Likelihood-based test statistic in Solar Axion searches

LA-UR-16-27151

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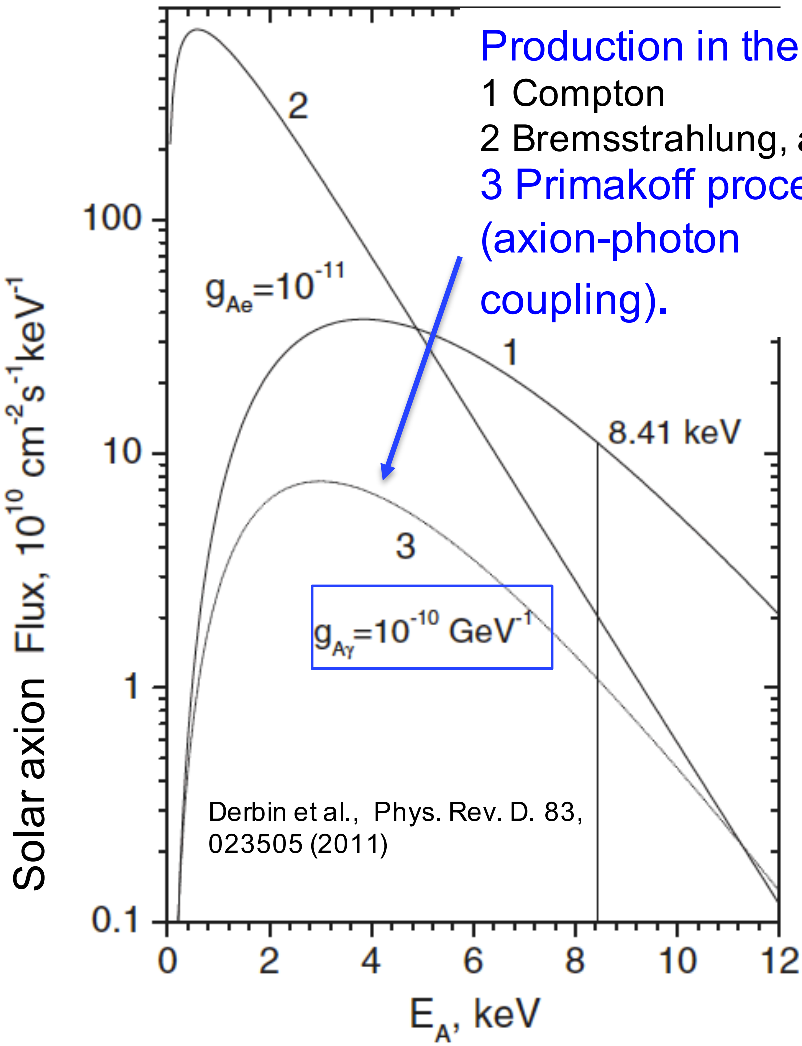
PhyStat-Nu workshop
Sept-20-16, FNAL

Solar Axions

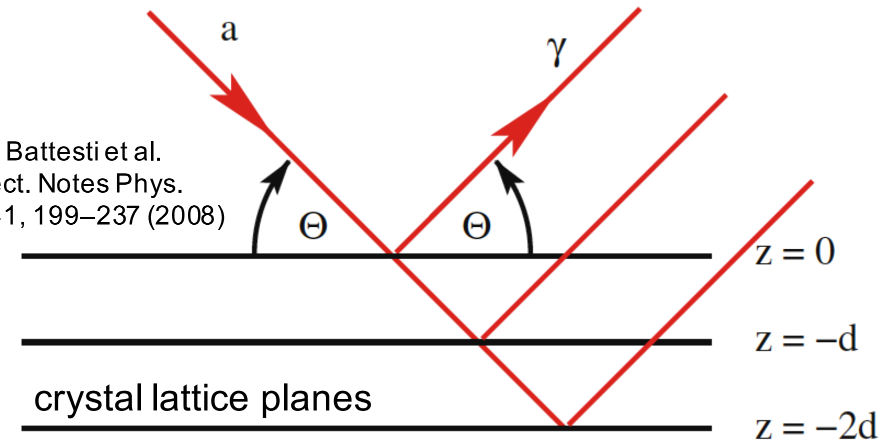
Axion: originated from the *strong CP problem*
Candidate for low mass dark matter



Detection in crystalline detectors

$Axion + \gamma_{virtual} \rightarrow \gamma$
Reversed Primakoff

R. Battesti et al.
Lect. Notes Phys.
741, 199–237 (2008)



Bragg diffraction: coherent scattering on the crystal planes, if $n\lambda=2d\sin\theta$

Signal pattern enhanced by Bragg diffraction at specific attacking angle (θ) with energy (λ)
→ **Need to know the crystal lattice plane angle $\phi \in [0, 90°)$**

This talk only considers experiments using crystals and Axion-photon coupling.
A recent, more comprehensive review:
Annu. Rev. Nucl. Part. Sci. 2015.65:485-514.

Example Experiments

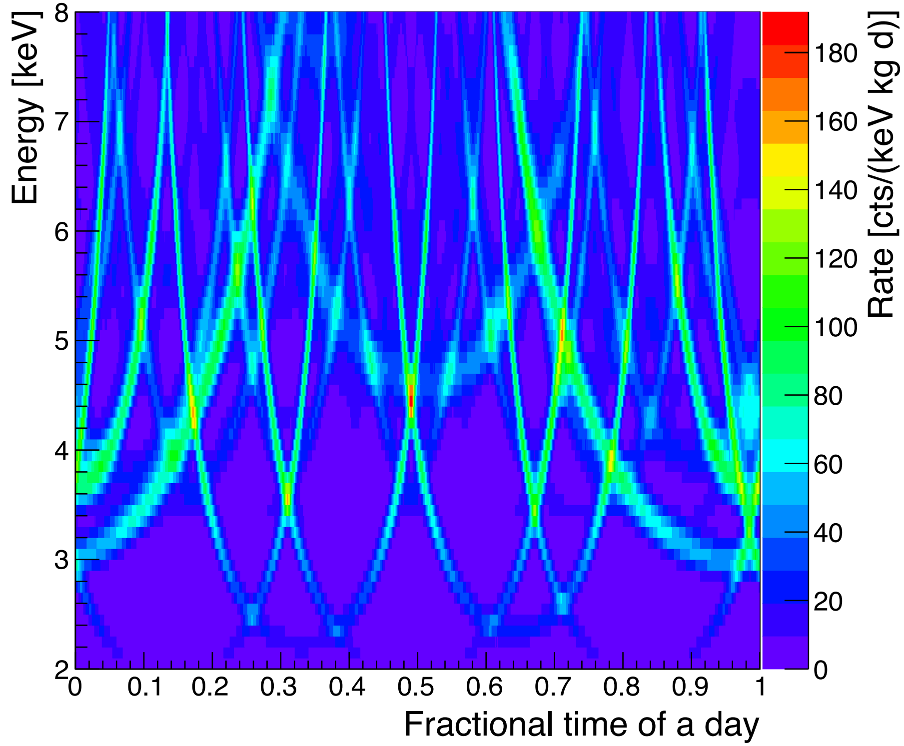
MAJORANA DEMONSTRATOR

EDELWEISS-III

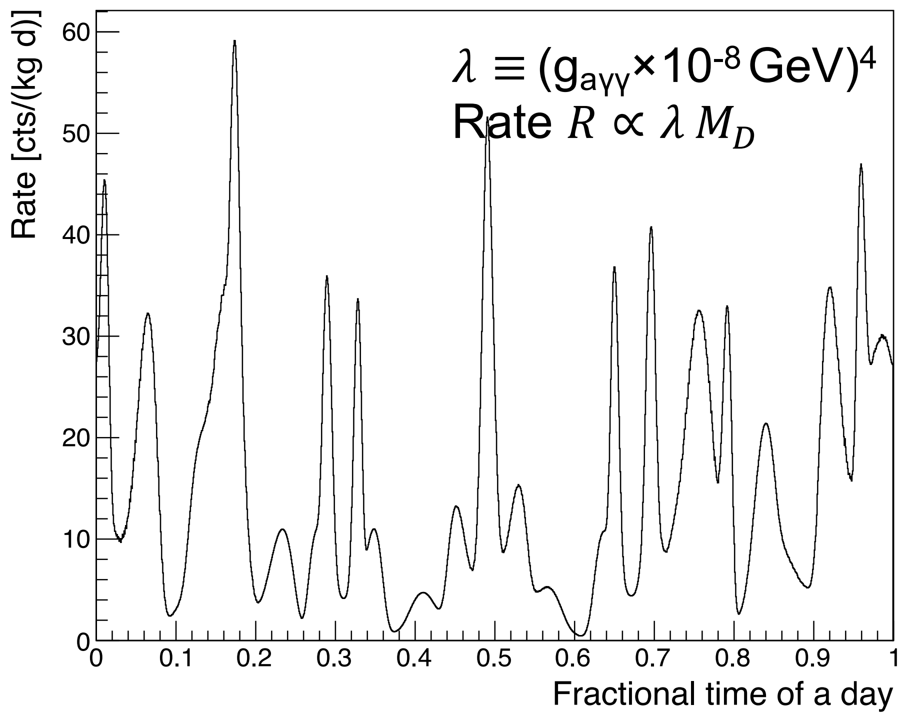
Array of High Purity Germanium (HPGe) detectors
Extremely clean, very low background rate

Axion Bragg Diffraction

Energy 2-8 keV possible for Bragg diffraction



Summed for 4.0-4.5keV



Motion of the Sun w.r.t. the Earth

→ Changing of attacking angles

→ Complicated signal patterns

Tables of Sun's trajectory

→ Numerical expression of signal rate

Calculated for λ=1 and M_D=1kg HPGe

Current limits: λ < around 1 x 10^{-3}

DAMA, Phys. Lett. B 515 (2001) 6, 90% CL; EDELWEISS JCAP 11 (2013) 067, 95% CL

$$R^{(j)} = R^{(j)}(\phi^{(j)})$$ depends on angle of the j^{th} detector

Calculation of Axion signal closely follows:
R. J. Creswick, et al., PLB 427 (1998) 235-240
SOLAX, Phys. Rev. Lett. 81, 5068 (1998)

The Statistical Problem

For the *j*-th detector (*j*=1, …, n_D), the likelihood function is

$$\mathcal{L}_D^{(j)} = e^{-m^{(j)}} \prod_{i=1}^{N^{(j)}} [b^{(j)} + R^{(j)}(t_i^{(j)}, E_i^{(j)})]$$

where *i* is the *i*-th event in this detector

Joint likelihood

$$\mathcal{L} = \prod_{j=1}^{n_D} \mathcal{L}_D^{(j)}$$

background $b^{(j)} = \bar{b} M_D^{(j)}$

Axion signal $R^{(j)} \propto \lambda M_D^{(j)}$

- **Assuming required angles of n_D detectors ($\vec{\phi}$) are all precisely known**
- **Parameter of interest: Axion-photon coupling constant (λ, universal)**
- **Nuisance parameter: background ($\bar{b}$, universal)**
- **Apply the profile likelihood method** Ref.:Rolke et al, NIMA 551 (2005) 493

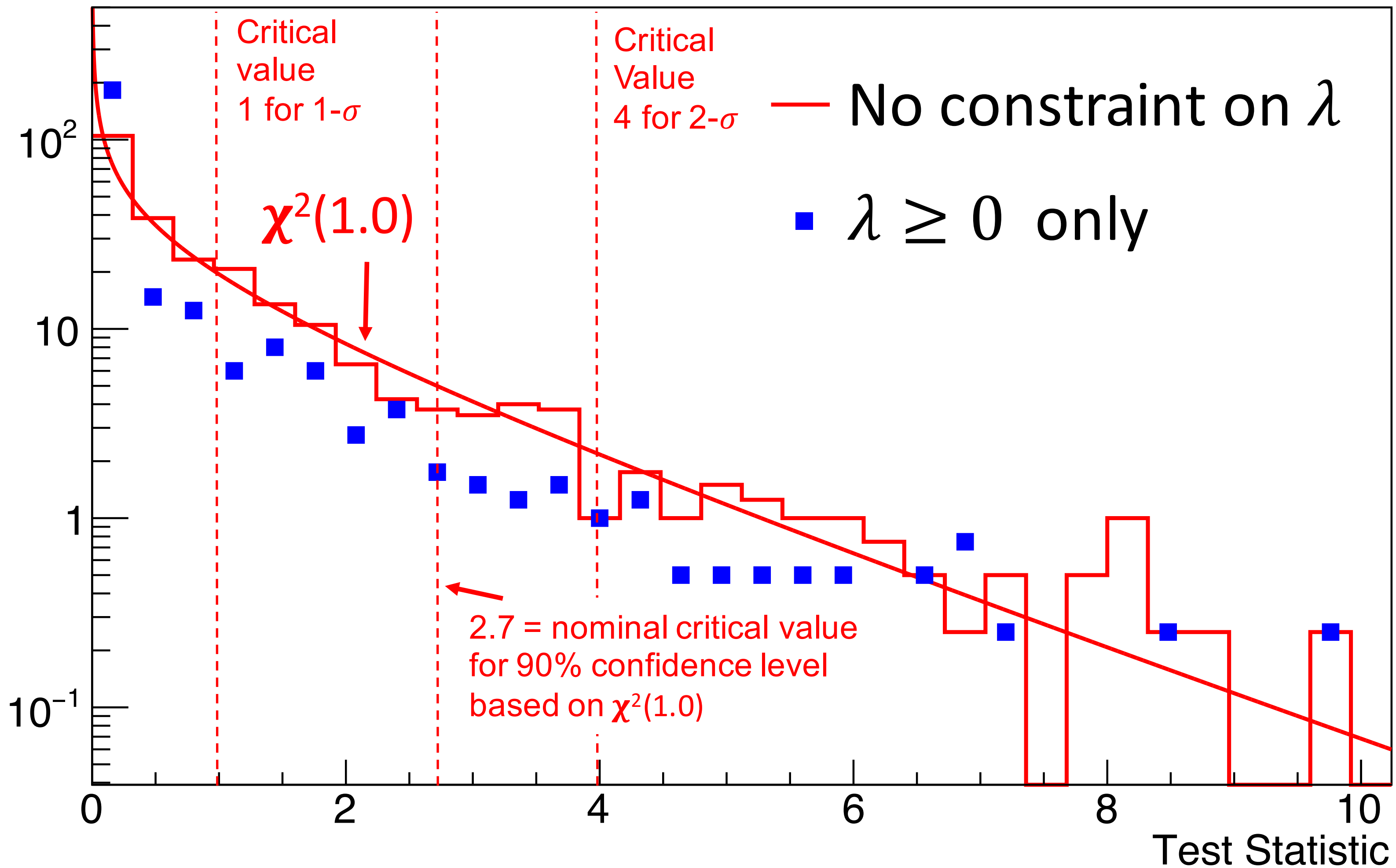
-2ln(profile likelihood) should approximately have a $\chi^2(1)$ distribution

- Monte Carlo simulations generated with λ_{true}
- 1000 simulations per ensemble
- Define a test statistic *D*, based on profile likelihood.

$$D \equiv -2\ln \mathcal{L}_p(\lambda_{true}) = -2\ln \frac{\sup\{\mathcal{L}(\lambda_{true}, \bar{b}); \bar{b}\}}{\sup\{\mathcal{L}(\lambda, \bar{b}); \lambda, \bar{b}\}}$$

Effect of Physical Boundary

- Coupling λ_{true}=0 used in Monte Carlo simulations, no Axion signals
- Physical boundary makes the test statistic distribution narrower than $\chi^2(1.0)$
  → Over-coverage with nominal critical value 2.7 for 90% confidence level
  → Proper critical value for 90% C.L. is smaller than 2.7



Detector Angles: Poorly Known

Comprehensive orientation surveys are required to obtain the absolute angle for every Ge detector.
Difficult to measure, and could suffer from large uncertainty ~ 15º, i.e. poorly known

Single Ge detector

Crystal axis angle measurable by Laue diffraction or charge carrier drift time (*in situ*)

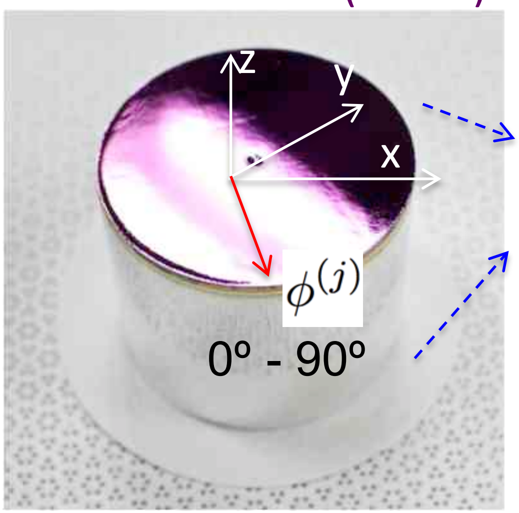
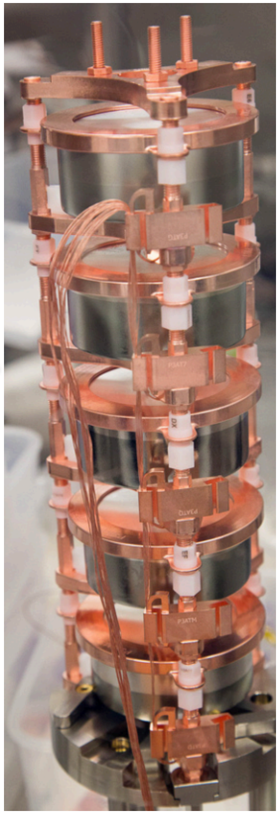
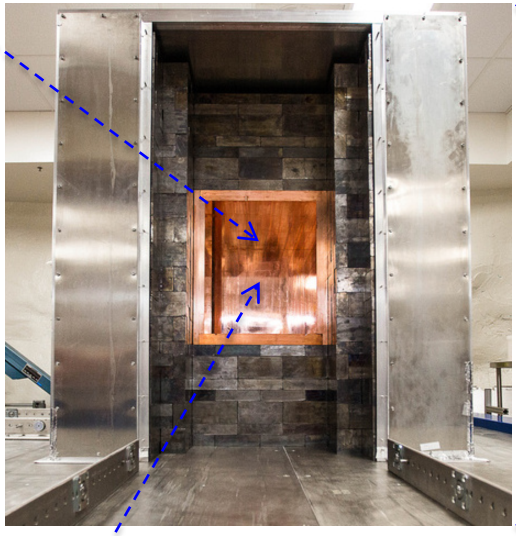


Figure of a HPGe Detector, Courtesy of James Loach, LBNL

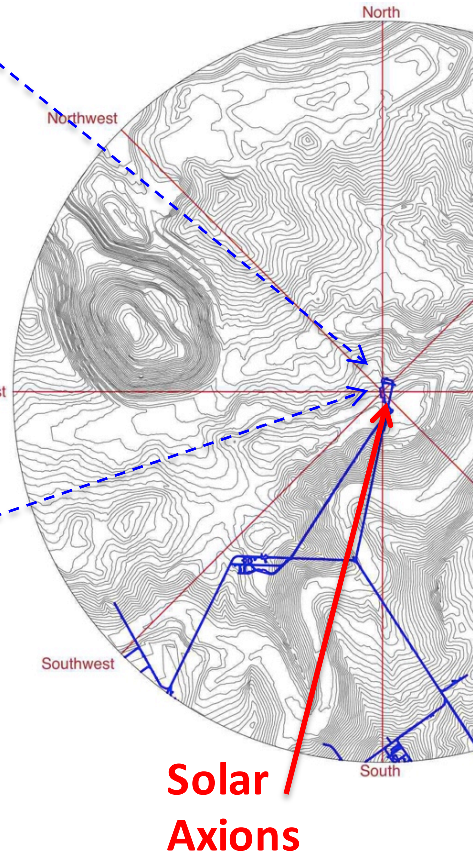
Detector Array



Experiment Apparatus



Laboratory



Figures of **MAJORANA DEMONSTRATOR** courtesy of sanfordlab.org.
Topological contours of Sanford Underground Research Facility from Journal of Physics: Conference Series **606** (2015) 012015

The Statistical Problem Revised

Parameter of interest: Axion-photon coupling constant (universal)
Nuisance parameters: background (universal) and
angles of n_D detectors (poorly known to within 15°)

Test statistic

$$D \equiv -2\ln\mathcal{L}_p(\lambda_{true})$$
$$= -2\ln\frac{\sup\{\mathcal{L}(\lambda_{true},\bar{b},\vec{\phi});\bar{b},\vec{\phi}\}}{\sup\{\mathcal{L}(\lambda,\bar{b},\vec{\phi});\lambda,\bar{b},\vec{\phi}\}}$$

- Supposedly the nuisance parameters can be profiled out from the data.
- Works for background

- Works for the detector angles only if the data itself has sensitivity on them.
- Axion signal depends on the detector angles; the background events don't.

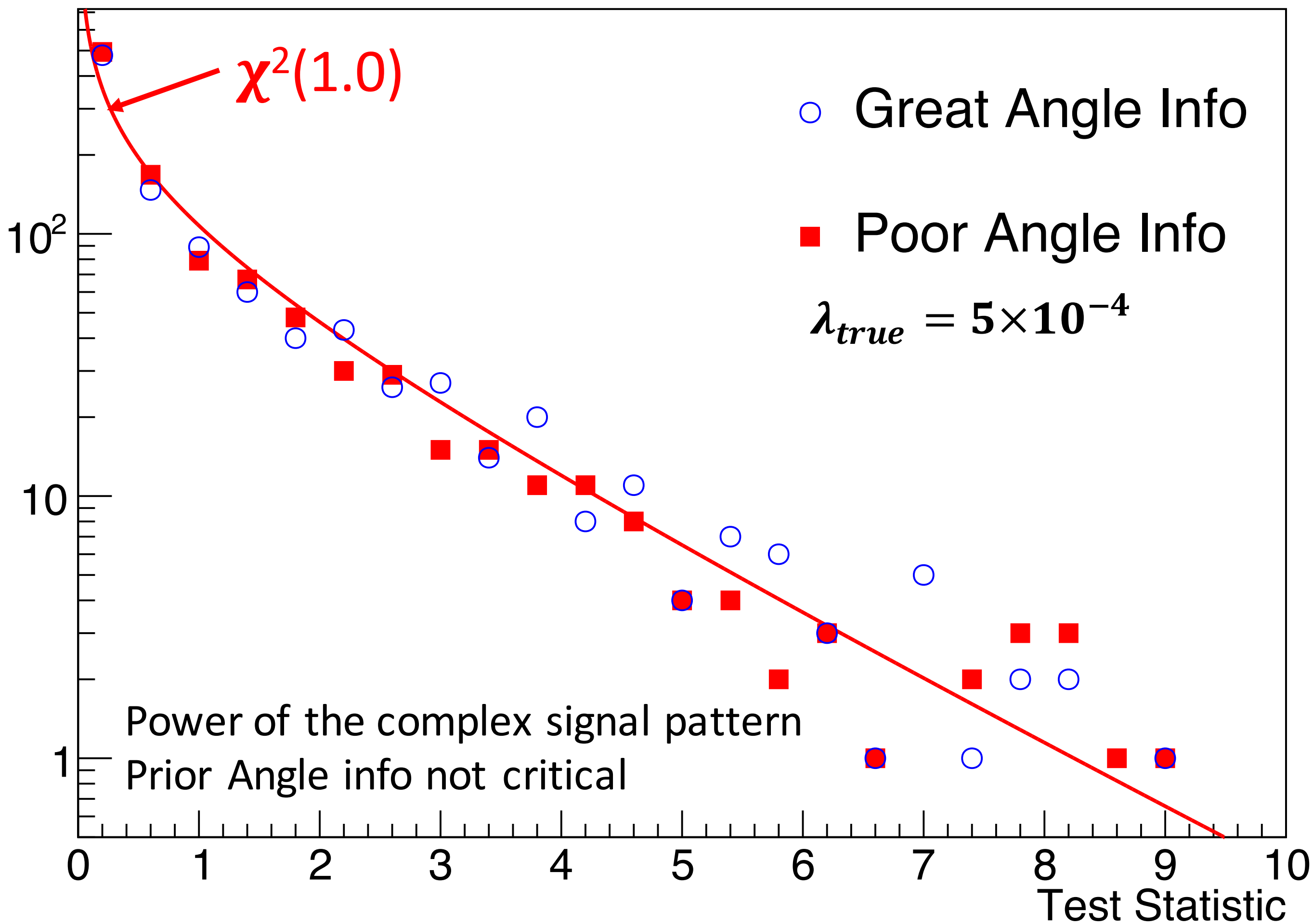
➔ None to few axion signals = none to little ability to profile out the angles.

Only considering four detectors, n_D=4
simulated 1000day kg exposure for each detector

Poor Angle Info + Big Coupling

Signal $\lambda_{true} = 5\times10^{-4}$, bakground $\bar{b} = 0.1$ cts/kg/day/keV

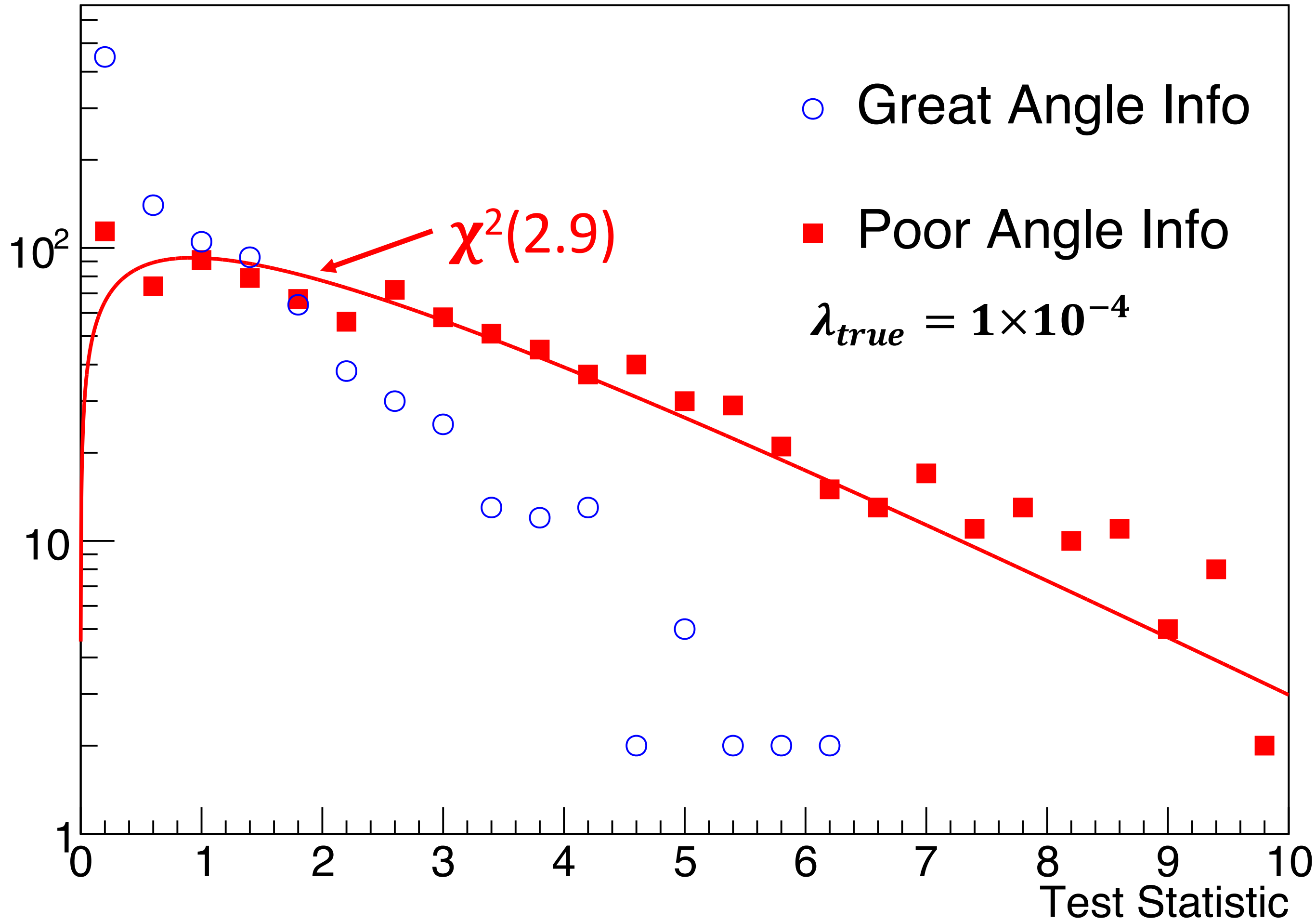
- Total signal ~ 0.1 cts/day/kg, total background ~ 0.6 cts/day/kg
- Signal large enough for the detector angles to be accurately profiled out from data. Nominal critical value 2.7 for 90% is OK



Poor Angle Info + Small Coupling

Signal $\lambda_{true} = 1\times10^{-4}$, bakground $\bar{b} = 0.1$ cts/kg/day/keV

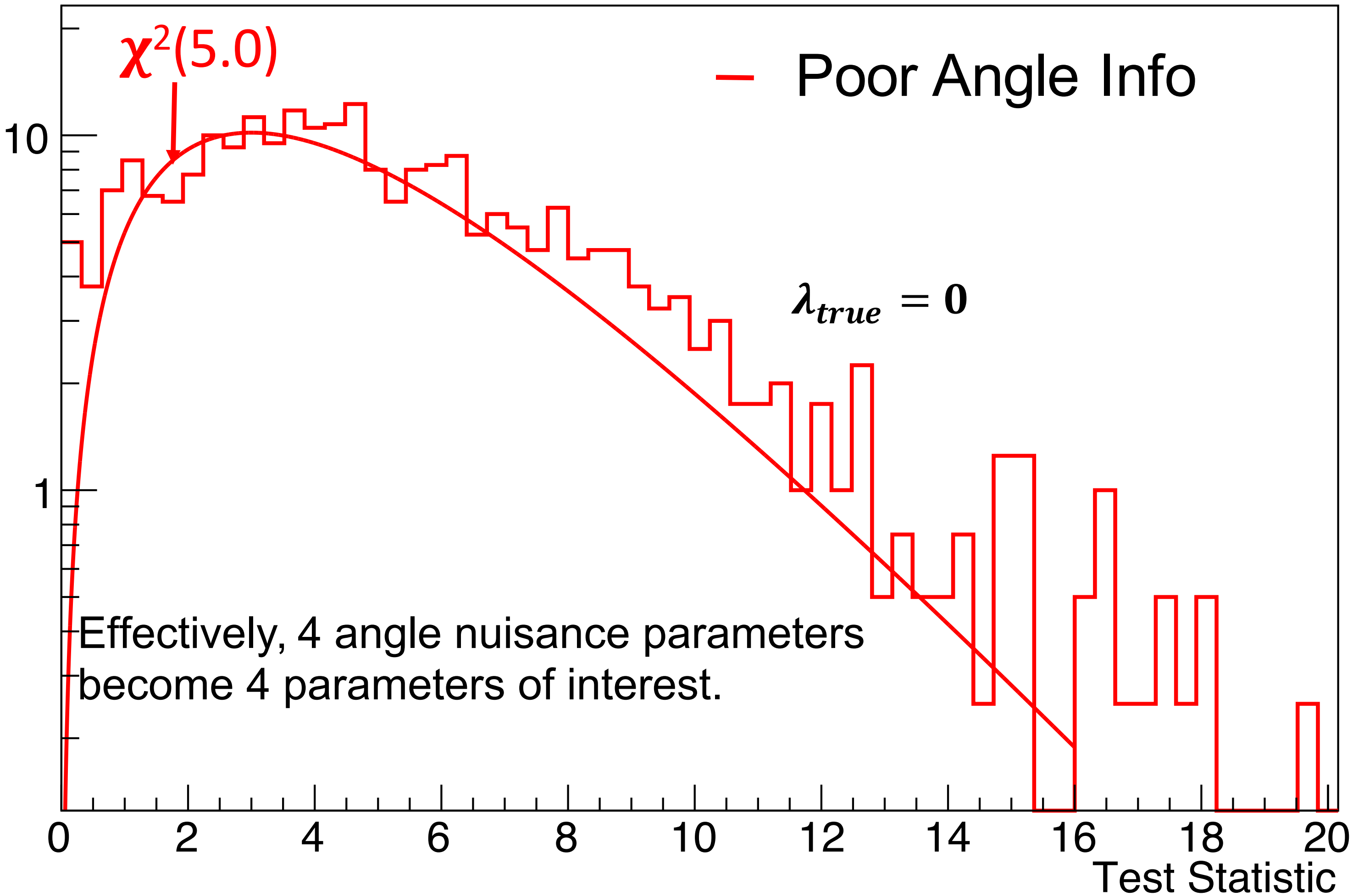
- Total signal ~ 0.02 cts/day/kg, total background ~ 0.6 cts/day/kg
- Signals not large enough for accurate profiling of angles, but still some sensitivity to angles. Nominal critical value 2.7 for 90%CL gives under-coverage



Poor Angle Info + Zero Coupling

Signal $\lambda_{true} = 0$, bakground $\bar{b} = 0.1$ cts/kg/day/keV

- No signals in MC data, no sensitivity on angles
- Test statistic distribution approximates $\chi^2(5.0)$
- 1 unknown coupling + 4 unknown angles of 4 detectors → 5 degrees of freedom.



χ^2 Degree of Freedom

- ***Very poorly constrained nuisance parameters could affect test statistic to deviate from $\chi 2(1.0)$, if they cannot be profiled out in the data.***
- ***Nominal critical values for $\chi^2(1.0)$ cause under-coverage.***

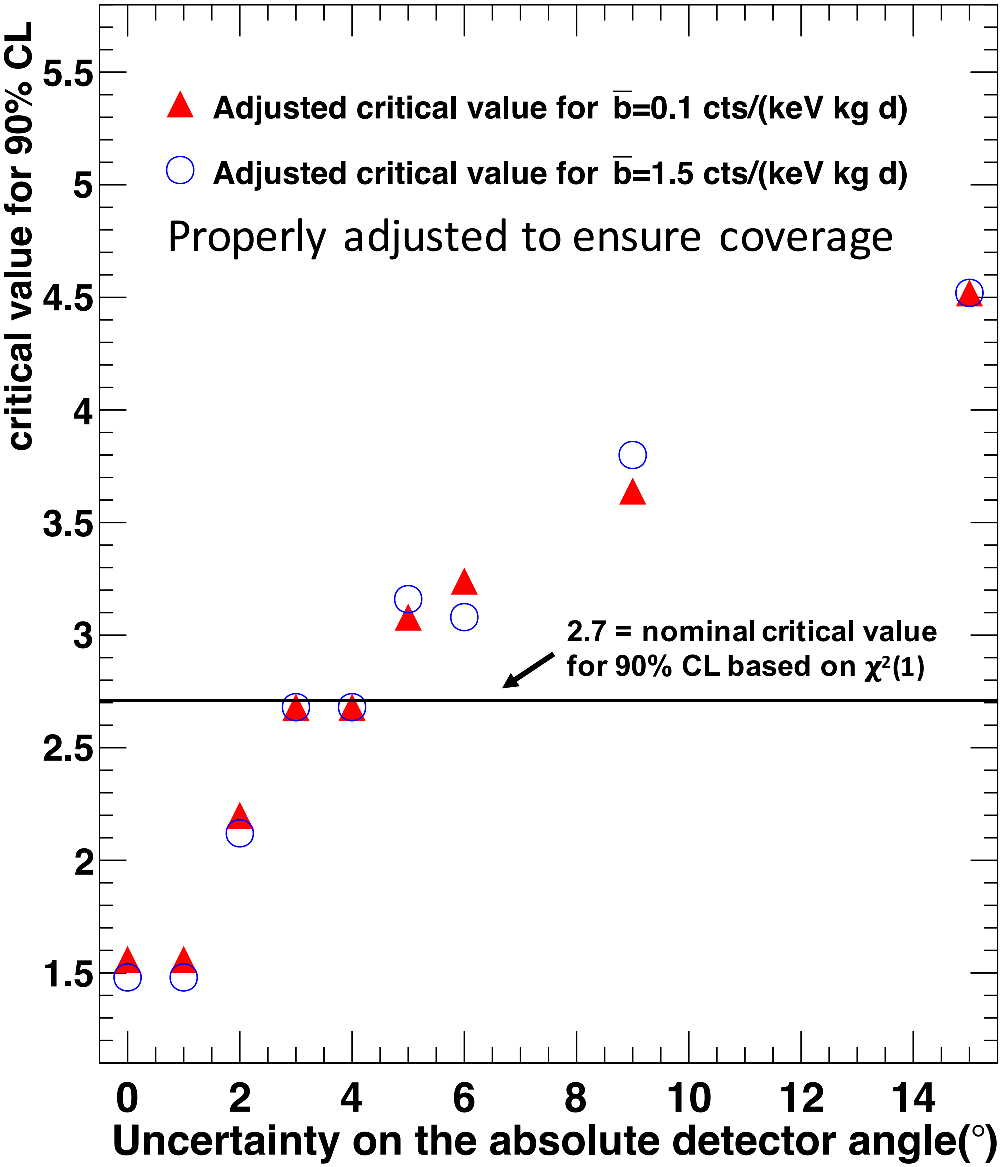
If $\lambda_{true} = 0$ and large uncertainty on n_D angles, almost χ^2 $(1+ n_D)$
How about tiny but non-zero λ_{true}?
Need extensive Monte Carlo simulations → Numerically obtain critical value

Is there a general solution without extensive Monte Carlo simulations?

Avoid the problem: measure all the angles
For one detector, *small enough* angular uncertainty
→ angle can be treated as known
→ Test statistic stays effectively χ^2 with 1 d.o.f.
→ For n_D detectors, all angles can be treated as known, still $\chi^2(1)$.

In this example: 15º uncertainty =17% of the allowed range of 90º, already too large
How small is small enough?

Maximum Angle Uncertainty

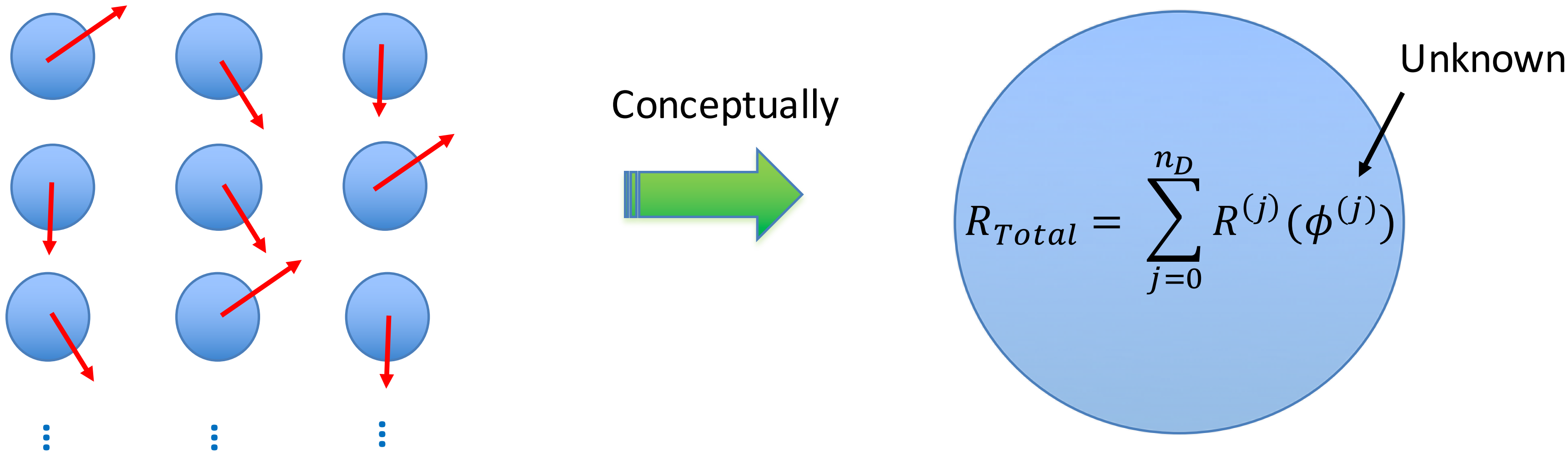


- Critical values for 90% C.L. determined based on the test statistic distribution in Monte Carlo simulations for 1 detector.
- 3º-4º is the maximum uncertainty for test statistic to approximate χ^2 (1)

Experiment	n_D (# of detectors)	Angle precision
SOLAX, DAMA, EDELWEISS-II	1 9 10	No angle constraint at all (90º)
CDMS (PRL2009)	19	3º
MAJORANA	~ 50	Study ongoing

Angles Unknown

Current and future experiments have many detectors.
Challenging to measure every detector angle to a 3º-4º precision. ***Alternatively,***



- n_D detectors with equal mass M_D
- Joint likelihood.

- One detector with mass $n_D \times M_D$
- Reduce the dimension of the problem
- At the cost of losing individual detector information
- Single likelihood based on R_{Total}

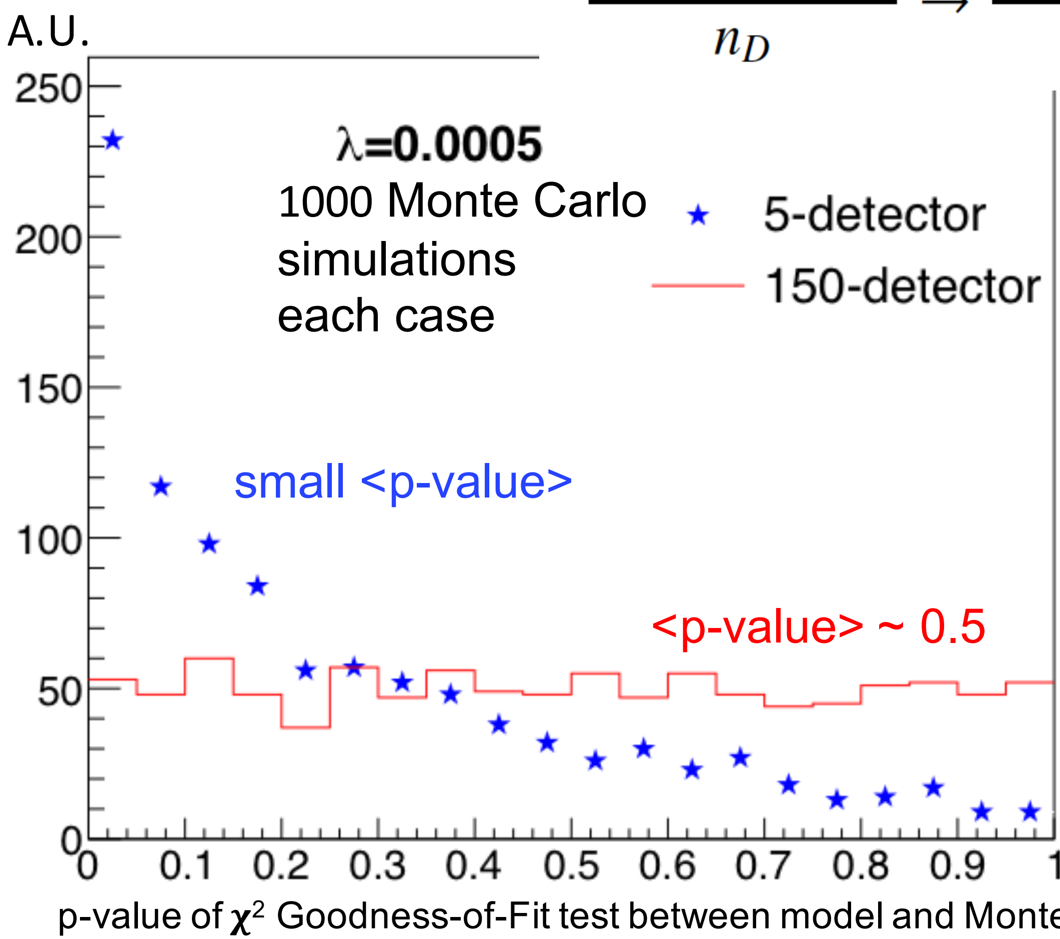
An Averaging Approximation

For numerous detectors, R_{Total} can be approximated using angle average (flat angle prior)

Unknown

$$\frac{\sum_{j=1}^{n_D} R^{(j)}(\phi^{(j)})}{n_D} \rightarrow \frac{\int_{-\pi/4}^{\pi/4} R(\phi)d\phi}{\pi/2}, \text{ if } n_D \rightarrow \infty$$

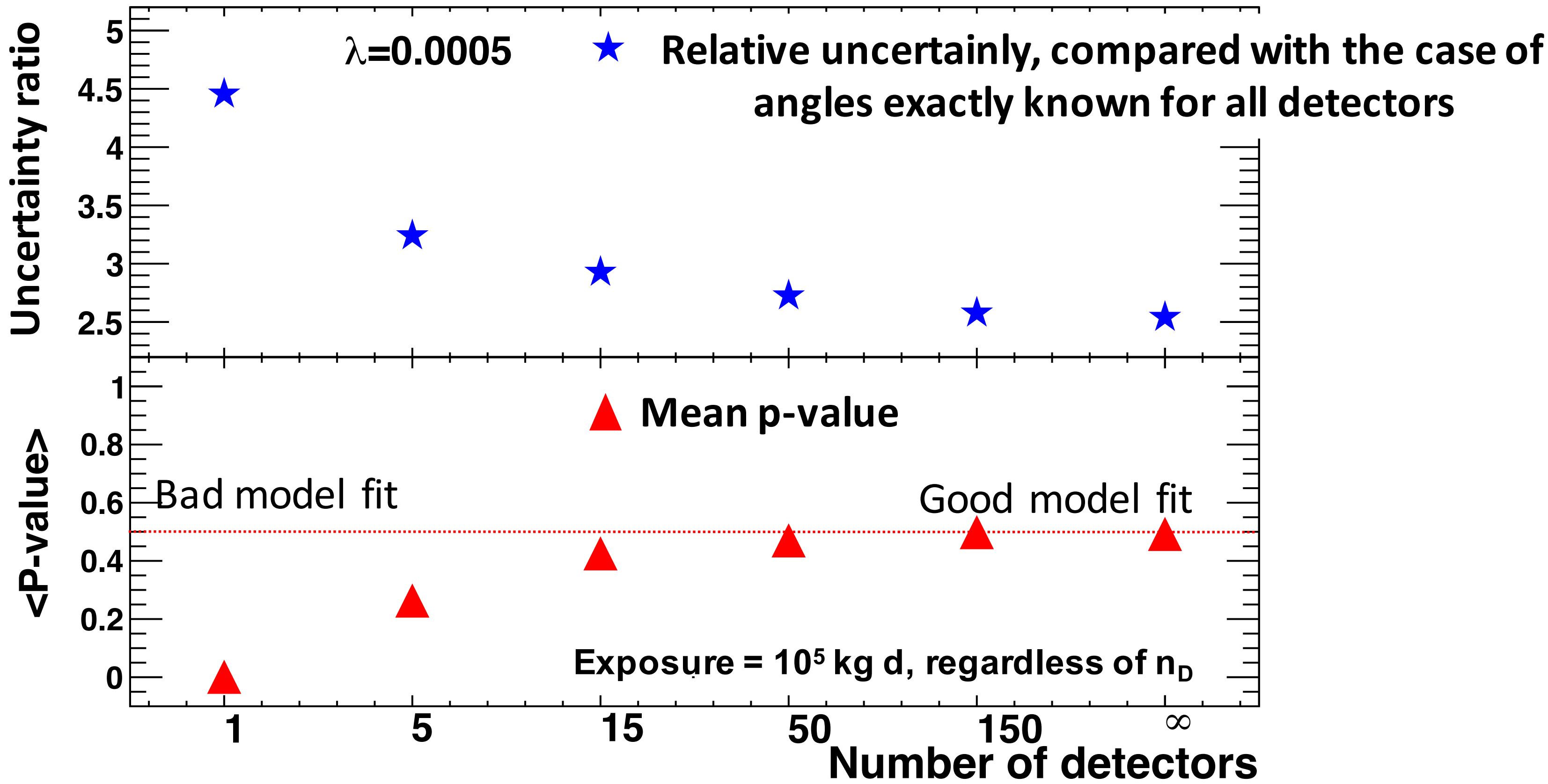
known



The model with approximation

- **well describes an experiment with 150 detectors**
- **does NOT describe an experiment with 5 detectors, model cannot be used for 5-detector experiment**

50-Detector Is Sufficient



- Close to 50 detectors is sufficient to use the averaging approximation in the model
- A factor of 2.5-3 increase of experimental uncertainty on the coupling constant, even for numerous detectors, due to loss of information.

Conclusions

Findings:

Very poorly constrained nuisance parameters cause likelihood-based test statistic to deviate from $\chi^2(1.0)$, if they cannot be profiled out in the data.

Physical boundary also affects the test statistic distribution as well.

In solar axion search:

Extensive Monte Carlo simulations + likelihood-based test statistic
→ Numerically obtain correct critical values for confidence intervals

OR measure angles to 3º-4º precision → test statistic ~ $\chi^2(1.0)$

OR use an averaging method with largely reduced sensitivity, if 50 or more detectors

Thank you!

Backup

Question: Both the axion-photon coupling and the angles can be treated as

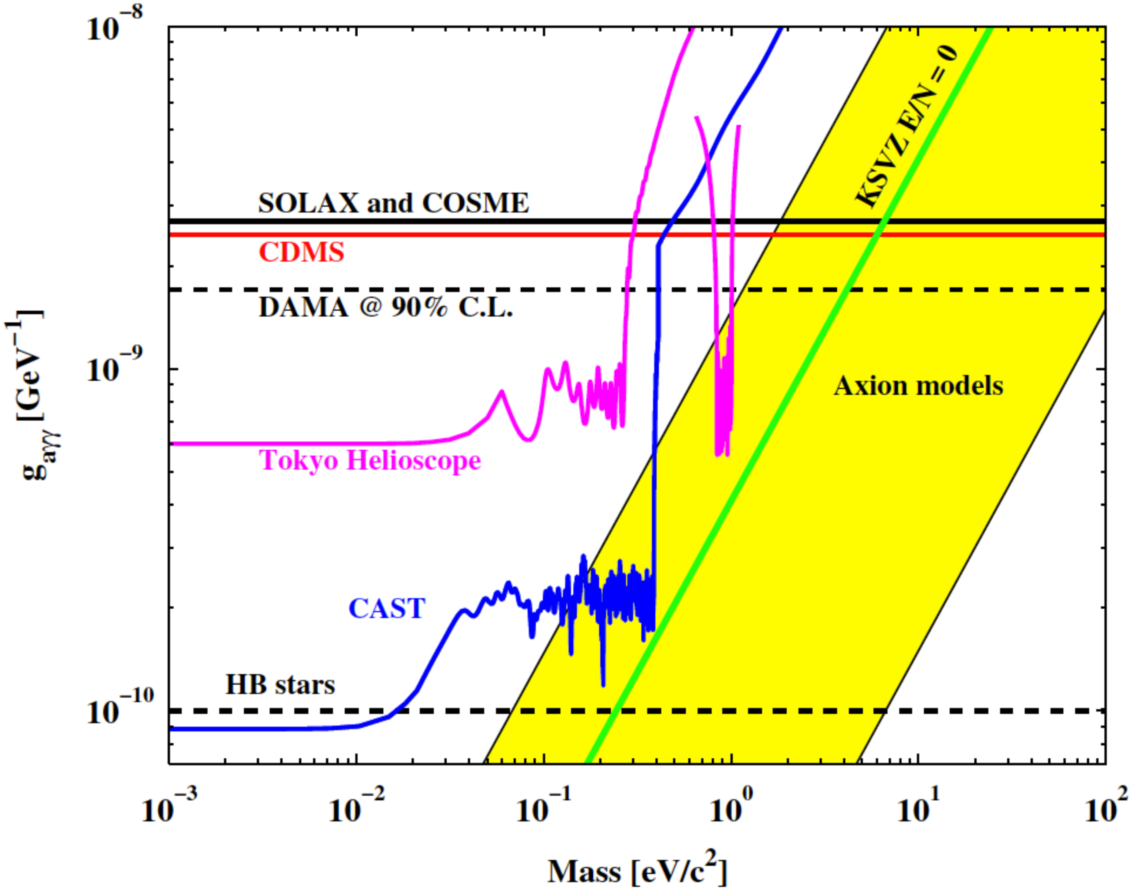
- parameters of interest
- little to none sensitivity from the data
- somewhat constrained otherwise (physical boundary, subsidiary measurements)

Statistical methods to treat this situation without extensive simulation?

Axions

Axion: originally postulated as the pseudo Nambu–Goldstone boson for the breaking of Peccei-Quinn U(1) symmetry, which is involved to explain the *strong CP problem*

Candidate for low mass dark matter



Upper limits on Axion-photon coupling constant.
Figure adapted CDMS Phys. Rev. Lett. 103, 141802 (2009)

Examples of search efforts:

Semi-conductor and solid scintillator (crystal) based:
SOLAX, CDMS, DAMA, EDELWEISS, TEXONO, **MAJORANA DEMONSTRATOR**

Man-made magnetic field based:
Tokyo Helioscope, CAST, ADMX, IAXO

Xenon based:
XMASS, XENON100

Space based:
XMM-Newton: Potential solar axion signatures in earth magnetic field
MNRAS **445,** 2146–2168 (2014)

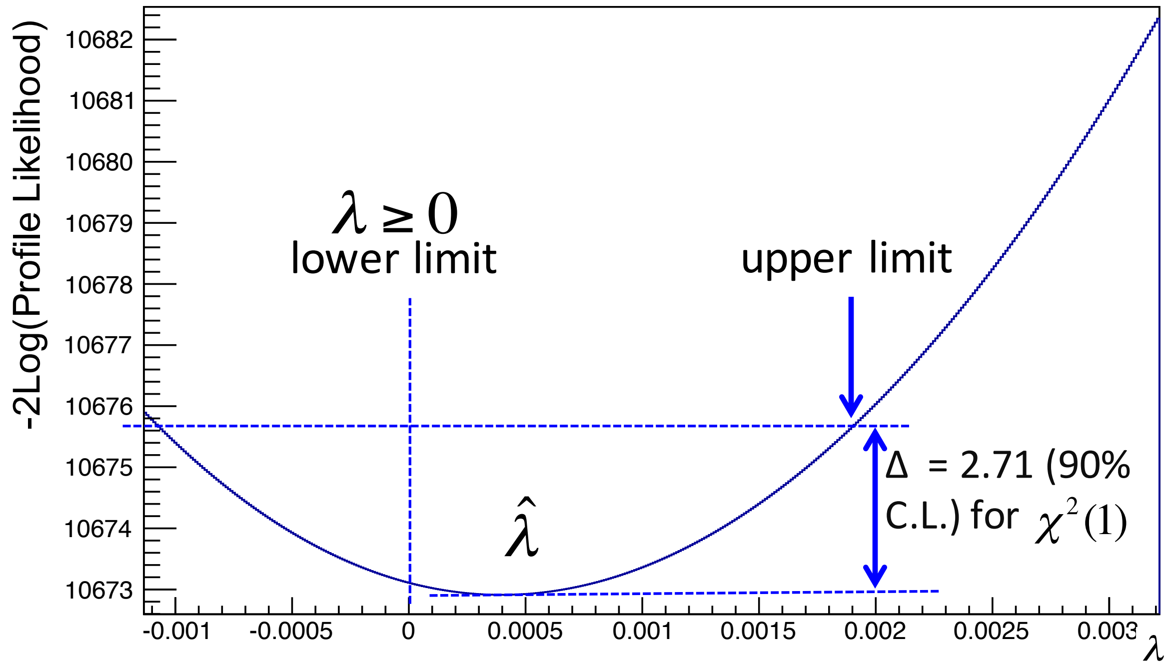
This talk only considers the Axion-photon coupling and experiments using crystals

Statistical Method

Construct likelihood function

$$L(\lambda,b,\vec{\phi}\,|\,\vec{X}) = \prod_{i=1}^{n} f(X_i\,|\,\lambda,b,\vec{\phi})$$

$\vec{\phi}$ are the azimuthal angles, and b is background level. They are the nuisance parameters



Profile Likelihood (Λ) = Maximized L against all nuisance parameters.

$$\Lambda(\lambda_0\,|\,\vec{X}) = \frac{\sup\{L(\lambda_0,b,\vec{\phi}\,|\,\vec{X});b,\vec{\phi}\}}{\sup\{L(\lambda,b,\vec{\phi}\,|\,\vec{X});\lambda,b,\vec{\phi}\}}$$

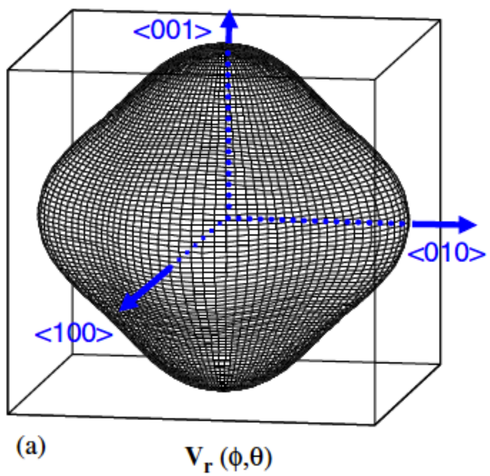
Rolke et al, NIMA 551 (2005) 493

Rolke et al, NIMA 551 (2005) 493:
“ $-2\log\Lambda$ has an approximated χ^2 distribution with 1 degree of freedom” i.e. $\chi^2(1)$

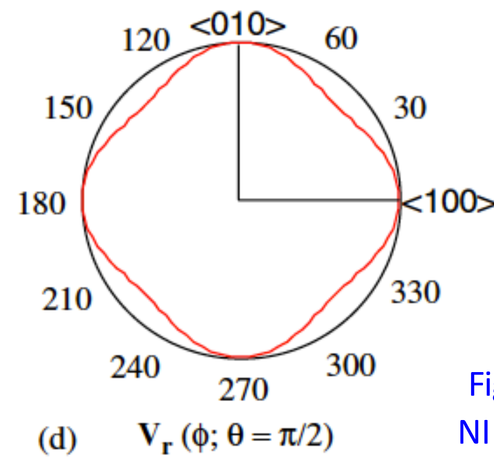
If $\chi^2(1)$ is indeed followed, change $-2\log\Lambda$ by 2.71 (3.84) for 90% (95%) confidence level.

- Physical limits and other factors will change the distribution of profile likelihood.
- Need adjustments to ensure coverage.

In-situ Angle Measurement



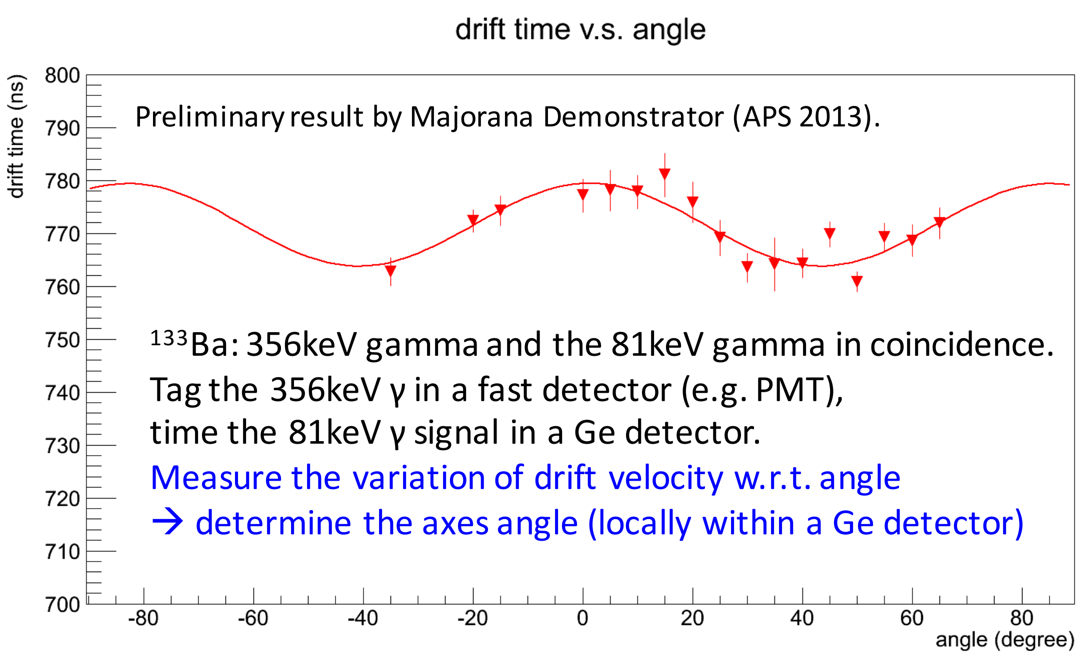
<001> always points up.



V_r is the radial component of the drift velocity.

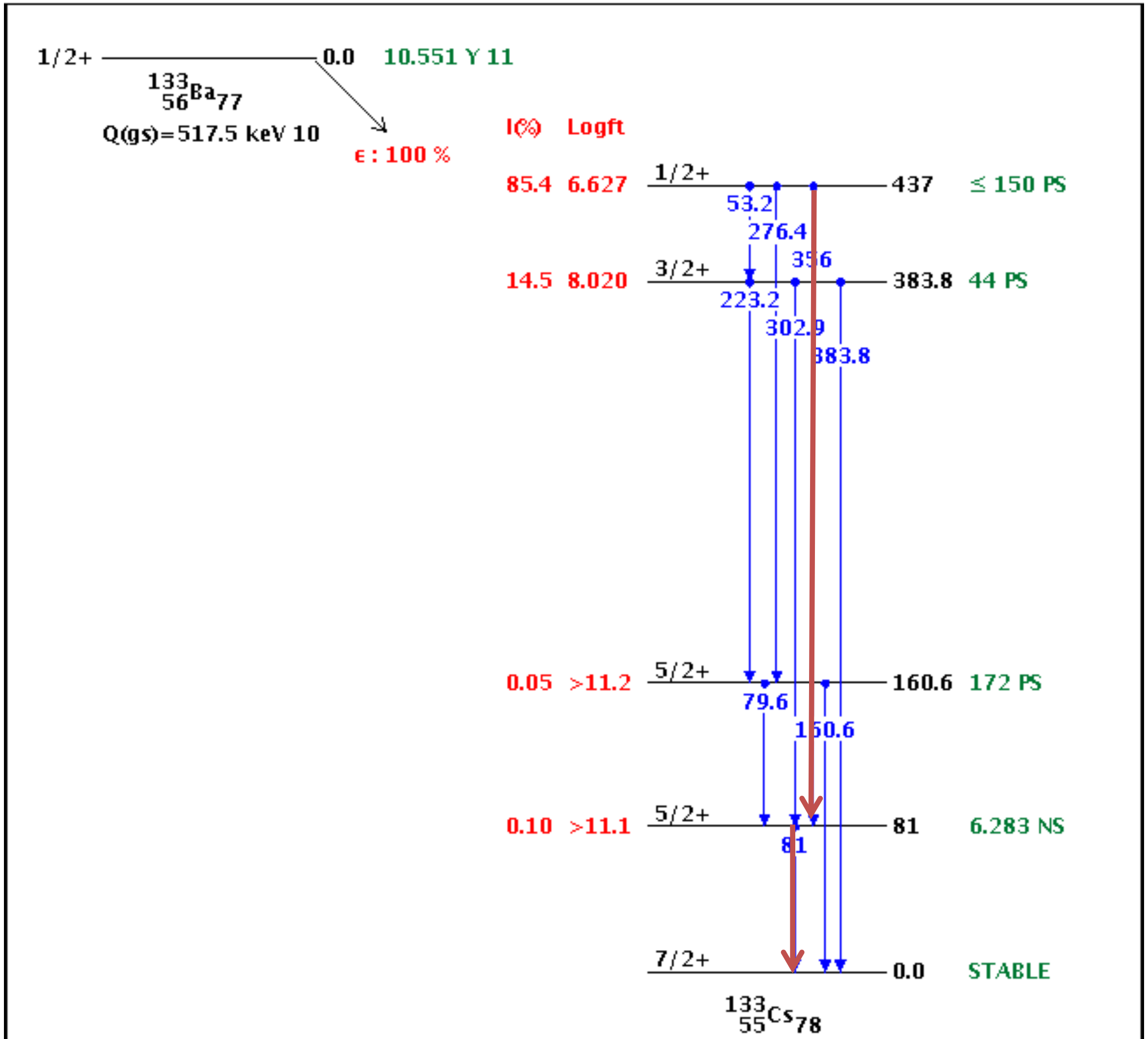
Figures adapted from NIM A 569 (2006) 764–773

The charge carrier velocity has an angular dependence.



Alternatively, use a Laue measurement to study the Ge detectors:
* Higher angular precision
* May introduce more surface time, undesired for underground Low background Ge experiment

Experiment design



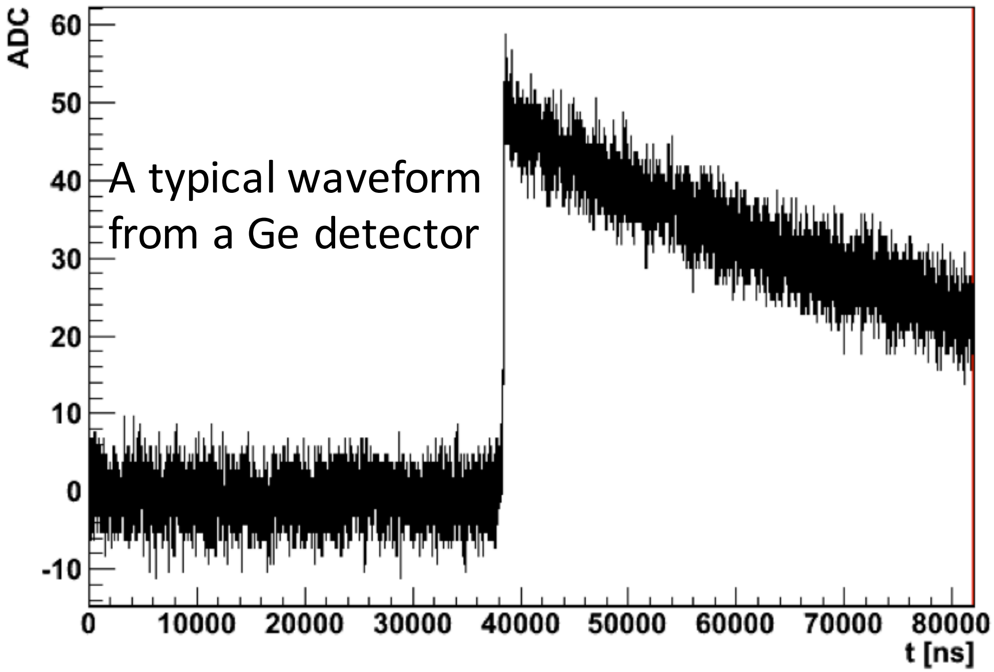
Utilize the coincidence between the 356keV gamma and the 81keV gamma of ^{133}Ba

For ^{133}Ba radioactive decay, the 356keV gamma is always immediately (~ns) followed by a 81keV gamma.

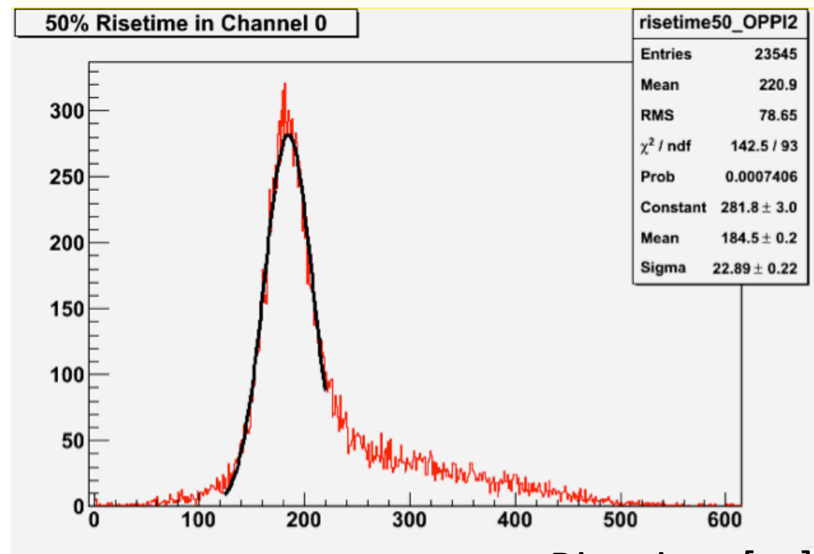
Use the 356keV gamma signal as a the starting time of an event.

Use the 81keV gamma to probe the Ge detector.
Most of this low energy gammas will not penetrate deep into Ge
→ similar drift distance
→ Avoids smearing in the drift time

Example efforts



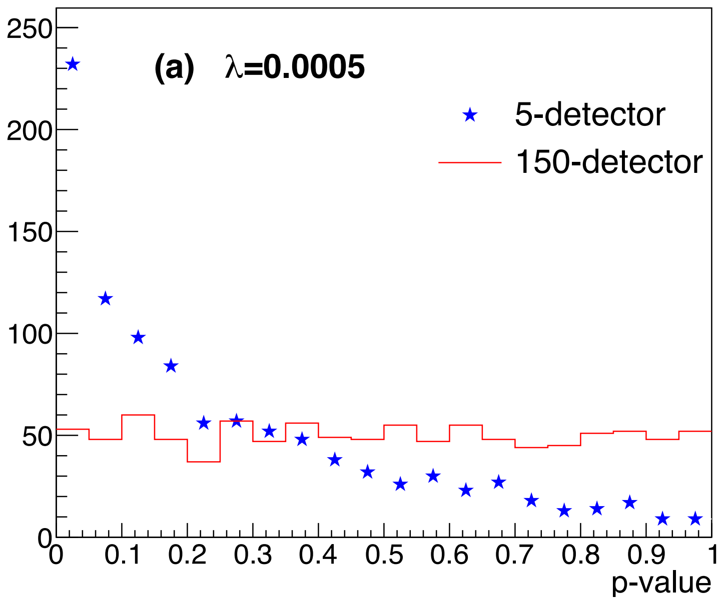
A typical 50% - 10% rise time distribution



Rise time [ns]

Exact treatment of waveform varies, typically including:
Baseline removal, smoothing (e.g. via averaging) and pole-zero correction.
Typically 50% or 90% rise time of the Ge detector signal are regarded as the end of the drift
→ Angle dependent
50% or 10% rise time of the scintillation signal are regarded as the start of the drift
→ Common for all angles on average

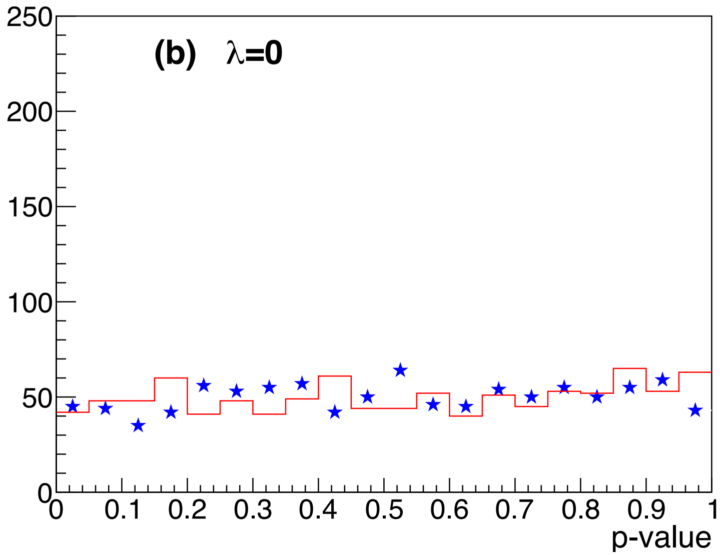
Goodness of Fit with Approx.



- Model’s single likelihood function constructed with the ***approximation.***
- To describe Monte Carlo data. Does it work?

Perform χ^2 Goodness of Fit test, convert χ^2 to p-value

If axion signals exist, the model (with the approximation)
well describes an experiment with 150 detectors,
does NOT describe an experiment with 5 detectors
model cannot be used for 5-detector expt.

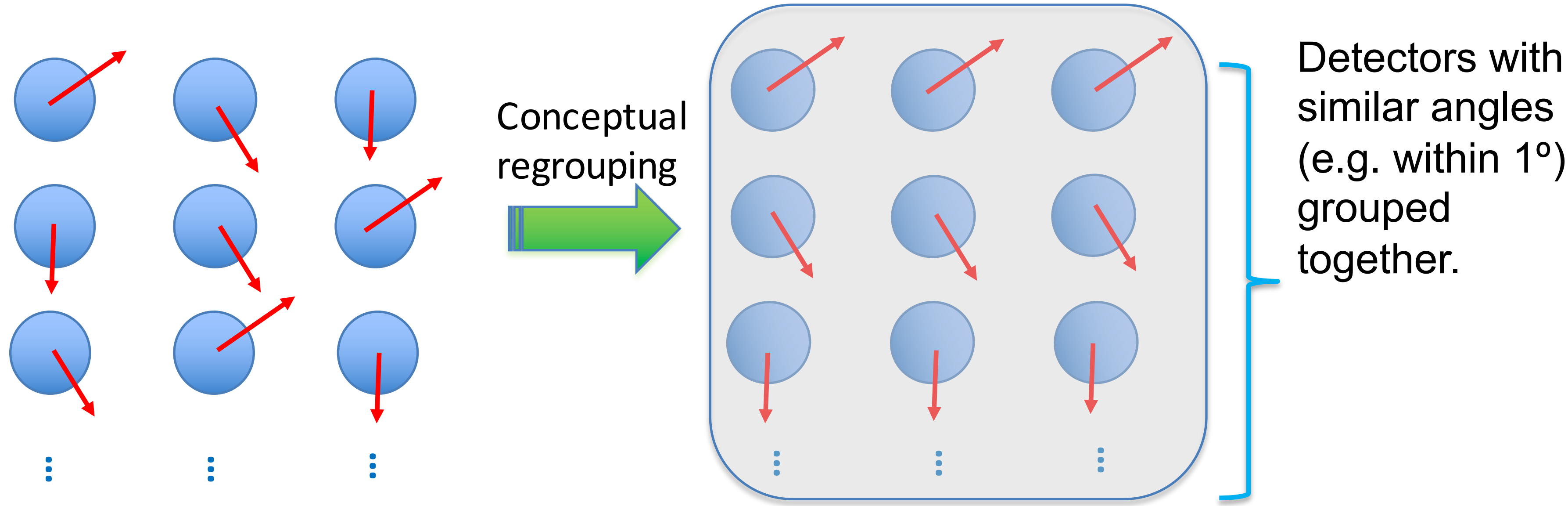


If no axion signal at all, the model
well describes an experiment regardless of the detectors.

No axion signals → expression of R_{Total} is irrelevant

Alternatively

Current and future experiments many detectors
Tremendous challenge to measure every detector angle to 3º-4º precision



n_G groups (e.g. $n_G = 90$), with equally n_D/n_G detectors in each group, if $n_D \to \infty$.
Flat angle prior assumed.

Treat n_D detectors with equal mass M_D as ndetector with mass $n_D \times M_D$

$$\frac{\sum_{j=1}^{n_D} R^{(j)}(\phi^{(j)})}{n_D} \to \frac{\int_{-\pi/4}^{\pi/4} R(\phi)d\phi}{\pi/2}, \text{ if } n_D \to \infty$$