Quark Fluctuations and Correlations at Mid-Rapidity in Heavy-Ion Collisions

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Outline

1) **Motivation:** Fluctuations in Heavy-Ion Collisions

2) **Background:** Quark Production in High-Energy QCD

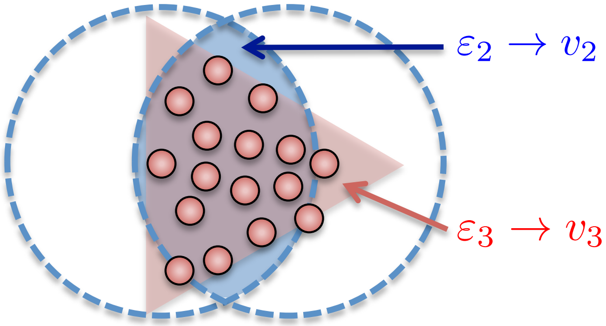
3) **Our Calculation:** Quark and Antiquark Correlations

4) **Preliminary Results:** Correlation Lengths in Heavy-Ion Collisions

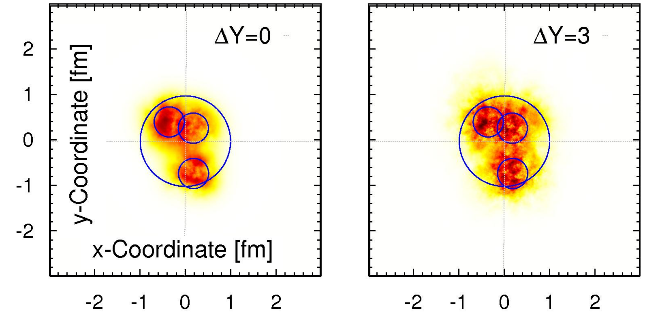
5) **Status and Outlook:** Progress and Future Extensions

The Importance of Fluctuations

- Many observables in heavy-ion collisions are driven by fluctuations
  - E.g., Triangular Flow (v_3)

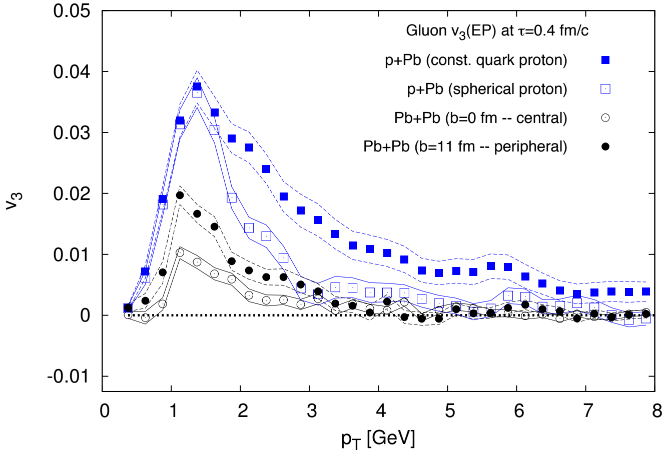


Schlichting & Schenke, Phys. Lett. **B739** (2014) 313



- Fluctuations occur at all length scales:
  - Geometrical fluctuations (Glauber)
  - Sub-nucleonic pQCD fluctuations

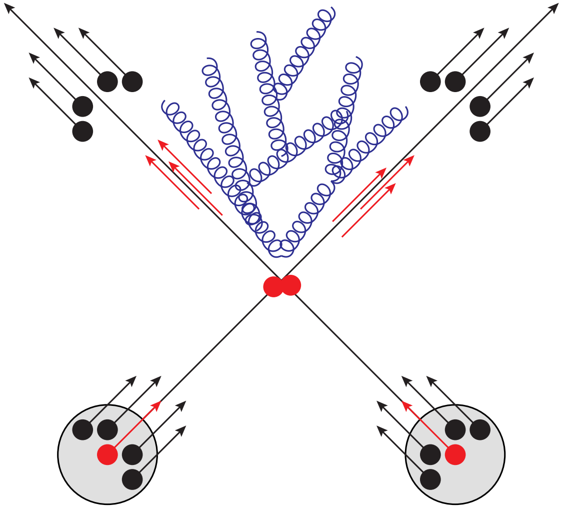
Schenke, Schlichting & Venugopalan, Phys. Lett. **B747** (2015) 76



- Initial-state fluctuations can leave an imprint on final-state observables
  - Dampened by viscous hydrodynamics

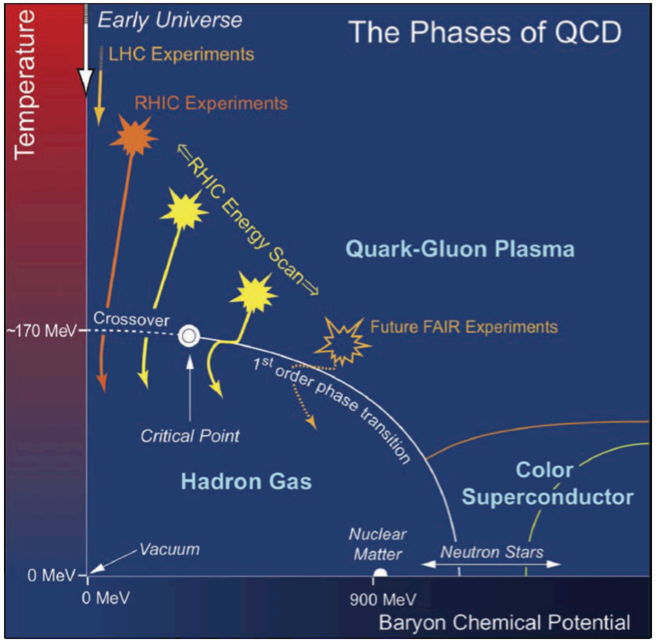
The Relevance of Baryon Number

- Production of net baryon number (baryon stopping) is suppressed at high energies
  - ➢ More important at lower energies



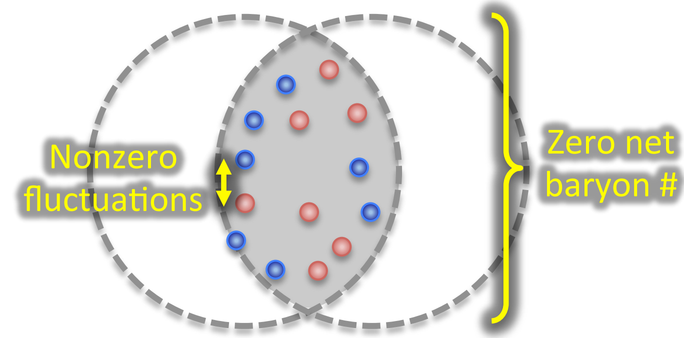
- With the RHIC Beam Energy Scan program, there is an emphasis on heavy-ion collisions including finite baryon number

- State-of-the-art hydrodynamic codes can propagate the effects of baryon number from the initial state to freeze-out.

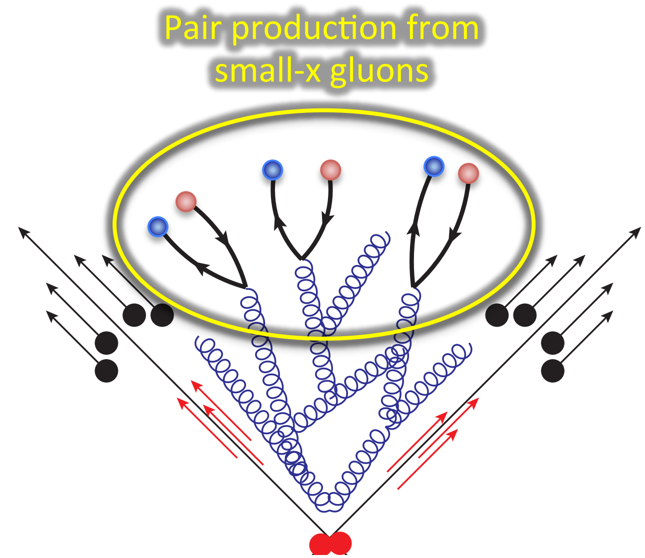


The Proposal

- Even at high energies where net baryon number vanishes, there are nonzero fluctuations in the initial state.



- Calculate the initial-state fluctuations and correlations of quarks and antiquarks
  - What are the typical scales of quark, antiquark, and baryon # fluctuations?

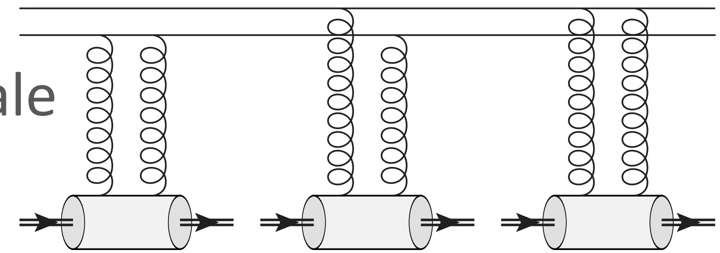


- How does this impact the final state?
  - Impact on hydrodynamic evolution?
  - Influence on soft particle production, hard probes, etc.?

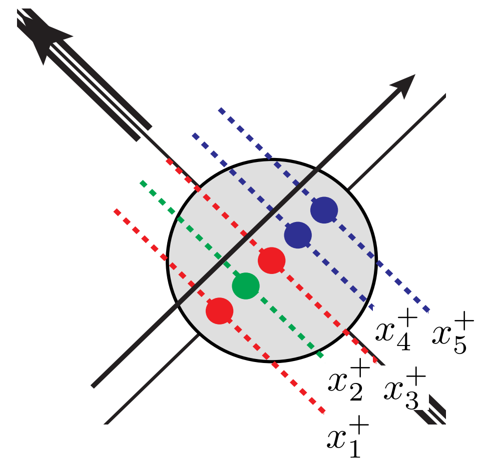
The Formalism: QCD at High Energies

- Dilute-Dense (pA) resummation of QCD
  - ➢ Dense target: classical gluon fields
  - ➢ High density sets hard momentum scale

$$\alpha_s^2 A^{1/3} \sim 1 \qquad Q_s^2 \sim \alpha_s \rho$$



- Light-cone “time”-ordered dynamics
  - ➢ Wave functions in light-front perturbation theory ($A^+=0$ gauge)
  - ➢ Eikonal scattering via Wilson lines



D. Wertepny, Ph. D. Thesis (2016) arXiv: 1608.08618

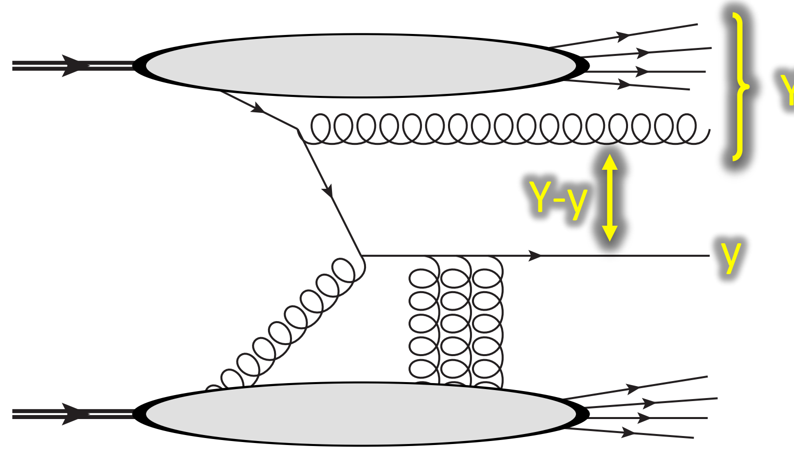
- “Heavy-light” (aA) paradigm incorporates projectile density order by order in perturbation theory
  - ➢ E.g.) ^{3}He+Au or Cu+Au
  - ➢ Moves toward the dense-dense limit

$$\alpha_s^2 A^{1/3} \sim 1$$
$$\alpha_s \ll \alpha_s^2 a^{1/3} \ll 1$$

Itakura et al., Nucl. Phys. **A730** (2004) 160

- Baryon stopping: transports valence quarks to mid-rapidity
  - Suppressed at high energies

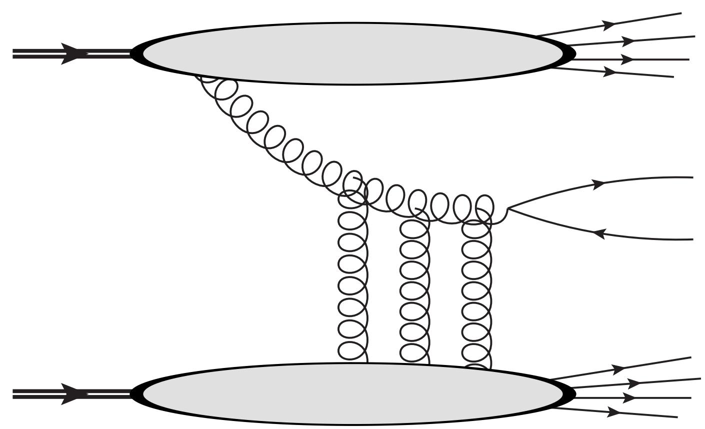
$$\frac{d\sigma^{val}}{dy} \sim e^{-(Y-y)} \qquad \langle n^q \rangle = \langle n^{\bar{q}} \rangle = 0$$



Levin et al., Sov. J. Nucl. Phys. **53** (1991) 657
Blaizot, Gelis, & Venugopalan, Nucl. Phys. **A743** (2004) 57
Kovchegov & Tuchin, Phys. Rev. **D74** (2006) 054014

- Pair production from small-x gluons
  - Only suppressed by α_s
  - Enhanced by strong gluon fields
  - Does not produce net baryon number, only fluctuations

$$\langle n^q(x_\perp)\, n^{\bar{q}}(y_\perp) \rangle \neq 0$$

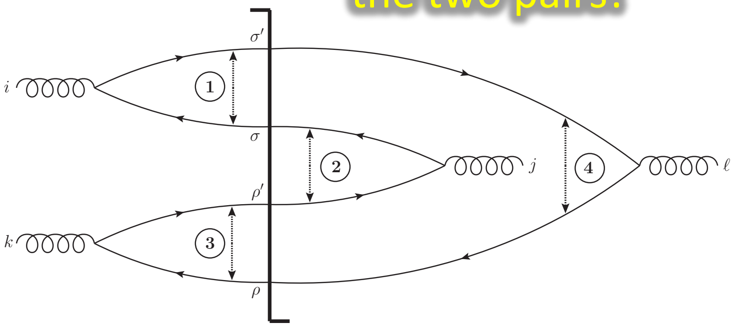


Sources of Quark-Quark Correlations

Altinoluk et al. (2016) arXiv: 1610.03020

- Quark-quark correlations via double pair production in heavy-light ion collisions
  - Enhanced by the density of the light ion

Entanglement between the two pairs!



- One source of correlations: quantum entanglement in the wave functions
  - Pauli blocking

$$\frac{dN_{correlated}}{d^2p_1\, dy_1\, d^2p_2\, dy_2} \propto -(\Delta y)^4\, e^{-\Delta y}$$

- Many other contributions to include
  - Other topologies beyond large N_c
  - Large number of time-orderings for the interactions

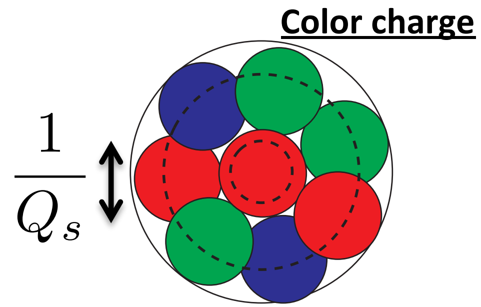
Our Calculation: Setup

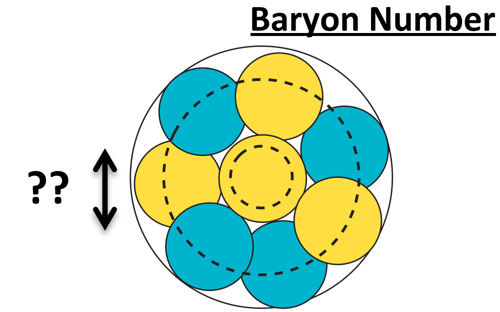
- **Goal**: Calculate (anti)quark correlations in heavy-light ion collisions
  - ➢ Coordinate space profile for hydro
  - ➢ What is the typical size of a baryon number domain?

$$\mathcal{C}^{q\bar{q}}(B_{1\perp}, Y_1; B_{2\perp}, Y_2) = \left\langle \frac{dn^q}{d^2B_1\, dY_1} \frac{dn^{\bar{q}}}{d^2B_2\, dY_2} \right\rangle - \left\langle \frac{dn^q}{d^2B_1\, dY_1} \right\rangle \left\langle \frac{dn^{\bar{q}}}{d^2B_2\, dY_2} \right\rangle$$

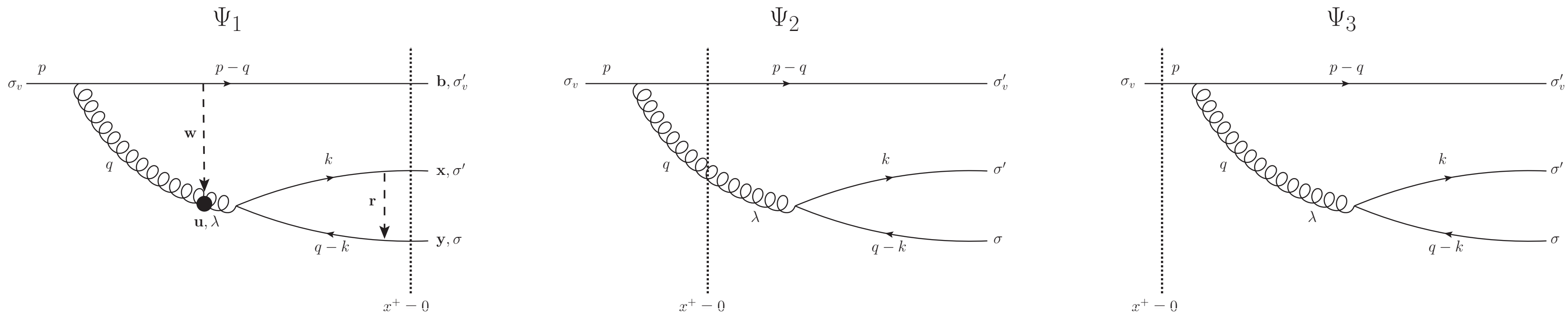
$$\left\langle \frac{dn^q}{d^2B_1\, dY_1} \frac{dn^{\bar{q}}}{d^2B_2\, dY_2} \right\rangle = \frac{1}{\sigma_{inel}} \frac{d\sigma^{q\bar{q}}}{d^2B_1\, dY_1\, d^2B_2\, dY_2}$$

$$\left\langle \frac{dn^q}{d^2B_1\, dY_1} \right\rangle = 0$$





- Various mechanisms dominate at various length scales
  - ➢ Work to LO in each regime
  - ➢ Avoid complicated rescattering corrections



- Light-front wave functions to radiate a quark-antiquark pair
  - Three time orderings for scattering in the target
  - Only two are linearly independent
  - Allow for the possibility of heavy quark production

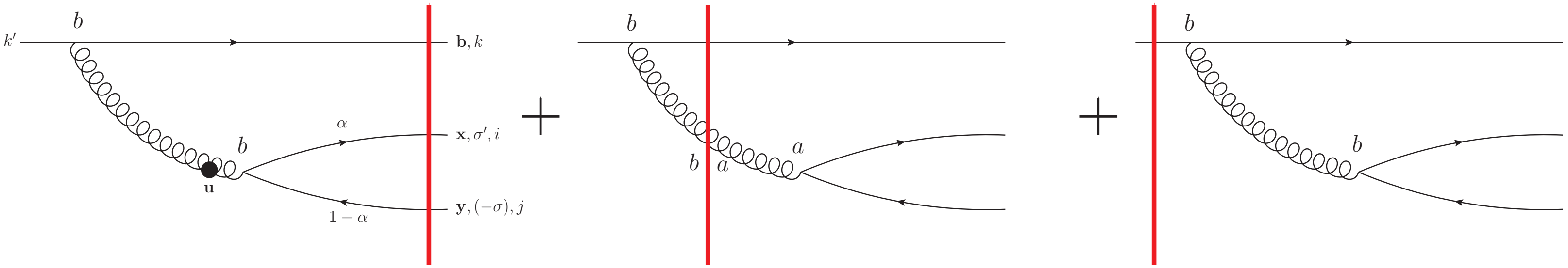
E.g.)

$$\Psi_2(w_\perp, r_\perp, \alpha) = -\frac{2\alpha_s}{\pi}\sqrt{\alpha(1-\alpha)}\left\{\delta_{\sigma,-\sigma'}\frac{m}{w_T}K_1(mr_T)\left[(1-2\alpha)\frac{\vec{w}_\perp \cdot \vec{r}_\perp}{w_T r_T} - i\sigma'\frac{\vec{w}_\perp \times \vec{r}_\perp}{w_T r_T}\right]\right.$$
$$\left. + i\sigma'\delta_{\sigma\sigma'}\frac{m}{w_T}K_0(mr_T)\left[\frac{w_\perp^1}{w_T} - i\sigma'\frac{w_\perp^2}{w_T}\right]\right\}$$

$$\Psi_3 = -\Psi_1 - \Psi_2$$

Time orderings: $\Psi = \Psi_i \qquad i = 1, 2$

Ingredients: Wilson Lines



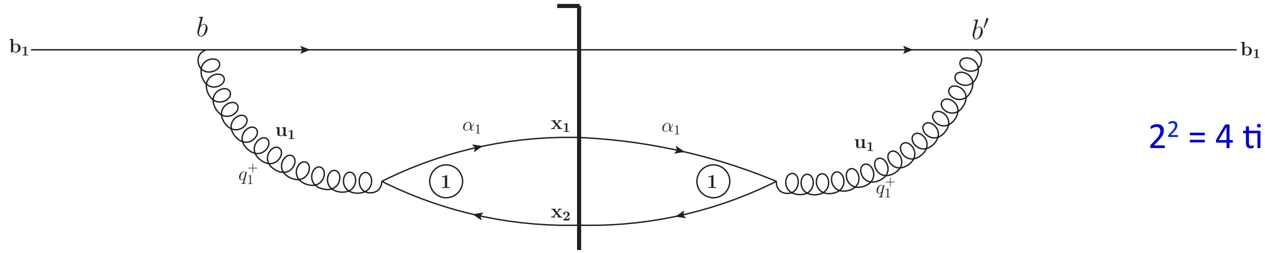
$$\mathcal{A}^{q\bar{q}} = (V_{b_\perp} t^b) \left[\left(V_{x_\perp} t^b V^\dagger_{y_\perp} - V_{b_\perp} t^b V^\dagger_{b_\perp} \right) \Psi_1 + \left(V_{u_\perp} t^b V^\dagger_{u_\perp} - V_{b_\perp} t^b V^\dagger_{b_\perp} \right) \Psi_2 \right]$$

- Wilson lines: Color rotation phase from multiple scattering
  - ➢ Use the quasi-classical approximation for now

$$V_{x_\perp} = \exp \left[ig \int dz^+ \hat{A}^-(0^-, z^+, x_\perp) \right] \qquad\qquad \alpha_s \Delta Y \leq 1$$

- Include all possible entangled topologies (finite N_c)
  - ➢ Obtain general results in terms of Wilson line multipoles that can be evaluated event-by-event using Monte Carlo

$$D_2(x_\perp, y_\perp) = \frac{1}{N_c} tr[V_{x_\perp} V^\dagger_{y_\perp}] \qquad\qquad D_4(x_\perp, y_\perp, z_\perp, w_\perp) = \frac{1}{N_c} tr[V_{x_\perp} V^\dagger_{y_\perp} V_{z_\perp} V^\dagger_{w_\perp}]$$



2^2 = 4 time orderings

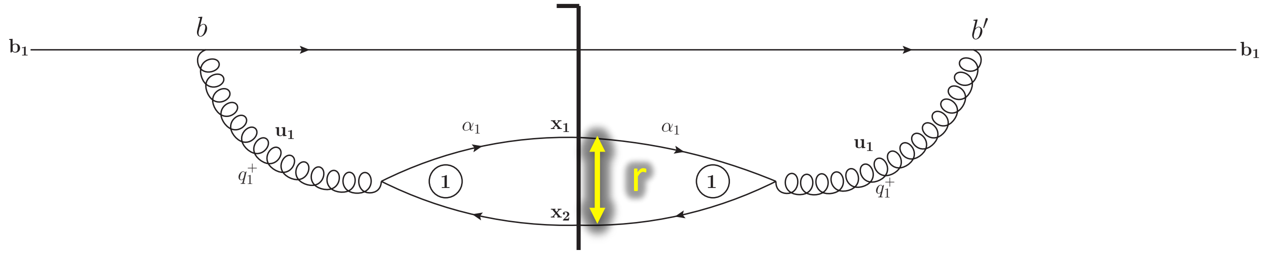
$$\mathcal{C}^{q\bar{q}} = \frac{1}{\sigma_{inel}} \frac{1}{(4\pi)^2} \int d^2B \, d^2b_1 \, T_a(b_1 - B \ ) \int d^2x_1 \, d^2x_2 \, dy_1 \, dy_2 \, \frac{1}{2N_c} tr[\Psi_i \Psi_j^\dagger] tr[W_i W_j^\dagger]$$

$$\delta^2(x_1 - B_1)\delta(y_1 - Y_1)\delta^2(x_2 - B_2)\delta(y_2 - Y_2)$$

$$tr[W_1 W_1^\dagger] = 2N_c C_F - \frac{N_c^2}{2} D_2(b_1, x_2) D_2(x_1, b_1) - \frac{N_c^2}{2} D_2(x_2, b_1) D_2(b_1) + \frac{1}{2} D_2(x_1, x_2) + \frac{1}{2} D_2(x_2, x_1)$$

- Single-pair production governs the short-distance correlations
  - Same as in proton-nucleus collisions
  - Physics driven by the single pair production WF
  - Interactions by Wilson line dipoles (and double dipoles)

Single Pair Production at Short Distances

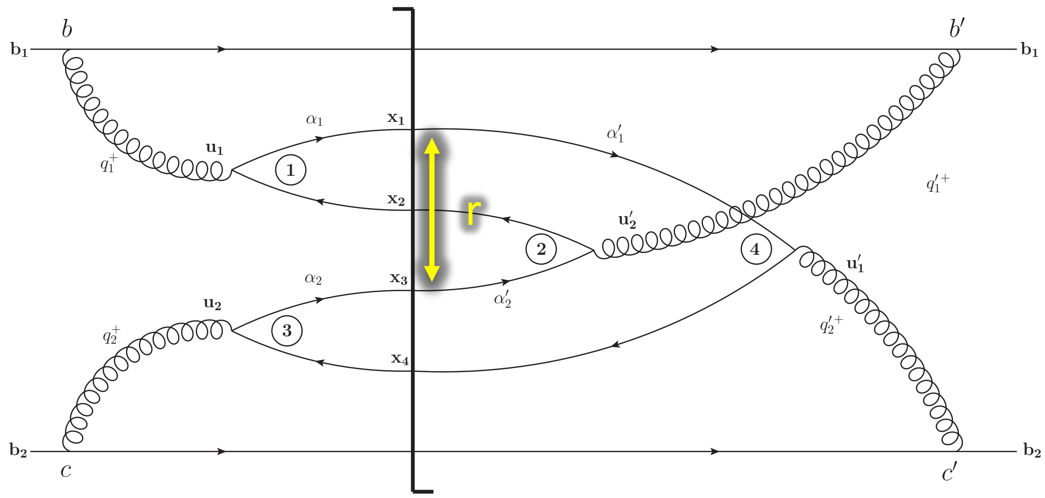


- Correlations are boost-invariant in the pair center of mass…
  - ➢ But nontrivial dependence on the rapidity difference

- The saturation scale Q_s acts as an intermediate scale, with some (but not all) interaction terms falling off exponentially with the separation:

For $r_T \gg 1/Q_s$ in the MV model:

$$tr[W_1 W_1^\dagger] = 2N_c C_F$$
$$tr[W_1 W_2^\dagger] = tr[W_2 W_1^\dagger] = \frac{1}{2} tr[W_2 W_2^\dagger] = N_c C_F \left(1 - e^{-\frac{1}{4}\frac{N_c}{C_F}|u_1 - b_1|_T^2 Q_s^2}\right)$$



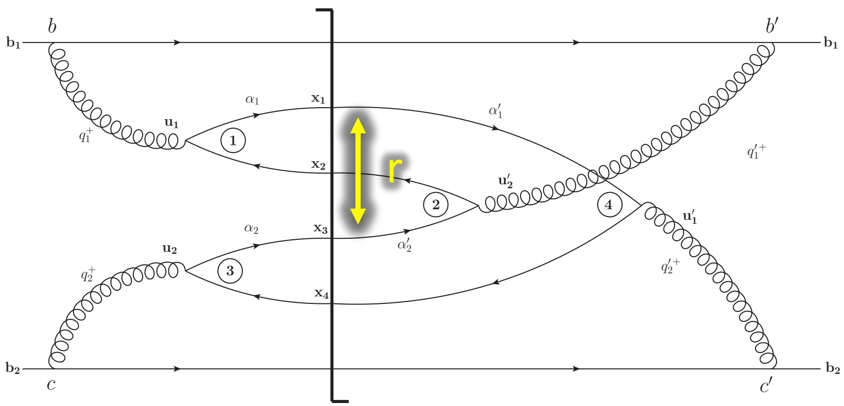
2^4 = 16 time orderings!

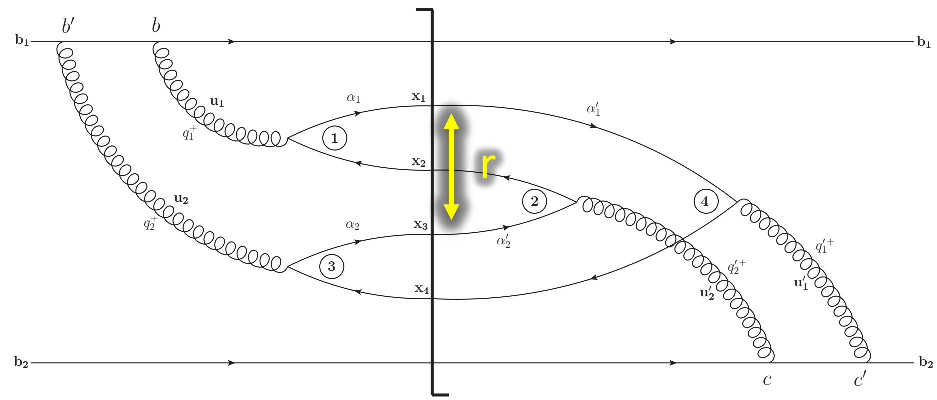
$$\mathcal{C}^{qq} = \frac{1}{\sigma_{inel}} \frac{1}{(4\pi)^4} \int d^2B \, d^2b_1 \, d^2b_2 \, T_a(b_1 - B) T_a(b_2 - B) \int d^2x_1 \, d^2x_2 \, d^2x_3 \, d^2x_4$$

$$\times \int dy_1 \, dy_2 \, dy_3 \, dy_4 \, \frac{1}{4N_c^2} tr[\Psi_i \Psi_j^\dagger \Psi_k \Psi_\ell^\dagger] \Omega_{ijk\ell}$$

$$\times \frac{1}{4} \Big[\delta^2(x_1 - B_1) \delta(y_1 - Y_1) \, \delta^2(x_3 - B_2) \delta(y_3 - Y_2) + \delta^2(x_3 - B_1) \delta(y_3 - Y_1) \, \delta^2(x_1 - B_2) \delta(y_1 - Y_2) \Big]$$

- (Anti)quark correlations at short distances require **double pair production**

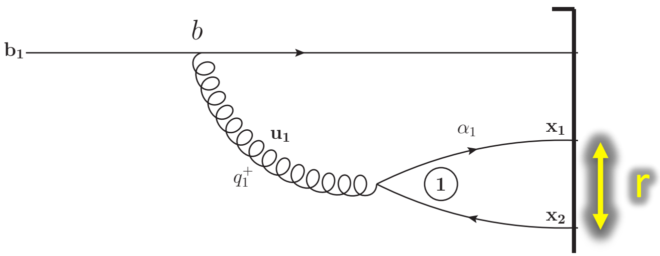


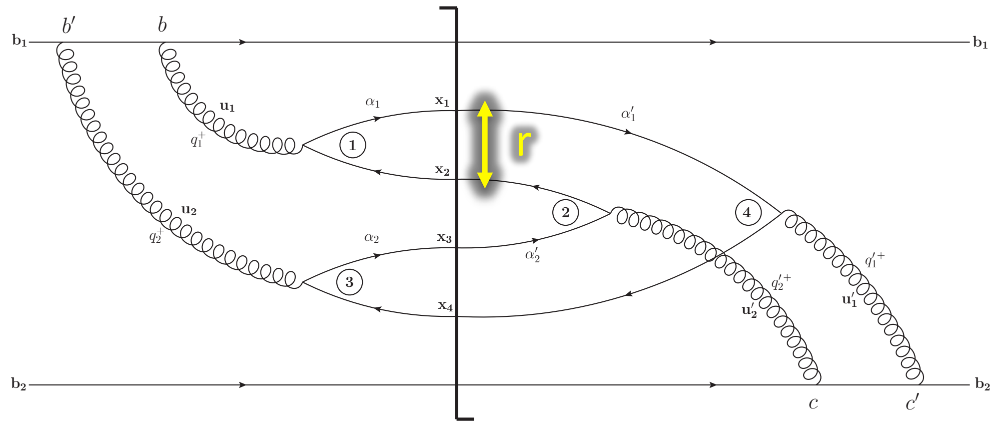


- Several independent channels (6) and time orderings (16)
- Complicated Wilson line interactions (~20-30 terms)
- Must contain the same physics of Pauli blocking

$$
\begin{aligned}
4\Omega_{1111}^{(\bar{q}\,ent)} &= 8C_F^2 N_c - 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{x}_2)\hat{D}_2(\boldsymbol{x}_1, \boldsymbol{b}_1) - 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{x}_4)\hat{D}_2(\boldsymbol{x}_1, \boldsymbol{b}_1) \\
&+ N_c^3 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{b}_2)\hat{D}_2(\boldsymbol{b}_2, \boldsymbol{x}_3)\hat{D}_2(\boldsymbol{x}_1, \boldsymbol{b}_1) - 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{x}_1)\hat{D}_2(\boldsymbol{x}_2, \boldsymbol{b}_1) \\
&- 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_2, \boldsymbol{x}_3)\hat{D}_2(\boldsymbol{x}_2, \boldsymbol{b}_2) + 4C_F^2 N_c \hat{D}_2(\boldsymbol{x}_2, \boldsymbol{x}_4) \\
&+ N_c^3 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{x}_1)\hat{D}_2(\boldsymbol{b}_2, \boldsymbol{b}_1)\hat{D}_2(\boldsymbol{x}_3, \boldsymbol{b}_2) - 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_2, \boldsymbol{x}_2)\hat{D}_2(\boldsymbol{x}_3, \boldsymbol{b}_2) \\
&- 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_2, \boldsymbol{x}_4)\hat{D}_2(\boldsymbol{x}_3, \boldsymbol{b}_2) - 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{x}_1)\hat{D}_2(\boldsymbol{x}_4, \boldsymbol{b}_1) \\
&- 2C_F N_c^2 \hat{D}_2(\boldsymbol{b}_2, \boldsymbol{x}_3)\hat{D}_2(\boldsymbol{x}_4, \boldsymbol{b}_2) + 4C_F^2 N_c \hat{D}_2(\boldsymbol{x}_4, \boldsymbol{x}_2) \\
&+ N_c^3 \hat{D}_2(\boldsymbol{x}_1, \boldsymbol{b}_1)\hat{D}_2(\boldsymbol{x}_3, \boldsymbol{b}_2)\hat{D}_4(\boldsymbol{b}_1, \boldsymbol{x}_2, \boldsymbol{b}_2, \boldsymbol{x}_4) \\
&+ N_c^3 \hat{D}_2(\boldsymbol{b}_1, \boldsymbol{x}_1)\hat{D}_2(\boldsymbol{b}_2, \boldsymbol{x}_3)\hat{D}_4(\boldsymbol{x}_4, \boldsymbol{b}_1, \boldsymbol{x}_2, \boldsymbol{b}_2) \\
&+ \cdots
\end{aligned}
$$

Pair Production Correlation Length

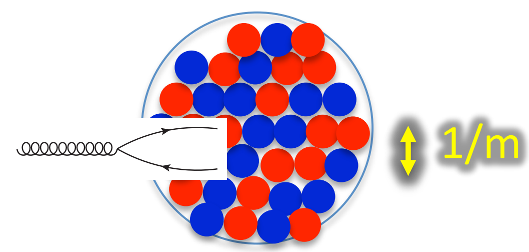




- The long-distance asymptotics of the splitting WF determine the correlation length of these channels:

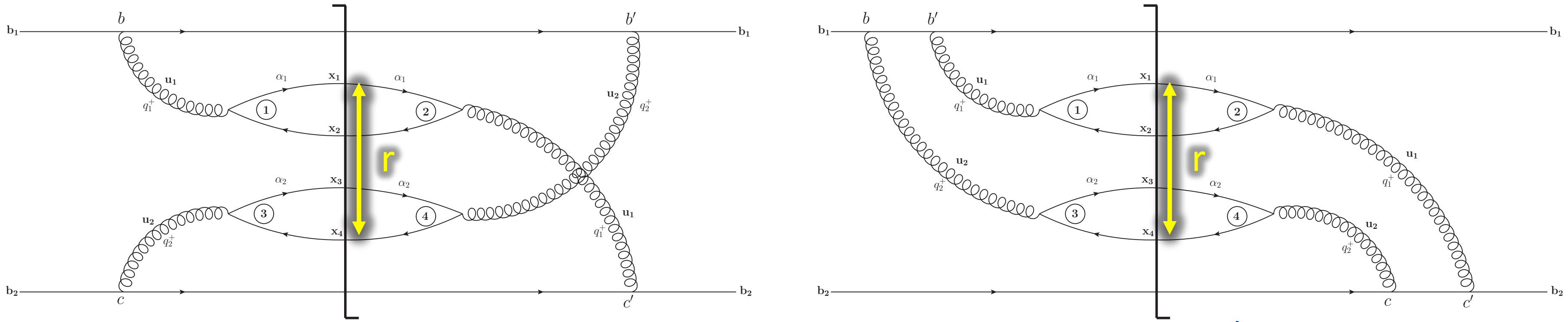
$$\Psi_i \sim e^{-m r_T}$$

$$tr[\Psi_i \Psi_j^\dagger] \sim tr[\Psi_i \Psi_j^\dagger \Psi_k \Psi_\ell^\dagger] \sim e^{-2 m r_T}$$



- For heavy quarks, the correlation length is set by the mass.
- For light quarks, must cut off by hand at Λ_{QCD}

Heavy Quarks at Intermediate Distances

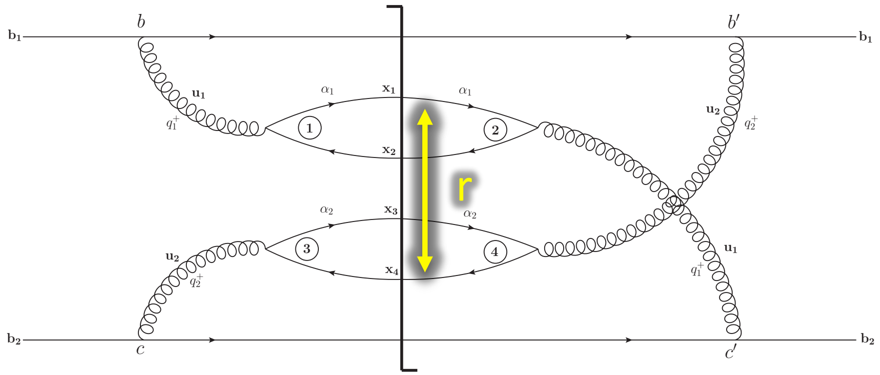


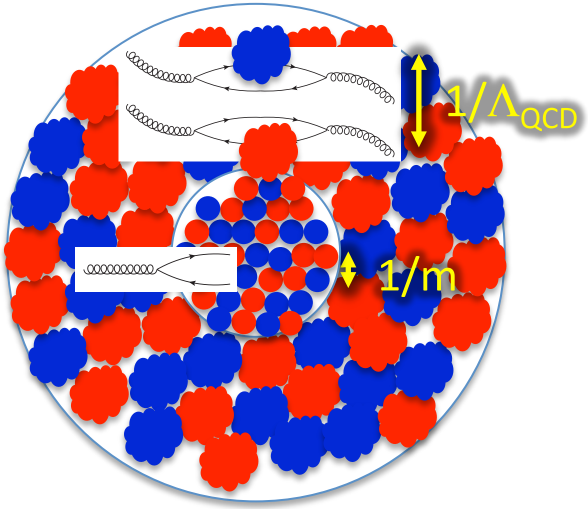
2^4 = 16 time orderings!

- Correlations over distances larger than 1/m are sensitive to double pair production (one particle from each pair)

$$r_T > \frac{1}{2m} \ln \frac{1}{\alpha_s^2 a^{1/3}}$$

- For heavy quarks, this occurs in the perturbative regime
  - Correlations remain from gluon entanglement and correlated scattering in the target field
  - Contributes to all (anti)quark correlations



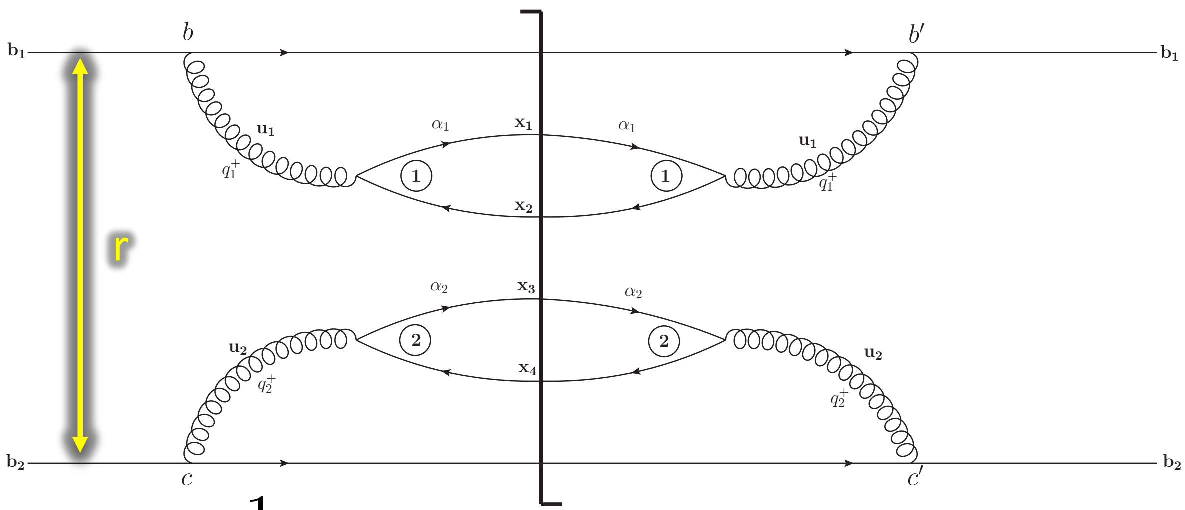


$$\frac{1}{2m}\ln\frac{1}{\alpha_s^2 a^{1/3}} < r_T < \frac{1}{\Lambda_{QCD}}$$

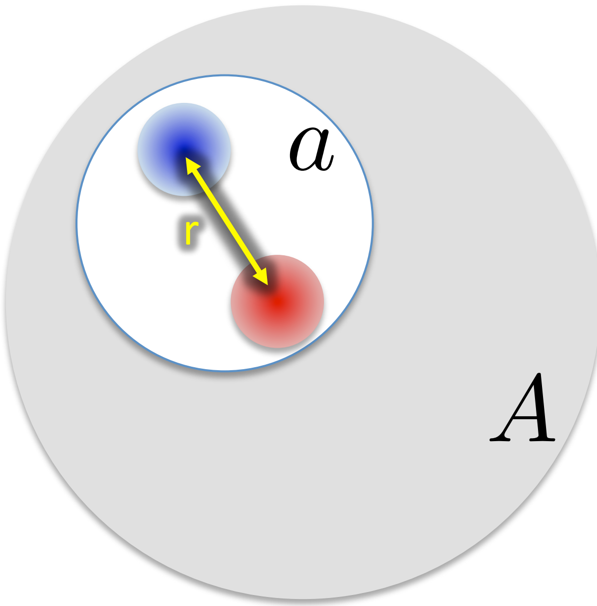
- Several independent channels (6) and time orderings (16)
- Still maximally complicated Wilson lines (~20-30 terms)

$$\begin{aligned}
4\Omega_{1112}^{(pair\ ent)} &= N_c^2 + N_c^2 \hat{D}_2(\boldsymbol{b}_1,\boldsymbol{b}_2)\hat{D}_2(\boldsymbol{b}_2,\boldsymbol{b}_1) - N_c^2\hat{D}_2(\boldsymbol{b}_1,\boldsymbol{u}_2)\hat{D}_2(\boldsymbol{u}_2,\boldsymbol{b}_1) \\
&- N_c^2\hat{D}_2(\boldsymbol{b}_2,\boldsymbol{u}_2)\hat{D}_2(\boldsymbol{u}_2,\boldsymbol{b}_2) - N_c^2\hat{D}_2(\boldsymbol{b}_1,\boldsymbol{x}_2)\hat{D}_2(\boldsymbol{x}_1,\boldsymbol{b}_1) + N_c^2\hat{D}_2(\boldsymbol{u}_2,\boldsymbol{x}_2)\hat{D}_2(\boldsymbol{x}_1,\boldsymbol{u}_2) \\
&- N_c^2\hat{D}_2(\boldsymbol{b}_2,\boldsymbol{x}_1)\hat{D}_2(\boldsymbol{x}_2,\boldsymbol{b}_2) - N_c^2\hat{D}_2(\boldsymbol{b}_1,\boldsymbol{x}_4)\hat{D}_2(\boldsymbol{x}_3,\boldsymbol{b}_1) - N_c^2\hat{D}_2(\boldsymbol{b}_2,\boldsymbol{x}_4)\hat{D}_2(\boldsymbol{x}_3,\boldsymbol{b}_2) \\
&+ N_c^2\hat{D}_2(\boldsymbol{u}_2,\boldsymbol{x}_4)\hat{D}_2(\boldsymbol{x}_3,\boldsymbol{u}_2) + N_c^2\hat{D}_2(\boldsymbol{x}_2,\boldsymbol{x}_4)\hat{D}_2(\boldsymbol{x}_3,\boldsymbol{x}_1) \\
&+ N_c^2\hat{D}_4(\boldsymbol{b}_1,\boldsymbol{u}_2,\boldsymbol{x}_3,\boldsymbol{b}_2)\hat{D}_4(\boldsymbol{b}_2,\boldsymbol{x}_4,\boldsymbol{u}_2,\boldsymbol{b}_1) + N_c^2\hat{D}_4(\boldsymbol{b}_2,\boldsymbol{x}_4,\boldsymbol{b}_1,\boldsymbol{x}_2)\hat{D}_4(\boldsymbol{x}_1,\boldsymbol{b}_1,\boldsymbol{x}_3,\boldsymbol{b}_2) \\
&- N_c^2\hat{D}_4(\boldsymbol{b}_2,\boldsymbol{x}_4,\boldsymbol{u}_2,\boldsymbol{x}_2)\hat{D}_4(\boldsymbol{x}_1,\boldsymbol{u}_2,\boldsymbol{x}_3,\boldsymbol{b}_2) + N_c^2\hat{D}_4(\boldsymbol{b}_1,\boldsymbol{u}_2,\boldsymbol{b}_2,\boldsymbol{x}_1)\hat{D}_4(\boldsymbol{x}_2,\boldsymbol{b}_2,\boldsymbol{u}_2,\boldsymbol{b}_1) \\
&- N_c^2\hat{D}_4(\boldsymbol{b}_1,\boldsymbol{u}_2,\boldsymbol{x}_3,\boldsymbol{x}_1)\hat{D}_4(\boldsymbol{x}_2,\boldsymbol{x}_4,\boldsymbol{u}_2,\boldsymbol{b}_1)
\end{aligned}$$

$$\begin{aligned}
&- \hat{D}_4(\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{x}_2,\boldsymbol{x}_1) - \hat{D}_6(\boldsymbol{x}_1,\boldsymbol{b}_1,\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{b}_1,\boldsymbol{x}_2) \\
&+ \hat{D}_6(\boldsymbol{x}_1,\boldsymbol{u}_2,\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{u}_2,\boldsymbol{x}_2) + \hat{D}_8(\boldsymbol{b}_1,\boldsymbol{u}_2,\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{u}_2,\boldsymbol{b}_1,\boldsymbol{x}_2,\boldsymbol{x}_1) \qquad (97)
\end{aligned}$$



$$r_T > \frac{1}{\Lambda_{QCD}}$$



- Nonperturbative correlations over distances larger than $1/\Lambda_{QCD}$
  - Simple but nonzero geometrical (Glauber-type) correlations due to sampling the projectile nucleus
  - Sensitive to separate pair production from independent nucleons

Long-Distance Correlations

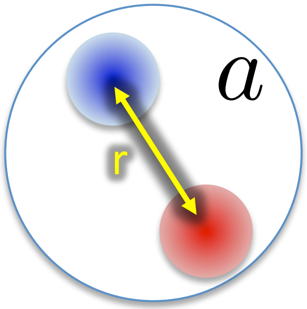
$$\mathcal{C}^{q\bar{q}} = \frac{1}{\sigma_{inel}} \int d^2B\, d^2x_1\, d^2x_2\, d^2x_3\, d^2x_4 \int dy_1\, dy_2\, dy_3\, dy_4$$

4 x 4 = 16 time orderings

$$\times \frac{1}{4}\Big[\delta^2(x_1 - B_1)\delta(y_1 - Y_1)\, \delta^2(x_4 - B_2)\delta(y_4 - Y_2) + \delta^2(x_3 - B_1)\delta(y_3 - Y_1)\, \delta^2(x_2 - B_2)\delta(y_2 - Y_2)\Big]$$

$$\times \frac{d\sigma^{q\bar{q}}}{d^2x_1\, dy_1\, d^2x_2\, dy_2\, d^2B} \, \frac{d\sigma^{q\bar{q}}}{d^2x_3\, dy_3\, d^2x_4\, dy_4\, d^2B}$$

$$\frac{1}{\Lambda_{QCD}} < r_T < R_a$$

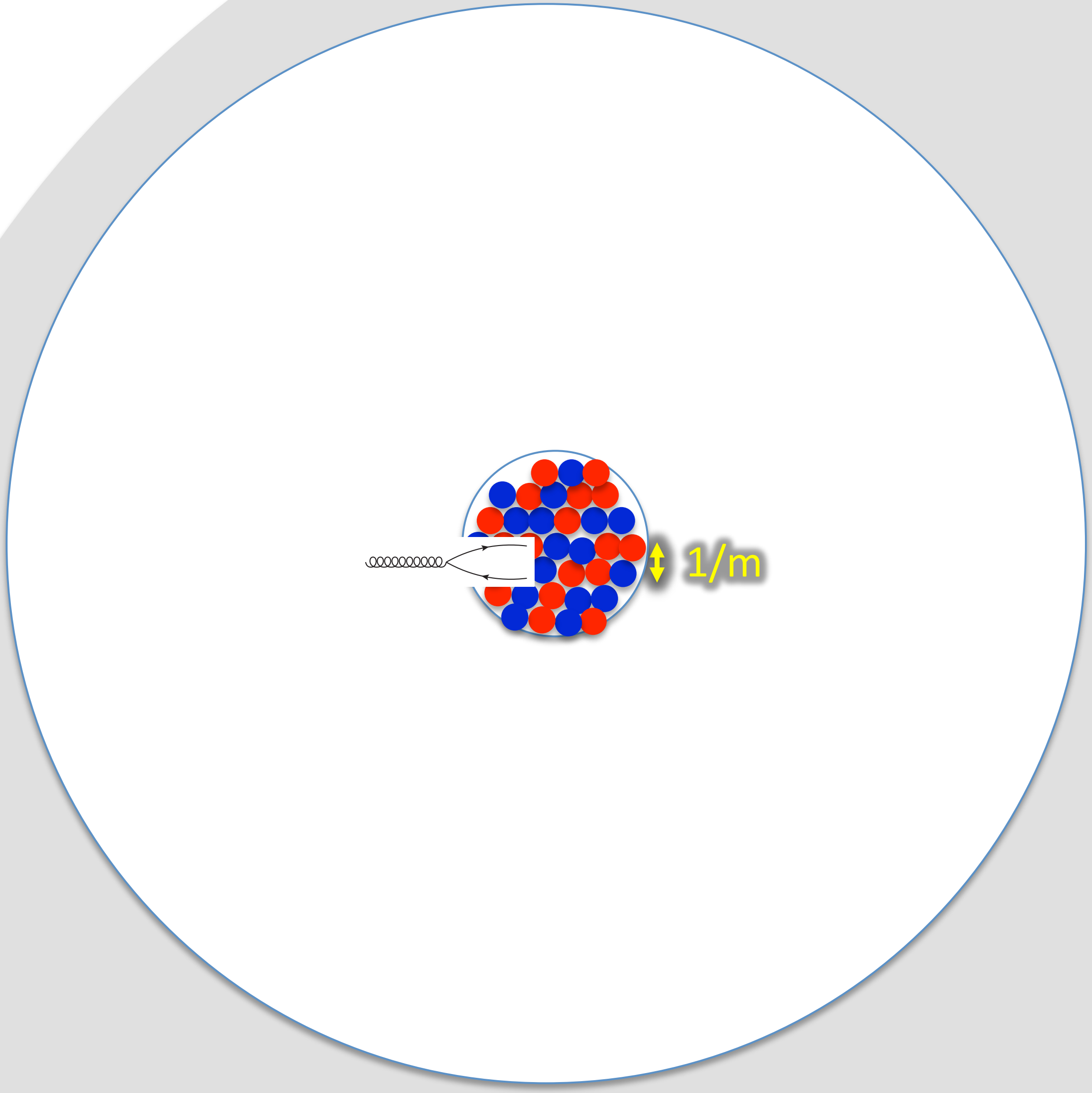


- Geometrical correlations are the only ones that remain
- No correlations from the interactions
  - Color fields uncorrelated over long distances
  - Scattering on a disjoint set of nucleons
- Factorizes into a convolution of single-pair cross sections

The Big Picture: Scales of Correlations

For quark-antiquark correlations:

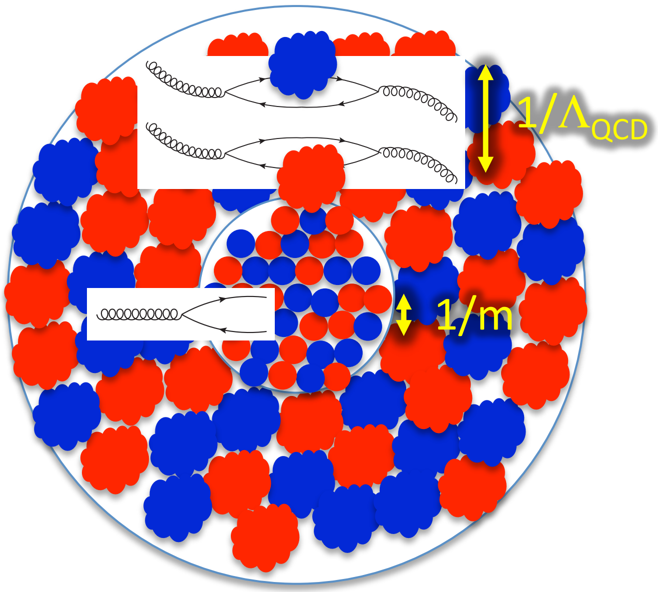
The Big Picture: Scales of Correlations



For quark-antiquark correlations:

- r < 1/m
  - Single pair production

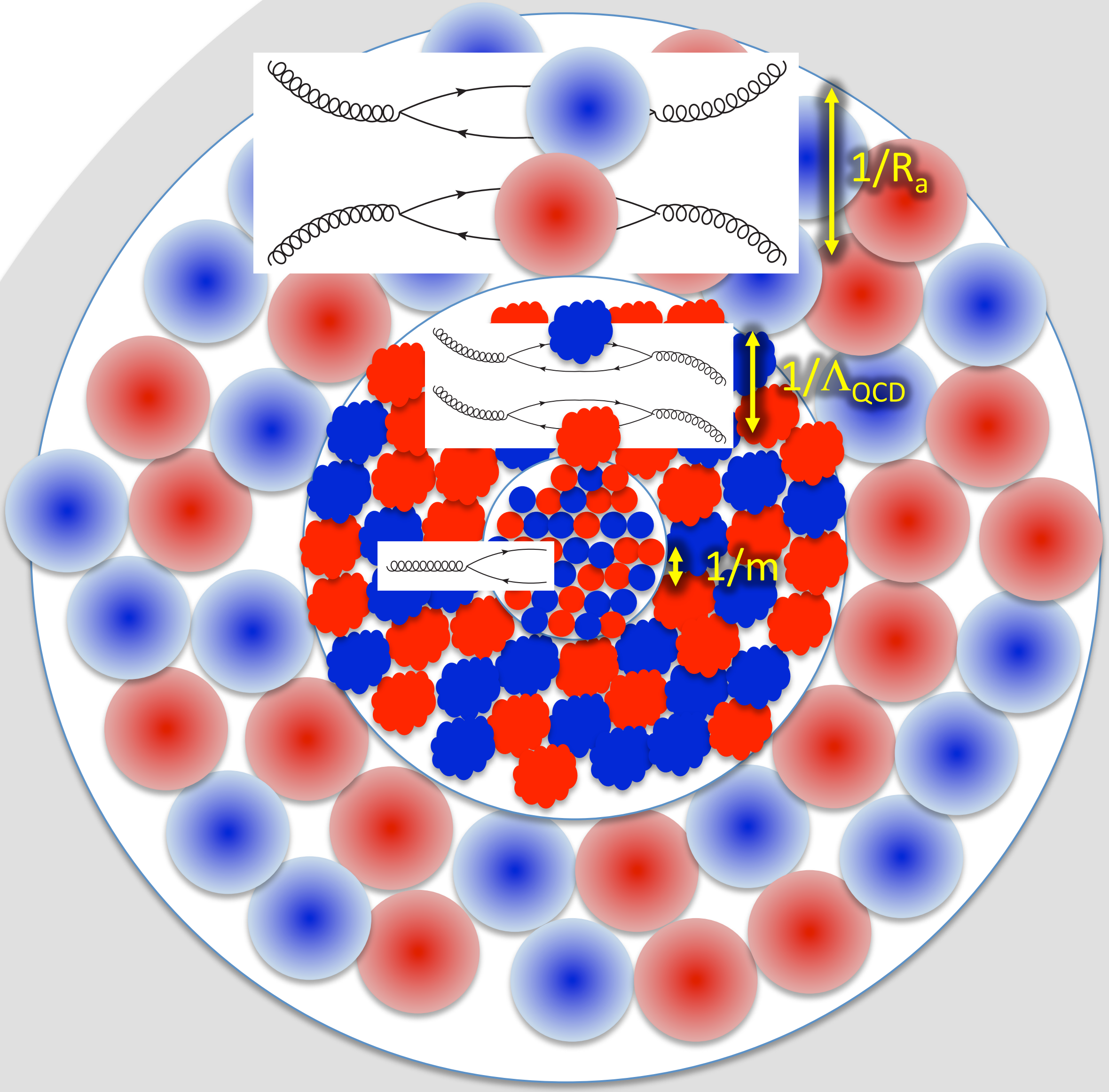
The Big Picture: Scales of Correlations



For quark-antiquark correlations:

- $r < 1/m$
  - Single pair production
- $1/m < r < 1/\Lambda_{QCD}$
  - Gluon entanglement

The Big Picture: Scales of Correlations



For quark-antiquark correlations:

- $r < 1/m$
  - Single pair production
- $1/m < r < 1/\Lambda_{QCD}$
  - Gluon entanglement
- $1/\Lambda_{QCD} < r < R_a$
  - Geometric Correlations

Project Status

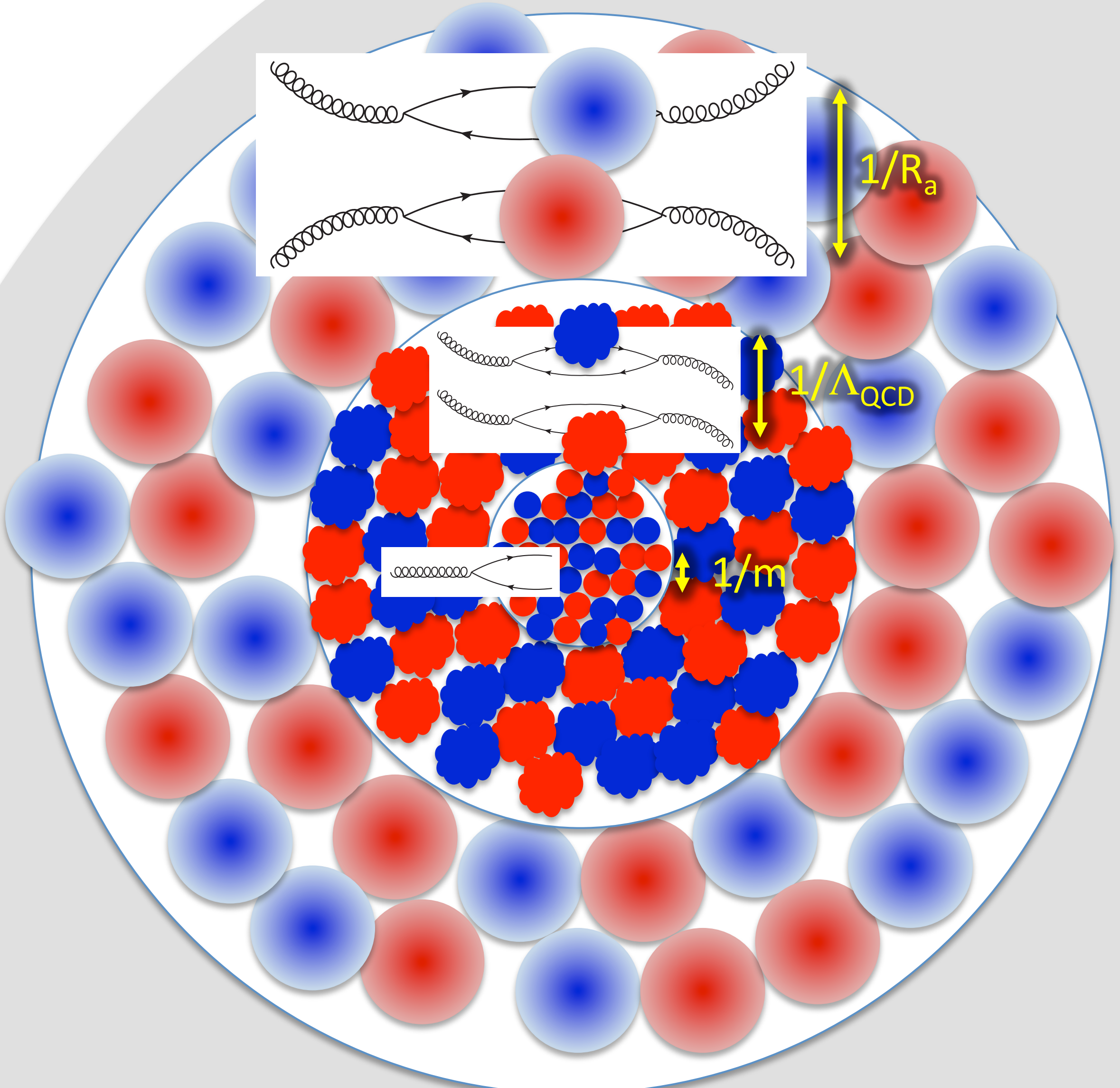
- ✓ We have derived a complete set of (anti)quark correlations in terms of operators suited to Monte Carlo simulation
  - ➢ Involve complicated Wilson line multipoles (up to D_8)
- ✓ We have analytic calculations of some channels in the MV model to illustrate the physical picture
  - ✓ (q qbar) at short distances (single-pair production)
  - ✓ All correlations at very long distances
- ❑ Some channels cannot presently be calculated analytically
  - ❑ High-order multipoles not currently available in the MV model (but can in principle be calculated)
  - ❑ Can approximate in the large-N_c limit, but this limits the contributions of certain topologies

Outlook

- Finishing this project:
  - ❑ Perform analytic calculations as much as presently possible
  - ❑ Include similar calculations of the local fluctuation spectrum

- Future extensions:
  - ➢ Perform Monte Carlo simulations of the operators
  - ➢ Couple to a hydrodynamics code to assess the impact on the particle spectrum
  - ➢ Beyond the quasi-classical approximation: small-x evolution
  - ➢ Moving toward the dense-dense limit: multiple scattering corrections in the light ion.

Conclusion: The Takeaway Message



For quark-antiquark correlations:

- $r < 1/m$
  - Single pair production
- $1/m < r < 1/\Lambda_{QCD}$
  - Gluon entanglement
- $1/\Lambda_{QCD} < r < R_a$
  - Geometric Correlations

Backup Slides: Wave Functions

$$\left[\tilde{\Psi}_i(\boldsymbol{w},\boldsymbol{r},\alpha)\right]_{\sigma',-\sigma} \equiv [\mathbb{1}]_{\sigma',-\sigma}\,\mathcal{U}_i(\boldsymbol{w},\boldsymbol{r},\alpha) + [\tau_3]_{\sigma',-\sigma}\,\mathcal{L}_i(\boldsymbol{w},\boldsymbol{r},\alpha) + [\boldsymbol{\tau}]_{\sigma',-\sigma} \times \boldsymbol{\mathcal{T}}_i(\boldsymbol{w},\boldsymbol{r},\alpha)$$

$$\begin{aligned}
\mathcal{U}_1(\boldsymbol{w},\boldsymbol{r},\alpha) &= \frac{g^2}{2\pi^2}\sqrt{\alpha(1-\alpha)}\Big[(1-2\alpha)\frac{\boldsymbol{w}\cdot\boldsymbol{r}}{w_T\, r_T}F_2(w_T,r_T,\alpha) \\
&\quad - 2\alpha(1-\alpha)F_0(w_T,r_T,\alpha)\Big] \\
\mathcal{L}_1(\boldsymbol{w},\boldsymbol{r},\alpha) &= \frac{g^2}{2\pi^2}\sqrt{\alpha(1-\alpha)}\left[-i\,\frac{\boldsymbol{w}\times\boldsymbol{r}}{w_T\, r_T}F_2(w_T,r_T,\alpha)\right] \\
\boldsymbol{\mathcal{T}}_1(\boldsymbol{w},\boldsymbol{r},\alpha) &= \frac{g^2}{2\pi^2}\sqrt{\alpha(1-\alpha)}\left[\left(\frac{m}{w_T}F_1(w_T,r_T,\alpha)\right)\boldsymbol{w}\right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_2(\boldsymbol{w},\boldsymbol{r},\alpha) &= \frac{g^2}{2\pi^2}\sqrt{\alpha(1-\alpha)}\left[-(1-2\alpha)\frac{\boldsymbol{w}\cdot\boldsymbol{r}}{w_T\, r_T}\,\frac{m}{w_T}K_1(m r_T)\right] \\
\mathcal{L}_2(\boldsymbol{w},\boldsymbol{r},\alpha) &= \frac{g^2}{2\pi^2}\sqrt{\alpha(1-\alpha)}\left[i\,\frac{\boldsymbol{w}\times\boldsymbol{r}}{w_T\, r_T}\,\frac{m}{w_T}K_1(m r_T)\right] \\
\boldsymbol{\mathcal{T}}_2(\boldsymbol{w},\boldsymbol{r},\alpha) &= \frac{g^2}{2\pi^2}\sqrt{\alpha(1-\alpha)}\left[\left(-\frac{m}{w_T^2}K_0(m r_T)\right)\boldsymbol{w}\right],
\end{aligned}$$

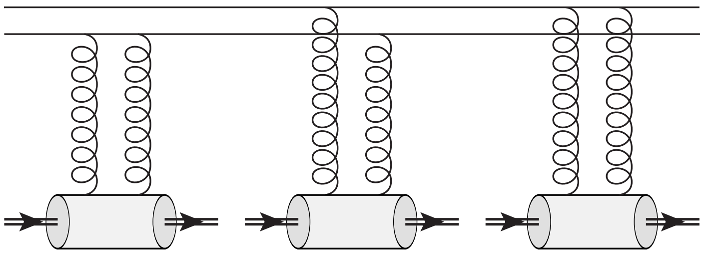
Backup Slides: Wave Function Spin Traces

$$\Psi^{(pairs)}_{ijk\ell}(①, ②, ③, ④) = 4 \left(\mathcal{U}_{i①}\mathcal{U}_{j②} - \mathcal{L}_{i①}\mathcal{L}_{j②} + \boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{j②} \right) \times \left(\mathcal{U}_{k③}\mathcal{U}_{\ell④} - \mathcal{L}_{k③}\mathcal{L}_{\ell④} + \boldsymbol{\mathcal{T}}_{k③} \cdot \boldsymbol{\mathcal{T}}_{\ell④} \right).$$

$$\begin{aligned}
\Psi^{(loop)}_{i,j,k,\ell}(①, ②, ③, ④) &= 2\mathcal{U}_{i①}\mathcal{U}_{j②}\mathcal{U}_{k③}\mathcal{U}_{\ell④} - 2\mathcal{U}_{i①}\mathcal{U}_{j②}\mathcal{L}_{k③}\mathcal{L}_{\ell④} + 2\mathcal{U}_{i①}\mathcal{U}_{j②}\boldsymbol{\mathcal{T}}_{k③} \cdot \boldsymbol{\mathcal{T}}_{\ell④} \\
&+ 2\mathcal{U}_{i①}\mathcal{L}_{j②}\mathcal{U}_{k③}\mathcal{L}_{\ell④} - 2\mathcal{U}_{i①}\mathcal{L}_{j②}\mathcal{L}_{k③}\mathcal{U}_{\ell④} - 2i\mathcal{U}_{i①}\mathcal{L}_{j②}\boldsymbol{\mathcal{T}}_{k③} \times \boldsymbol{\mathcal{T}}_{\ell④} \\
&+ 2\mathcal{U}_{i①}\mathcal{U}_{k③}\boldsymbol{\mathcal{T}}_{j②} \cdot \boldsymbol{\mathcal{T}}_{\ell④} - 2i\mathcal{U}_{i①}\mathcal{L}_{k③}\boldsymbol{\mathcal{T}}_{j②} \times \boldsymbol{\mathcal{T}}_{\ell④} + 2\mathcal{U}_{i①}\mathcal{U}_{\ell④}\boldsymbol{\mathcal{T}}_{j②} \cdot \boldsymbol{\mathcal{T}}_{k③} \\
&- 2i\mathcal{U}_{i①}\mathcal{L}_{\ell④}\boldsymbol{\mathcal{T}}_{j②} \times \boldsymbol{\mathcal{T}}_{k③} - 2\mathcal{L}_{i①}\mathcal{U}_{j②}\mathcal{U}_{k③}\mathcal{L}_{\ell④} + 2\mathcal{L}_{i①}\mathcal{U}_{j②}\mathcal{L}_{k③}\mathcal{U}_{\ell④} \\
&+ 2i\mathcal{L}_{i①}\mathcal{U}_{j②}\boldsymbol{\mathcal{T}}_{k③} \times \boldsymbol{\mathcal{T}}_{\ell④} - 2\mathcal{L}_{i①}\mathcal{L}_{j②}\mathcal{U}_{k③}\mathcal{U}_{\ell④} + 2\mathcal{L}_{i①}\mathcal{L}_{j②}\mathcal{L}_{k③}\mathcal{L}_{\ell④} \\
&- 2\mathcal{L}_{i①}\mathcal{L}_{j②}\boldsymbol{\mathcal{T}}_{k③} \cdot \boldsymbol{\mathcal{T}}_{\ell④} + 2i\mathcal{L}_{i①}\mathcal{U}_{k③}\boldsymbol{\mathcal{T}}_{j②} \times \boldsymbol{\mathcal{T}}_{\ell④} - 2\mathcal{L}_{i①}\mathcal{L}_{k③}\boldsymbol{\mathcal{T}}_{j②} \cdot \boldsymbol{\mathcal{T}}_{\ell④} \\
&+ 2i\mathcal{L}_{i①}\mathcal{U}_{\ell④}\boldsymbol{\mathcal{T}}_{j②} \times \boldsymbol{\mathcal{T}}_{k③} - 2\mathcal{L}_{i①}\mathcal{L}_{\ell④}\boldsymbol{\mathcal{T}}_{j②} \cdot \boldsymbol{\mathcal{T}}_{k③} + 2\mathcal{U}_{j②}\mathcal{U}_{k③}\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{\ell④} \\
&- 2i\mathcal{U}_{j②}\mathcal{L}_{k③}\boldsymbol{\mathcal{T}}_{i①} \times \boldsymbol{\mathcal{T}}_{\ell④} + 2\mathcal{U}_{j②}\mathcal{U}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{k③} - 2i\mathcal{U}_{j②}\mathcal{L}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \times \boldsymbol{\mathcal{T}}_{k③} \\
&+ 2i\mathcal{L}_{j②}\mathcal{U}_{k③}\boldsymbol{\mathcal{T}}_{i①} \times \boldsymbol{\mathcal{T}}_{\ell④} - 2\mathcal{L}_{j②}\mathcal{L}_{k③}\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{\ell④} + 2i\mathcal{L}_{j②}\mathcal{U}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \times \boldsymbol{\mathcal{T}}_{k③} \\
&- 2\mathcal{L}_{j②}\mathcal{L}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{k③} + 2\mathcal{U}_{k③}\mathcal{U}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{j②} - 2i\mathcal{U}_{k③}\mathcal{L}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \times \boldsymbol{\mathcal{T}}_{j②} \\
&+ 2i\mathcal{L}_{k③}\mathcal{U}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \times \boldsymbol{\mathcal{T}}_{j②} - 2\mathcal{L}_{k③}\mathcal{L}_{\ell④}\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{j②} + 2(\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{j②})(\boldsymbol{\mathcal{T}}_{k③} \cdot \boldsymbol{\mathcal{T}}_{\ell④}) \\
&- 2(\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{k③})(\boldsymbol{\mathcal{T}}_{j②} \cdot \boldsymbol{\mathcal{T}}_{\ell④}) + 2(\boldsymbol{\mathcal{T}}_{i①} \cdot \boldsymbol{\mathcal{T}}_{\ell④})(\boldsymbol{\mathcal{T}}_{j②} \cdot \boldsymbol{\mathcal{T}}_{k③}). \qquad (19)
\end{aligned}$$

Backup Slides: Operators in the MV Model

- MV Model: Gaussian charge density functional
  - 2 Gluons / nucleon
- Construct matrix of possible color states
  - 2-gluon kernel
  - Diagonalize / exponentiate matrix kernel



- E.g.) Double Dipole Dominguez, Marquet, & Wu, Nucl. Phys. **A823** (2009) 99

$$\left\langle \hat{D}_2(\boldsymbol{x},\boldsymbol{y})\hat{D}_2(\boldsymbol{u},\boldsymbol{v})\right\rangle = e^{-\frac{1}{4}|\boldsymbol{x}-\boldsymbol{y}|_T^2 Q_s^2}\, e^{-\frac{1}{4}|\boldsymbol{u}-\boldsymbol{v}|_T^2 Q_s^2}\, e^{-\frac{1}{4}\frac{N_c}{2C_F}(\boldsymbol{x}-\boldsymbol{u})\cdot(\boldsymbol{y}-\boldsymbol{v})Q_s^2}$$
$$\times\left[\left(\frac{(\boldsymbol{x}-\boldsymbol{u})\cdot(\boldsymbol{y}-\boldsymbol{v})+(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}}{2\,(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}} - \frac{1}{N_c^2}\frac{(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{u}-\boldsymbol{v})}{(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}}\right) e^{+\frac{1}{4}\frac{N_c}{2C_F}(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}Q_s^2}\right.$$
$$\left. - \left(\frac{(\boldsymbol{x}-\boldsymbol{u})\cdot(\boldsymbol{y}-\boldsymbol{v})-(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}}{2\,(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}} - \frac{1}{N_c^2}\frac{(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{u}-\boldsymbol{v})}{(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}}\right) e^{-\frac{1}{4}\frac{N_c}{2C_F}(\frac{2C_F}{Q_s^2})\mu^2\sqrt{\Delta}Q_s^2}\right] \tag{170b}$$

$$\left(\frac{2C_F}{Q_s^2}\right)\mu^2\sqrt{\Delta} = \sqrt{\left[(\boldsymbol{x}-\boldsymbol{u})\cdot(\boldsymbol{y}-\boldsymbol{v})\right]^2 + \frac{4}{N_c^2}\left[(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{u}-\boldsymbol{v})\right]\left[(\boldsymbol{x}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{y})\right]}$$

Backup Slides: Doing (Some of) the Integrals

$$\mathcal{C}^{(q\bar{q})}_{single}(\boldsymbol{r_{12}}, \boldsymbol{u_1}; Y_1, Y_2) = \frac{a}{\sigma_{inel}} \frac{\alpha_s^2 C_F}{\pi^3} \frac{\alpha_1(1-\alpha_1)}{r_{12\,T}^2} \Bigg\{ \left(\alpha_1^2 + (1-\alpha_1)^2\right) \mathcal{G}_1(m r_{12\,T})$$
$$+ 4\alpha_1(1-\alpha_1)\mathcal{G}_2(m r_{12\,T}) + m^2 r_{12\,T}^2 \mathcal{G}_3(m r_{12\,T})$$
$$+ \frac{1}{2} m^2 r_{12\,T}^2 \ln\frac{Q_s^2}{\Lambda^2} \left[\left(\alpha_1^2 + (1-\alpha)_1^2\right) K_1^2(m r_{12\,T}) + K_0^2(m r_{12\,T})\right]$$
$$- m^2 r_{12\,T}^2 e^{+\frac{C_F}{N_c}\frac{1}{\alpha_1(1-\alpha_1)}\frac{m^2}{Q_s^2}} \left[\left(\alpha_1^2 + (1-\alpha_1)^2\right) \frac{1}{m r_{12\,T}} K_1(m r_{12\,T}) \mathcal{G}_4(m r_{12\,T}, Q_s r_{12\,T})\right.$$
$$\left. + K_0(m r_{12\,T}) \mathcal{G}_5(m r_{12\,T}, Q_s r_{12\,T})\right] \Bigg\}, \quad (177)$$

$$\mathcal{G}_1(m r_{12\;T}) = \int_{\zeta_{min}}^{\infty} d\zeta \, \frac{\zeta^3}{\zeta^2 - m^2 r_{12\;T}^2} K_1^2(\zeta)$$

$$\mathcal{G}_2(m r_{12\;T}) = \int_{\zeta_{min}}^{\infty} d\zeta \, \zeta \, K_0^2(\zeta)$$

$$\mathcal{G}_3(m r_{12\;T}) = \int_{\zeta_{min}}^{\infty} d\zeta \, \frac{\zeta}{\zeta^2 - m^2 r_{12\;T}^2} K_0^2(\zeta)$$

$$\mathcal{G}_4(m r_{12\;T}, Q_s r_{12\;T}) = \int_{\zeta_{min}}^{\infty} d\zeta \, \frac{\zeta^2}{\zeta^2 - m^2 r_{12\;T}^2} K_1(\zeta) \, e^{-\frac{C_F}{N_c}\frac{1}{\alpha_1(1-\alpha_1)}\frac{\zeta^2}{Q_s^2 r_{12\,T}^2}}$$

$$\mathcal{G}_5(m r_{12\;T}, Q_s r_{12\;T}) = \int_{\zeta_{min}}^{\infty} d\zeta \, \frac{\zeta}{\zeta^2 - m^2 r_{12\;T}^2} K_0(\zeta) \, e^{-\frac{C_F}{N_c}\frac{1}{\alpha_1(1-\alpha_1)}\frac{\zeta^2}{Q_s^2 r_{12\,T}^2}}$$