Heat Load on a Pillbox Cavity

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Abstract

Derivation of the power distribution from RF fields on the inner surfaces of a pillbox cavity made of multiple materials.

Introduction

Consider a cylindrical pillbox cavity with radius a and length L. The electromagnetic field amplitudes in SI units are given in cylindrical coordinates (r, ϕ, z) by

$$\begin{aligned} \vec{E}(r) &= E_0 \; J_0(j_{01} \, \frac{r}{a}) \; \hat{\mathbf{z}} \\ \vec{B}(r) &= \frac{E_0}{c} \; J_1(j_{01} \, \frac{r}{a}) \; \hat{\boldsymbol{\phi}} \end{aligned}$$

in terms of Bessel functions of the first kind J_0, J_1 with

$$\begin{aligned} \omega &= c \, \frac{j_{01}}{a} = 2\pi \, f \\ J_0(j_{01}) &= 0 \end{aligned}$$

where

- E_0 is the peak on-axis gradient
- $c \simeq 2.9979 \times 10^8$ m/s is the speed of light,
- f is the resonant frequency of the cavity and
- $j_{01} \simeq 2.4048$ is the smallest root of J_0

Stored energy

The electromagnetic energy density in the cavity volume is

$$w = \frac{1}{2}\left[\epsilon_0\, E(r,t)^2 + \frac{1}{\mu_0}\, B(r,t)^2\right]$$

and with the full time-dependent fields expressed as

$$\begin{aligned}\vec{E}(r,t) &= \vec{E}(r)\ \cos\omega t \\ \vec{B}(r,t) &= \vec{B}(r)\ \sin\omega t\end{aligned}$$

the total stored energy can be found when $\omega t = 0$ or π through

$$U = \epsilon_0\, L \int_0^a dr\ 2\pi\, r\ E^2(r) = \frac{L}{\mu_0} \int_0^a dr\ 2\pi\, r\ B^2(r)$$

using

$$\int_0^1 dx\ x\ J_0^2(j_{01}\, x) = \int_0^1 dx\ x\ J_1^2(j_{01}\, x) = \frac{1}{2}\ J_1^2(j_{01})$$

as

$$U = J_1^2(j_{01})\ (\pi\, a^2\, L)\left[\frac{1}{2\,\mu_0}\ \left(\frac{E_0}{c}\right)^2\right]$$

where $J_1^2(j_{01}) \simeq 0.26951$ showing that the (spatial) average energy density is about 27% of the peak in this simple structure and does not depend on the aspect ratio.

Surface currents and power dissipation

In a cavity made of material with finite conductivity, power dissipation is due to a thin layer of current under the surface confined to the skin depth δ. The current density $\vec{J}$ is given by

$$\begin{aligned}\vec{J}_{cr} &= \frac{B_\phi(a)}{\mu_0\, \delta_{cr}}\ \hat{\mathbf{z}}, \quad -\frac{L}{2} < z < \frac{L}{2} \\ \vec{J}_{e1} &= \frac{B_\phi(r)}{\mu_0\, \delta_{e1}}\ \hat{\mathbf{r}}, \quad z = -\frac{L}{2} \\ \vec{J}_{e2} &= \frac{B_\phi(r)}{\mu_0\, \delta_{e2}}\ \hat{\mathbf{r}}, \quad z = \frac{L}{2}\end{aligned}$$

where

- $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of vacuum and
- the subscripts cr, e1 and e2 refer to the center ring and the two endplates respectively.

The corresponding (rms) volume and surface power density can be written as

$$\begin{aligned} \frac{dP}{dV} &= \frac{1}{2}\,\frac{1}{\sigma}\,J^2 \\ \frac{dP}{dA} &= \delta\,\frac{dP}{dV} = \frac{1}{2}\,\frac{\delta}{\sigma}\,J^2 \end{aligned}$$

where σ is the conductivity. This yields

$$\begin{aligned} \frac{dP_{cr}}{dA} &= \frac{1}{2}\,\frac{B_\phi^2(a)}{\mu_0^2\,\delta_{cr}\,\sigma_{cr}} = \frac{1}{2}\,R_{cr}\,\left[\frac{E_0}{\mu_0\,c}\,J_1(j_{01})\right]^2 \\ \frac{dP_{e1}}{dA} &= \frac{1}{2}\,\frac{B_\phi^2(r)}{\mu_0^2\,\delta_{e1}\,\sigma_{e1}} = \frac{1}{2}\,R_{e1}\left[\frac{E_0}{\mu_0\,c}\,J_1(j_{01}\,\frac{r}{a})\right]^2 = \frac{R_{e1}}{R_{cr}}\,\left[\frac{J_1(j_{01}\,\frac{r}{a})}{J_1(j_{01})}\right]^2\,\frac{dP_{cr}}{dA} \\ \frac{dP_{e2}}{dA} &= \frac{1}{2}\,\frac{B_\phi^2(r)}{\mu_0^2\,\delta_{e2}\,\sigma_{e2}} = \frac{1}{2}\,R_{e2}\left[\frac{E_0}{\mu_0\,c}\,J_1(j_{01}\,\frac{r}{a})\right]^2 = \frac{R_{e2}}{R_{e1}}\,\frac{dP_{e1}}{dA} \end{aligned}$$

with the surface (sheet) resistance R, skin depth and conductivity related through

$$\delta \simeq \sqrt{\frac{2}{\mu_0\,\omega\,\sigma}}, \qquad R = \frac{1}{\delta\,\sigma} \simeq \sqrt{\frac{\mu_0\,\omega}{2\,\sigma}}$$

The total heat deposited on the surface follows from integration of these as

$$\begin{aligned} P_{cr} &= (2\pi\,a\,L)\,\frac{dP_{cr}}{dA} = \pi\,a\,L\,R_{cr}\,\left[\frac{E_0}{\mu_0\,c}\,J_1(j_{01})\right]^2 \\ P_{e1,2} &= \frac{R_{e1,2}}{R_{cr}}\,\frac{a}{2\,L}\,P_{cr} \\ P &= P_{cr} + P_{e1} + P_{e2} \end{aligned}$$

Quality factor

Using the expressions for U and P above, the quality factor of the cavity resonator is found as

$$Q = \omega\,\frac{U}{P} = \frac{\sigma_{cr}\,R_{cr}\,a}{1 + \frac{R_{e1}+R_{e2}}{2\,R_{cr}}\,\frac{a}{L}} = \frac{a}{\delta_{cr}}\,\frac{1}{1 + \frac{\delta_{e1}+\delta_{e2}}{2\,\delta_{cr}}\,\frac{a}{L}}$$

We can express this in terms of the individual components as

$$\frac{1}{Q} = \frac{1}{Q_{cr}} + \frac{1}{Q_{e1}} + \frac{1}{Q_{e2}}$$

where

$$Q_{cr} = \frac{\omega\, U}{P_{cr}} = \frac{a}{\delta_{cr}}, \qquad Q_{e1,2} = \frac{\omega\, U}{P_{e1,2}} = \frac{2\, L}{\delta_{e1,2}}$$

Effect of the interface

Consider additional power dissipation P_g and corresponding Q_g at the interfaces (eg. gaskets) between the endplates and the center ring. The resulting Q is

$$\frac{1}{Q} = \frac{1}{Q_{cr}} + \frac{1}{Q_{g1}} + \frac{1}{Q_{e1}} + \frac{1}{Q_{g2}} + \frac{1}{Q_{e2}}$$

Dissipation at the interface is not easy to model. If the endplates and interfaces are the same on both sides, the unknown Q_g can be calculated using the measured cavity Q and the known surface properties.

Effect of coating

A thin layer of TiN is often used as a multipacting barrier. This leads to additional power dissipation on the surface. For a layer of thickness t much less than the skin depth δ_t, the equivalent surface resistance would be

$$R_t = \frac{1}{\sigma_t\, \delta_t}\, \frac{t}{\delta_t}$$

If placed over a substrate with power dissipation P_s, the additional power consumed by the TiN layer can be expressed as

$$\frac{dP_t}{dA} = \frac{R_t}{R_s}\, \frac{dP_s}{dA}$$

and the effective Q of the surface changes accordingly

$$Q_s \rightarrow \left(1 + \frac{R_t}{R_s}\right)^{-1} Q_s$$

Numerical example

Consider the MTA Modular Cavity with

$$
\begin{aligned}
a &= 14.1 \text{ cm} \\
L &= 10.3 \text{ cm} \\
f &= 805 \text{ MHz}
\end{aligned}
$$

The center ring is made of copper and endplates made of both copper and beryllium are available.

$$
\begin{aligned}
\sigma_{Cu} &= 6.0 \times 10^7 \ (\Omega\text{m})^{-1} \\
\frac{\sigma_{Be}}{\sigma_{Cu}} &= 0.43 \\
\frac{\sigma_{TiN}}{\sigma_{Be}} &= 0.10 \\
\frac{t}{\delta_t} &= 0.01
\end{aligned}
$$

The table below shows various quantities to aid in exploring different endplate configurations. Cavity Q refers to the unloaded quality factor. No correction for the TiN coating is included.

Theory		
Peak energy density		[4.42 $(\mathrm{J/m^3})/(\mathrm{MV/m})^2$] E_0^2
Average energy density		[1.19 $(\mathrm{J/m^3})/(\mathrm{MV/m})^2$] E_0^2
Stored energy	U	[7.65 $\mathrm{mJ}/(\mathrm{MV/m})^2$] E_0^2
Assumed		
Skin depth (Cu)	δ_{Cu}	2.29 μm
Skin depth (Be)	δ_{Be}	3.49 μm
Surface resistance (Cu)	R_{Cu}	7.28 mΩ
Surface resistance (Be)	R_{Be}	11.1 mΩ
Surface resistance (TiN)	R_{TiN}	0.35 mΩ
Calculated		
Center ring power	P_{cr}	[630 $\mathrm{W}/(\mathrm{MV/m})^2$] E_0^2
Endplate power (Cu)	P_e	[431 $\mathrm{W}/(\mathrm{MV/m})^2$] E_0^2
Endplate power (Be)	P_e	[658 $\mathrm{W}/(\mathrm{MV/m})^2$] E_0^2
Cavity power (2 Cu ep, no i/f)	P	[1.49 $\mathrm{kW}/(\mathrm{MV/m})^2$] E_0^2
Cavity power (1 Cu + 1 Be ep)	P	[1.72 $\mathrm{kW}/(\mathrm{MV/m})^2$] E_0^2
Cavity power (2 Be ep)	P	[1.95 $\mathrm{kW}/(\mathrm{MV/m})^2$] E_0^2
Component Q (center ring)	Q_{cr}	61.6k
Component Q (Cu endplate)	Q_e	89.8k
Component Q (Be endplate)	Q_e	58.9k
Cavity Q (all Cu, no interface)		26.0k
Cavity Q (1 Cu & 1 Be endplate)		22.5k
Cavity Q (2 Be endplates)		19.9k
Measured		
Measured cavity Q (2 Cu endplates)		21.2k
Measured cavity Q (2 Be endplates)		13.5k
Inferred		
Interface Q (Cu-ep, calculated)	Q_g	231k
Interface Q (Be-ep, calculated)	Q_g	83.8k
Interface power (Cu-ep)	P_g	[168 $\mathrm{W}/(\mathrm{MV/m})^2$] E_0^2
Interface power (Be-ep)	P_g	[463 $\mathrm{W}/(\mathrm{MV/m})^2$] E_0^2
Predicted		
Expected cavity Q (1 Cu & 1 Be ep)		16.5k
Cavity power (Cu+Cu ep)	P	[1.83 $\mathrm{kW}/(\mathrm{MV/m})^2$] E_0^2
Cavity power (Be+Cu ep)	P	[2.35 $\mathrm{kW}/(\mathrm{MV/m})^2$] E_0^2
Cavity power (Be+Be ep)	P	[2.87 $\mathrm{kW}/(\mathrm{MV/m})^2$] E_0^2