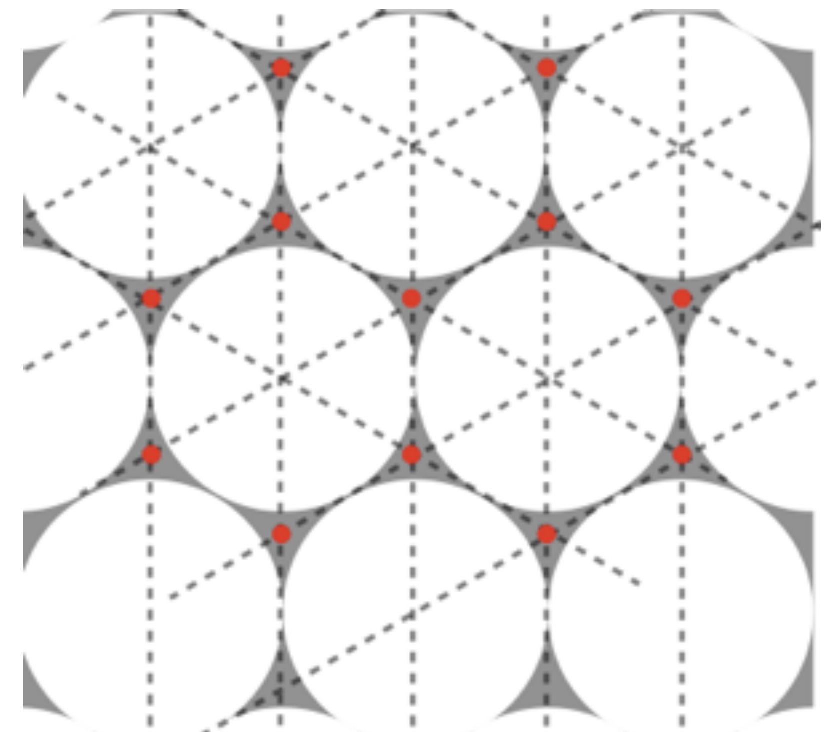
$(H^{(2)}(\tau), \eta(\tau)^3)$ is such a pair with k=1/2

This definition looks quite technical, but is a great improvement over the original presentation of these by Ramanujan who just wrote down q-series with little explanation of their properties.

Umbral Moonshine

Mock modular forms ⟷ Niemeier finite groups

Deep holes of a sphere packing: points at maximum distance to centers of spheres. They also form a lattice.



The Leech lattice has 23 different types of deep holes leading to 23 Niemeier lattices. There are 23 finite groups constructed from the symmetries of these lattices, they are the groups of Umbral Moonshine. The first group is a sporadic group called M24.

Mock modular forms or theta functions fail to transform well under SL(2,Z) in a way specified by another modular form called the shadow (umbra in latin).

Almost all of the mock theta functions in Ramanujan's last letter to Hardy and in the "lost notebook" have now been linked to the representation theory of Niemeier groups. Hence Umbral Moonshine.

At the forefront of research are attempts to explain this structure using string theory and to prove a number of conjectures made in the work on Umbral Moonshine (the main conjecture has now been proven by Duncan, Griffin and Ono).

What is the connection to gravity? They are indirect so far, but still quite suggestive.

One connection involves the computation of Black Hole entropy in string theory which involves counting supersymmetric BPS states of given charges and mass and comparing their degeneracy to that of the corresponding black holes.

Black Holes and Mock Modular Forms

"My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions ... But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further."

Freeman Dyson (1987 Ramanujan Centenary Conference)

Dabholkar, Murthy and Zagier (2012):

The degeneracies of single centered (immortal) black holes with magnetic charge invariant $M^2/2 = m$ are Fourier coefficients of a mock Jacobi form of index m.

Thus mock modular forms have a direct physical interpretation in black hole counting problems, and more generally in the study of the elliptic genus of non-compact sigma models. However so far the mock modular forms of Umbral Moonshine do not arise in physical models.

A second connection to gravity is inspired by the AdS/CFT correspondence. The (S)CFTs with Monster and Conway symmetry are special CFTs. Are they dual to special theories of gravity in AdS3 as suggested by Witten? The jury is still out, but there is something very suggestive in all examples of moonshine:

Every known example of Moonshine involves (mock) modular objects which are Rademacher sums

Rademacher $$J(\tau) = \mathrm{Regularize}\left[\mathrm{Average}_{SL(2,\mathbb{Z})}\left(q^{-1}\right)\right]$$

Generalized by Knapp, Niebur, Manschot, Moore, Duncan, Frenkel, Cheng, Dabholkar, Gomes, Murthy, …..

These sums like semi-classical sums over Euclidean AdS3 geometries whose conformal boundary geometry is that of a torus and thus has an action of SL(2,Z). However it is not always clear what underlying “gravity” theory is giving rise to these sums.

Physics prospects for moonshine (conservative):

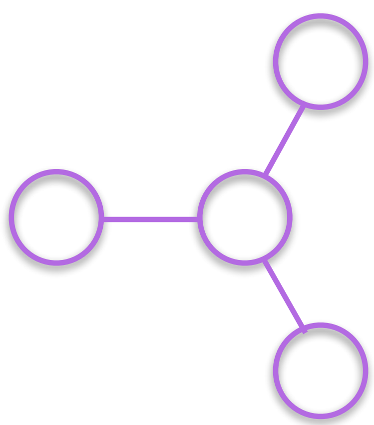
We may discover new kinds of Conformal Field Theories related to Umbral Moonshine.

We will certainly learn more about interesting new finite symmetry groups of Calabi-Yau manifolds and string compactifications which should be interesting for mathematics and physics. For example the “special group” SL(2,7) has appeared in models of neutrino masses (Chen, Perez, Ramond).

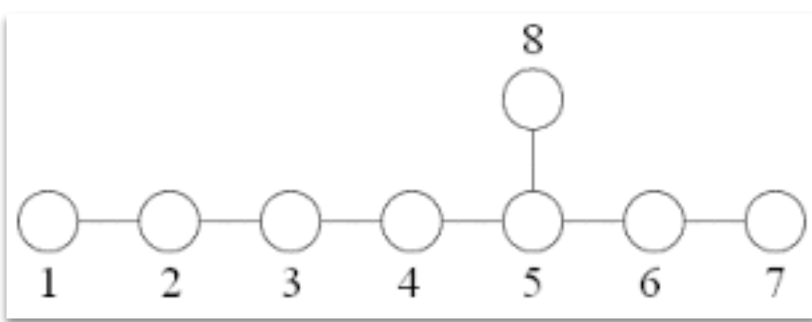
It seems likely to shed new light on the structure of gravity in three dimensions.

Wild-eyed prospects

The existence of string theory, particularly superstring theory and even more heterotic string theory relies on all sort of special structures:



Triality of the SO(8) Dynkin diagram



Special properties of the Lie group E8

Sporadic groups are also linked to exceptional structures like the Leech lattice and Golay code.

Moonshine is linked to special structures appearing in compactifications of string theory. Is there perhaps something to be learned here about the vacuum selection problem in string theory?

Why is string theory so beautiful and exceptional in 10 dimensions and so ugly and multitudinous in its compactifications?

THANK YOU