MICHIGAN STATE UNIVERSITY

Dissipative Transport

Jeffrey Schenker

Dissipative Transport in the Localized Regime

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NMP17
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Localization

Waves in a disordered environment may be *trapped* by disorder.

$$H_\omega = \sum_{x \sim y} |x\rangle\langle y| + \lambda \sum_x \omega(x) |x\rangle\langle x|$$

on $\mathbb{Z}^d$ with $\omega(x)$ uniform in $[-1, 1]$.

Then

$$\mathbb{E}\left(\left|\langle x| \mathrm{e}^{-\mathrm{i}tH_\omega} |y\rangle\right|\right) \le \mathrm{e}^{-\mu(\lambda,d)|x-y|}$$

for all $t > 0$,provided

- $d = 1$ (or 2?) or
- $\lambda >> 1$

Anderson 1958 + thousands of later papers

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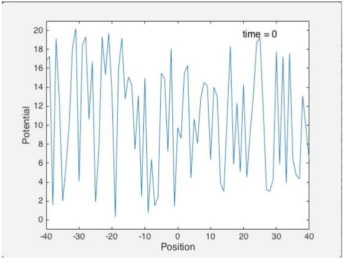
What if the potential fluctuates?

Question

What are the long time dynamics for solutions to

$$\partial_t U(t, t_0) \;=\; -\mathrm{i}H(t)U(t, t_0),$$

with $U(t_0, t_0) = I$ and $H(t) \;=\; H_\omega + u \sum_x w(x, t) \left|x\right\rangle \left\langle x\right|$?



Fluctuating potential

Fairly generally, the answer is “diffusive”, i.e.,

$$\mathbb{E}\left(\left|\langle x|\, U(t,t_0)\,|y\rangle\right|^2\right) \approx \frac{C}{|t-t_0|^{\frac{d}{2}}}\mathrm{e}^{-\frac{|x-y|^2}{2D|t-t_0|}},$$

provided w fluctuates stochastically.

Literature.

- Ovchinnikov and Erikhman (JETP 1974)
- Pillet (CMP 1985)
- Tcheremchantsev (CMP 1997, CMP 1998)
- Fischer, Leshke, Müller (Ann. Phys. 1998)
- Kang & S. (JSP 2009); Hamza, Kang & S. (LMP 95 2010); S. (CMP 2015)
- Applications:
  - To signal propagation in optical fibers (Mitra & Stark, Nature 411 (2001); Green, Littlewood, Mitra, Wegener PRE 66 (2002))
  - Other applications discussed in A. Amir, Y. Lahini and H. B. Perets, (PRE 2009)

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A general Theorem

Theorem (S. 2015)

Let $w(x,t) = v(\theta_x(t))$ where $\theta_x(t)$ are independent Brownian motions on the circle and v is a non-constant function. If $u > 0$ then solutions to

$$\mathrm{i}\partial_t |\psi_t\rangle \;=\; H(t)\,|\psi_t\rangle$$

satisfy

$$\lim_{t\to\infty} \sum_x f\left(\frac{x}{\sqrt{Dt}}\right) \mathbb{E}\left(|\langle x|\psi_t\rangle|^2\right) \;=\; C_d \int f(\mathbf{r}) \mathrm{e}^{-\frac{d}{2}\mathbf{r}^2} \mathrm{d}^d\mathbf{r},$$

with a positive diffusion constant $D = D(\lambda, u)$.

Comments

This result is:

1. *Non-perturbative* result (no small parameter).
2. Purely *qualitative*.
   - Estimating, or computing D, takes more work.
3. Dimension independent.
4. Rigorous.
   - Fundamental assumption is that $w(x,t)$ is a functional of a Markov process with exponential return to equilibrium.

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Quantitative analysis: fast diffusion

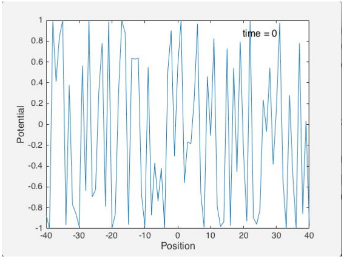
Theorem(Kang and S. 2009)

For

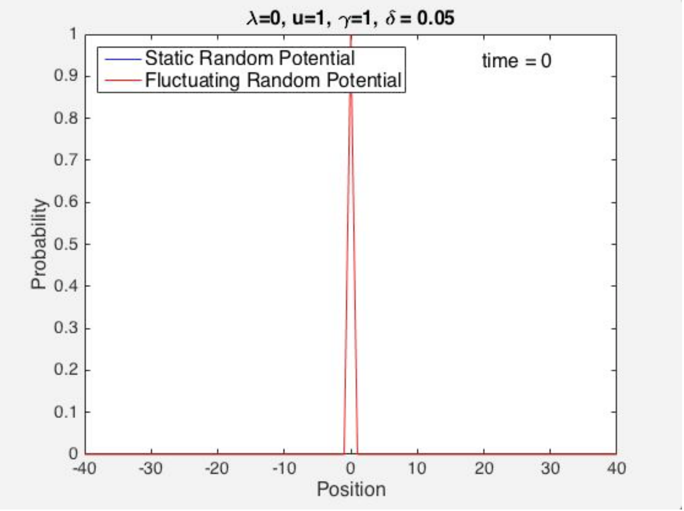
$$H(t) = \sum_{x \sim y} |x\rangle\langle y| + u \sum_{x} w(x,t) |x\rangle\langle x|$$

we have

$$D = \frac{1}{u^2} D_0 + O(1/u).$$



Fast diffusion



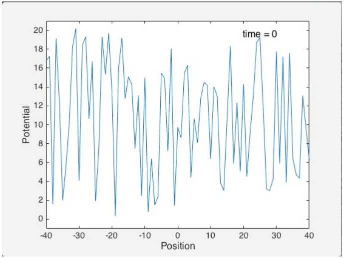
Quantitative analysis: slow diffusion

Theorem (Schenker 2015)

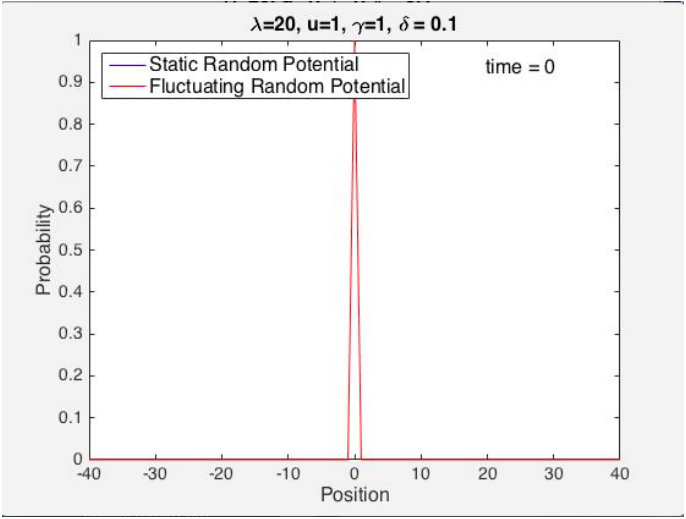
If $H(t) \; = \; H_\omega + u \sum_x w(x,t) \left|x\right\rangle \left\langle x\right|$, where H_ω exhibits localization, then we have

$$D \; = \; Fu^2 + o(u^2),$$

where $0 < F < \infty$.



Slow Diffusion



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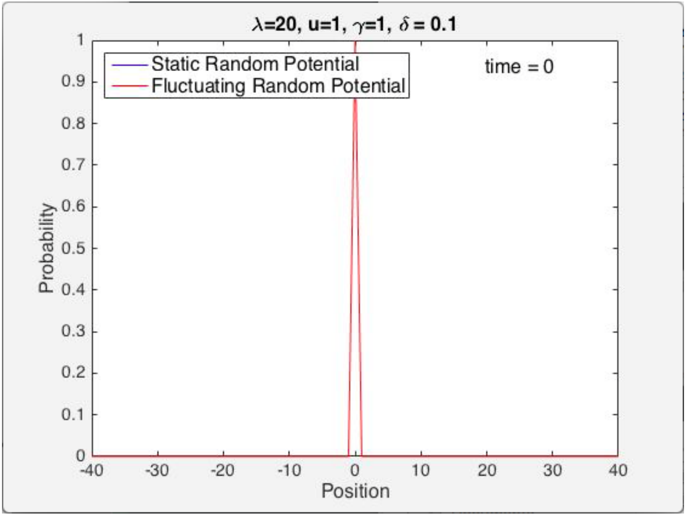

Comments on the proof

- "Augmented space formalism" with disorder variables in the Hilbert space:

$$\mathbb{E}\left(|\langle x|\psi_t\rangle|^2\right) = \langle \delta_x \times \delta_x \times 1|\, \mathrm{e}^{-t\mathcal{G}}\, |\psi_0 \times \overline{\psi_0} \times 1\rangle_{\mathcal{H}\times\mathcal{H}\times L^2(\Omega)}$$

  - $\mathcal{G} = \mathrm{i}\,[H, \cdot] + B$,
  - B = Markov process generator.

- Generator $\mathcal{G}$ commutes with translations $\delta_x \times \delta_y \times f(\omega) \mapsto \delta_{x+\xi} \times \delta_{y+\xi} \times f(S_\xi \omega)$.
- After a Bloch-Floquet transform the analysis rests on controlling matrix elements of $(\eta + \mathcal{G})^{-1}$ in the zero "momentum" fiber.

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Open Quantum System

What about diffusion for a quantum particle interacting with a thermal bath?

- Vast literature going back to Mott.
- Most work relies on quantum Markov formalism ("Fermi Golden Rule").
- Some recent mathematical physics literature:
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Open Quantum System in the Markov Approximation

Theorem (Fröhlich S. 2016)

$$\partial_t \rho_t = -\mathrm{i}\,[H_\omega, \rho_t] + u\mathcal{L}(\rho_t),$$

with a suitable Lindbladian $\mathcal{L}$. Then

$$D = \lim_{t\to\infty} \frac{1}{t} \sum_x |x|^2 \mathbb{E}\left(\langle x| \rho_t |x\rangle\right)$$

exists and satisfies $0 < D < \infty$.

1. If $\lambda = 0$ (no disorder), then $D = \frac{C}{u}$ for all $u > 0$.
2. If H_ω exhibits localization, then

$$D = \Delta u + o(u)$$

where $0 < \Delta < (\text{loc. length})^2$.

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The Lindblad generator

Describes a hopping process for the particle momentum:

$$\rho^W(X,\mathbf{p}) = \sum_{\xi} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\xi} \left\langle \frac{X+\xi}{2} \middle| \rho \middle| \frac{X-\xi}{2} \right\rangle,$$

$$\mathcal{L}\rho^W(X,\mathbf{p}) = \int \widehat{r}(\mathbf{p},\mathbf{q}) \left[\rho^W(X,\mathbf{q}) - \rho^W(X,\mathbf{p}) \right] \mathrm{d}q,$$

$$\widehat{r}(\mathbf{p},\mathbf{q}) = \widehat{r}(\mathbf{q},\mathbf{p}),$$

and

$$\iint \widehat{r}(\mathbf{p},\mathbf{q}) \left| f(\mathbf{p}) - f(\mathbf{q}) \right|^2 \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q} \ \geq \ c \iint \left| f(\mathbf{p}) - f(\mathbf{q}) \right|^2 \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}.$$

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Conclusions

- Diffusion is universal in the presence of time dependent fluctuations.
- Diffusion is quantifiably slow for weak fluctuations around a localized system.

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Open problems

1. Diffusion in an open quantum system without the Markov approximation.
   - How to *prove* that decoherence emerges and memory in the bath decays?
2. Diffusion for weak disorder (*without fluctuations*)
   - *Recurrence* is the problem.
   - Can fluctuations help?

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