Frequency Shift for a Pillbox Cavity under Vacuum

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Abstract

We estimate the frequency shift of a pillbox cavity due to deflection of endplates from differential pressure.

Fields

Consider a cylindrical pillbox cavity with radius a and length L. The electromagnetic field amplitudes for the TM_{010} mode in SI units are given in cylindrical coordinates (r, ϕ, z) by

$$\begin{aligned}\vec{E}(r) &= E_0\, J_0(j_{01}\,\frac{r}{a})\;\hat{\mathbf{z}} \\ \vec{B}(r) &= \frac{E_0}{c}\, J_1(j_{01}\,\frac{r}{a})\;\hat{\boldsymbol{\phi}}\end{aligned}$$

in terms of Bessel functions of the first kind J_0, J_1 with

$$\begin{aligned}\omega &= c\,\frac{j_{01}}{a} = 2\pi\, f \\ J_0(j_{01}) &= 0\end{aligned}$$

where

- E_0 is the peak on-axis gradient
- $c \simeq 2.9979 \times 10^8$ m/s is the speed of light,
- f is the resonant frequency of the cavity and
- $j_{01} \simeq 2.4048$ is the smallest root of J_0

Stored energy

The electromagnetic energy density in the cavity volume is

$$w = \frac{1}{2} \left[\epsilon_0 \ E(r,t)^2 + \frac{1}{\mu_0} \ B(r,t)^2 \right]$$

and with the full time-dependent fields expressed as

$$\begin{aligned} \vec{E}(r,t) &= \vec{E}(r) \ \cos \omega t \\ \vec{B}(r,t) &= \vec{B}(r) \ \sin \omega t \end{aligned}$$

the total stored energy can be found when $\omega t = 0$ or π through

$$U = \epsilon_0 \ L \int_0^a dr \ 2\pi \, r \ E^2(r) = \frac{L}{\mu_0} \int_0^a dr \ 2\pi \, r \ B^2(r)$$

using

$$\int_0^1 dx \ x \ J_0^2(j_{01} \, x) = \int_0^1 dx \ x \ J_1^2(j_{01} \, x) = \frac{1}{2} \ J_1^2(j_{01})$$

as

$$U = J_1^2(j_{01}) \ (\pi \, a^2 \, L) \left(\frac{1}{2} \, \epsilon_0 \, E_0^2 \right)$$

where $J_1^2(j_{01}) \simeq 0.26951$ showing that the (spatial) average energy density is about 27% of the peak in this simple structure and does not depend on the aspect ratio.

Deflection

If there's vacuum on one side of an endplate (inside the cavity) and pressure p on the other side (outside), the deflection $d(r)$ of the endplate, under the thin-plate approximation, is given by

$$d(r) = \frac{3}{16} \ \frac{p}{E} \ \frac{a^4}{t^3} \ (1 - \nu^2) \left[1 - \left(\frac{r}{a} \right)^2 \right] \left[\left(\frac{5+\nu}{1+\nu} \right) - \left(\frac{r}{a} \right)^2 \right]$$

where t is the plate thickness and E and ν are the Young's modulus and Poisson's ratio respectively. This can be rewritten as

$$\begin{aligned} d(r) &= d_0 \left[1 - \left(\frac{r}{a} \right)^2 \right] \left[1 - \alpha \ \left(\frac{r}{a} \right)^2 \right], \quad \alpha \equiv \frac{1+\nu}{5+\nu} \\ d_0 &= \frac{3}{16} \ \frac{p}{E} \ \frac{a^4}{t^3} \ (1-\nu) \ (5+\nu) \end{aligned}$$

in terms of the maximum deflection $d_0 \equiv d(0)$.

Frequency shift

Slater's theorem gives the frequency shift $\delta\omega$ when a portion ΔV of the cavity volume V is removed due to a shape deformation as

$$\frac{\delta\omega}{\omega} = \frac{\int_{\Delta V}(\frac{1}{\mu_0}\,B^2 - \epsilon_0\,E^2)\,dV}{\int_V(\frac{1}{\mu_0}\,B^2 + \epsilon_0\,E^2)\,dV} = \frac{1}{4\mu_0\,U}\int_{\Delta V}\left[B^2 - \left(\frac{E}{c}\right)^2\right]\,dV$$

where the integrals are calculated using the unperturbed fields. Using the volume removed due to deflection of the endplate, we expand the integral to find

$$\begin{aligned} A &\equiv \int_{\Delta V}\left[B^2 - \left(\frac{E}{c}\right)^2\right]\,dV \\ &= \left(\frac{E_0}{c}\right)^2 \int_0^a dr\;2\pi\,r \int_0^{d(r)} dz\left[J_1^2(j_{01}\,\frac{r}{a}) - J_0^2(j_{01}\,\frac{r}{a})\right] \\ &= 2\pi\left(\frac{E_0}{c}\right)^2 \int_0^a dr\;r\;d(r)\;\left[J_1^2(j_{01}\,\frac{r}{a}) - J_0^2(j_{01}\,\frac{r}{a})\right] \end{aligned}$$

Plugging in the deflection formula yields

$$\begin{aligned} A &= 2\pi\,a^2\,d_0\left(\frac{E_0}{c}\right)^2 \int_0^1 dx\;x\;(1-x^2)\;(1-\alpha\;x^2)\left[J_1^2(j_{01}\,x) - J_0^2(j_{01}\,x)\right] \\ \frac{\delta\omega}{\omega} &= -\frac{1}{J_1^2(j_{01})}\,\frac{d_0}{L}\int_0^1 dx\;x\;(1-x^2)\;(1-\alpha\;x^2)\left[J_0^2(j_{01}\,x) - J_1^2(j_{01}\,x)\right] \end{aligned}$$

Using $\nu = 0.1$ for Be and evaluating the integral numerically, we get

$$\frac{\delta\omega}{\omega} = (-0.17768)\,\frac{d_0}{L}$$

Numerical example

Substituting 804.5 MHz for the center frequency and $L = 10.4$ cm for the length (MTA Modular Cavity),

$$\delta f = (-1.374\text{ kHz}/\mu\text{m})\;d_0 = (-34.9\text{ kHz/mil})\;d_0$$

The other relevant parameters for the Modular Cavity are $a = 14.1$ cm, $t = 1.27$ cm and $E = 290$ GPa for Be. For $p = 1$ atm, the corresponding maximum deflection d_0 and frequency shift δf are 58 μm and -78.6 kHz respectively. For 2 atm across one plate, we get 116 μm and -157.3 kHz.

Note that

- the accuracy of the estimate can be improved by using the deflection results from finite element analysis simulation directly in the perturbation integral (unlikely to make much difference) or using fancier perturbation theory with additional modes in the expansion (total waste of time)
- the normal operating condition (1 atm across each of the 2 plates) is equivalent to the 2 atm case above if deformation of the center ring is ignored; to get an estimate for the center ring contribution, we start with the hoop stress on the ring

  $$\sigma_{\phi\phi} = \frac{p\,a}{t_{cr}}$$

  and the corresponding radial strain

  $$\frac{\delta a}{a} = \frac{\sigma_{\phi\phi}}{E_{cr}}$$

  to get

  $$\frac{\delta f}{f} = \frac{\delta a}{a} = \frac{p\,a}{E_{cr}\,t_{cr}} \simeq 8.346 \times 10^{-6} \to \delta f \simeq 6.72 \text{ kHz}$$

  for $p = 1$ atm using the thickness of the ring $t_{cr} = 1.46$ cm and $E_{cr} = 117$ GPa for Cu
- the other effect on frequency shift when evacuating the cavity comes from the change in relative permittivity and is given by

  $$\frac{\delta f}{f} = \frac{1}{2}\left[(\epsilon_r - 1) + (\mu_r - 1)\right] \to \delta f \simeq 237.5 \text{ kHz}$$

  using $\epsilon_r - 1 = 5.9 \times 10^{-4}$ and $\mu_r - 1 = 3.7 \times 10^{-7}$ for air; thus, the frequency is expected to go up by 86.9 kHz after the cavity is evacuated and down by 78.6 kHz if one Be endplate is subjected to an additional load of 1 atm (gauge) external pressure (for a net change of +8.3 kHz)
- if the frequency measurement is to be used for monitoring deflection, the temperature should be controlled or monitored; this correction can be estimated as

  $$\frac{\delta f}{f} = \frac{\delta a}{a} \simeq -\kappa\,\Delta T$$

where $\kappa = 1.65 \times 10^{-5}$ is the coefficient of thermal expansion for Cu which gives

$$\delta f = (-13.3 \text{ kHz/K}) \, \Delta T$$