# $z$-Expansion Formalism

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GENIE $z$-Expansion Mini-Workshop

# While I have your attention. . .

Thanks to everyone who helped put the workshop together and for coming to listen!

# Outline

Theory

- Introduction - why should we study nucleon cross sections?
- $z$-expansion formalism
- Breakdown

Vector form factor status

Deuterium Bubble Chamber Fits

- Re-fit of bubble chamber data using $z$-expansion

CKM Matrix Elements

# Motivation

$$\Phi(E_\nu) = \frac{\mathcal{N}(E_\nu)}{\sigma_A(E_\nu)}$$

Oscillation experiments monitor flux by counting interactions assuming cross section, near/far detector do not perfectly cancel

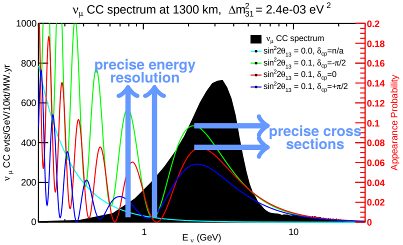
$\implies$ Measurements of neutrino oscillation depend on precise knowledge of neutrino cross section

$$\sigma_A \sim \sigma_{CCQE} \otimes \text{(nucl. models)}$$

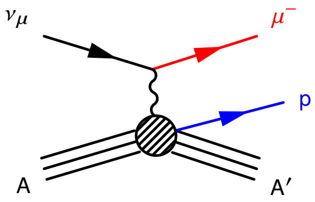
($\sigma_{CCQE}(E_\nu, Q^2)$ is quadratic function of form factors)

- Large nuclear targets $\implies$ measurements of oscillation parameters depends on nuclear models
- Nuclear effects entangled with nucleon amplitudes $\implies$ factorization is oversimplification
- Model-dependent shape parameterization introduces systematic uncertainties and underestimates errors

# Cross Sections




*(Figure from LBNE, 1307.7335 [hep-ex])*



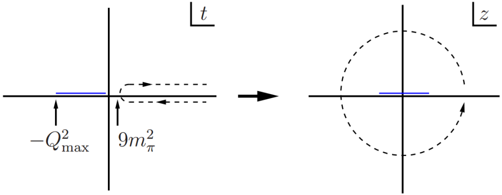

*Charge Current QE scattering*

- Measurements of neutrino parameters require precise knowledge of cross sections
- Nuclear cross sections obtained using nucleon amplitudes as input to nuclear models
- Uncertainty on $F_A(Q^2)$ is primary contribution to systematic errors
  - $F_{1V}$, $F_{2V}$ known from $e-p$ scattering
  - $F_P$ suppressed by lepton mass in cross sections
- Focus on $F_A$, but $z$-expansion could be applied to any of form factors

# $z$-Expansion

The z-Expansion (1108.0423 [hep-ph]) is a conformal mapping which takes the kinematically allowed region ($t = -Q^2 \leq 0$) to within $|z| < 1$

$$z(t;t_0,t_c) = \frac{\sqrt{t_c - t} - \sqrt{t_c - t_0}}{\sqrt{t_c - t} + \sqrt{t_c - t_0}} \qquad F_A(z) = \sum_{k=0}^{\infty} a_k z^k \qquad t_c = 9m_\pi^2$$




$$z(t = t_c) = -1 \qquad z(t = t_0) = 0 \qquad z(t = -\infty) = 1$$

# $z$-Expansion: $t_c$

$$z(t; t_0, t_c) = \frac{\sqrt{t_c - t} - \sqrt{t_c - t_0}}{\sqrt{t_c - t} + \sqrt{t_c - t_0}} \qquad t_c = 9m_\pi^2$$

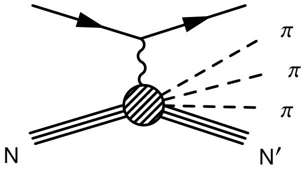
$t_c$ is the 4-momentum cutoff for particle production

$F_A \implies 3\pi$ threshold
(1$\pi$ forbidden by kinematics, 2$\pi$ forbidden by $G$-parity)

$G$-parity:

$$\eta_G = (-1)^{L+I+S}$$

(both $N$ multiplet and $\pi$ multiplet have charge $-1$)



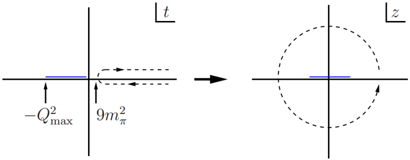
# $z$-Expansion: $t_0$

$$z(t; t_0, t_c) = \frac{\sqrt{t_c - t} - \sqrt{t_c - t_0}}{\sqrt{t_c - t} + \sqrt{t_c - t_0}} \qquad z(t = t_0) = 0$$

$t_0$ is an unphysical parameter which can be used to improve convergence

The best $t_0$ is determined by the interesting kinematic range:

$$t_0^{\text{opt}}(Q^2_{\text{max}}) = t_c\left(1 - \sqrt{1 + Q^2_{\text{max}}/t_c}\right)$$




| $Q^2_{\text{max}}[\text{GeV}^2]$ | $t_0[\text{GeV}^2]$ | $\|z\|_{\text{max}}$ |
|---|---|---|
| 1.0 | 0 | 0.44 |
| 3.0 | 0 | 0.62 |
| 1.0 | $t_0^{\text{opt}}(1.0\ \text{GeV}^2) = -0.28$ | 0.23 |
| 3.0 | $t_0^{\text{opt}}(1.0\ \text{GeV}^2) = -0.28$ | 0.45 |
| 3.0 | $t_0^{\text{opt}}(3.0\ \text{GeV}^2) = -0.57$ | 0.35 |

# $z$-Expansion: Sum Rules

In practice, only finite order expansion

$$F_A(z) = \sum_{k=0}^{k_{\max}} a_k z^k$$

We can use sum rules to enforce behavior of the form factor in certain limits

- $F_A(Q^2 = 0) = g_A \quad \Longrightarrow \quad a_0$ fixed
- $F_A(Q^2 = \infty) = 0 \quad \Longrightarrow \quad a_{k_{\max}}$ fixed

The second of these rules comes from the perturbative QCD requirement:

$$\lim_{t\to-\infty} F_A(t) = a_0 + a_1 + \ldots \quad \propto (-t)^{-2} \qquad \left(\lim_{t\to-\infty} z(t) = 1\right)$$

More sum rules can be obtained by taking derivatives:

$$\lim_{t\to-\infty} \frac{d}{dz} F_A(z(t)) = \frac{dt}{dz}\frac{d}{dt} F_A(z(t)) \propto (-t)^{-3/2}$$

$$\lim_{t\to-\infty} \frac{d^2}{dz^2} F_A(z(t)) \propto (-t)^{-1} \qquad \lim_{t\to-\infty} \frac{d^3}{dz^3} F_A(z(t)) \propto (-t)^{-1/2}$$

These relations fix $a_k$ for $k \in \{0, k_{\max}-3, k_{\max}-2, k_{\max}-1, k_{\max}\}$

# Coefficient Bounds

Finiteness of the coefficients can be shown by defining a norm:

$$||F_A||_p \equiv \left(\sum_k |a_k|^p\right)^{\frac{1}{p}}$$

$p = 2$ can be related directly to the dispersion integral integrated around the unit circle:

$$||F_A||_2 = \left(\sum_k |a_k|^2\right)^{\frac{1}{2}} = \left(\oint \frac{dz}{z} |F_A|^2\right)^{\frac{1}{2}}$$

This integral can be shown to be finite, so the $a_k$ must be bounded and decreasing for large $k$

$p = \infty$ is a special case,

$$||F_A||_\infty = \lim_{p\to\infty} \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} \to \max\{|a_k|\}$$

with $||F_A||_\infty \leq ||F_A||_2$, so $||F_A||_2$ can overestimate relevant coefficient size

# Order Unity Coefficients

Order unity coefficients can be motivated by an ansatz for the form factor

For instance, axial vector meson dominance ansatz with Breit-Wigner form:

$$F_A \sim \frac{m_{a_1}^2}{m_{a_1}^2 - t - i\Gamma_{a_1} m_{a_1}} \equiv -\frac{m_{a_1}^2}{b(t)}$$

This is recovered by assuming the dispersion relation

$$\mathrm{Im} F_A(t + i0) = \frac{\mathcal{N} m_{a_1}^3 \Gamma_{a_1}}{|b(t)|^2} \theta(t - t_c)$$

Given this ansatz, can get analytical expression for $z$-expansion coefficient bound

From $||F_A||_\infty$, estimate ratio of order-0 and largest $z$-expansion coefficients:

$$\left|\frac{a_k}{a_0}\right| \leq \frac{2|\mathcal{N}|}{|F_A(t_0)|} \mathrm{Im}\left(\frac{-m_{a_1}^2}{b(t_c) + \sqrt{(t_c - t_0) b(t_c)}}\right) \qquad \text{for all } k$$

This expression demonstrates that we can expect coefficients of order unity

| | $t_0 = 0$ | $t_0 = t_0^{\mathrm{opt}}(1.0\ \mathrm{GeV}^2)$ |
|---|---|---|
| $\lvert\lvert F_A \rvert\rvert_2 / \lvert F_A(t_0) \rvert$ | 1.5-1.7 | 1.9-2.3 |
| $\lvert\lvert F_A \rvert\rvert_\infty / \lvert F_A(t_0) \rvert$ | 1.0-1.4 | 1.4-1.8 |

(arXiv 1108.0423[hep-ph])

# Theory Summary

$z$-expansion is a model-independent description of the axial form factor

- Motivated by analyticity arguments
- Allows quantification of systematic errors

Coefficients in the $z$-expansion fall off as the order $k$ increases

- Only a few coefficients needed to represent the form factor
- Good convergence of form factor
- Provides a prescription for introducing more parameters as data improves

Good control of form factor outside of kinematically interesting region

- Expansion parameter $|z| < 1$ for entire kinematic region ($t \in [0, \infty)$), can determined the required number of coefficients in the expansion *a priori*
- Convergence can be improved by varying $t_0$
- Sum rules to control large-$Q^2$ behavior, enforce falloff required by pertrubative QCD (up to log corrections)

# Backup

# CCQE Cross section

(Formaggio, Zeller 1305.7513[hep-ex])

$$\sigma_{CCQE}(E_\nu, Q^2) \propto \frac{1}{E_\nu^2}\left(A(Q^2) \mp \left(\frac{s-u}{M_N^2}\right)B(Q^2) + \left(\frac{s-u}{M_N^2}\right)^2 C(Q^2)\right)$$

$$s - u = 4M_N E_\nu - Q^2 - m_\ell^2 \qquad \eta \equiv \frac{Q^2}{4M_N^2}$$

$$A(Q^2) = \frac{m_\ell^2 + Q^2}{M_N^2} \times \Big[(1+\eta)F_A^2 - (1-\eta)(F_1^2 + \eta F_2^2) + 4\eta F_1 F_2 - \frac{m_\ell^2}{4M_N^2}\left((F_1+F_2)^2 + (F_A + 2F_P)^2 - 4(1+\eta)F_P^2\right)\Big]$$

$$B(Q^2) = 4\eta F_A (F_1 + F_2) \qquad C(Q^2) = \frac{1}{4}\left(F_A^2 + F_1^2 + \eta F_2^2\right)$$