

Symmetry Breaking in Haloscope Microwave Cavities

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UNIVERSITY *of* FLORIDA

2nd Workshop on
Microwave Cavities and
Detectors for Axion Research

Theory of Cavity Resonators

- Standing waves (TE/TM/TEM)

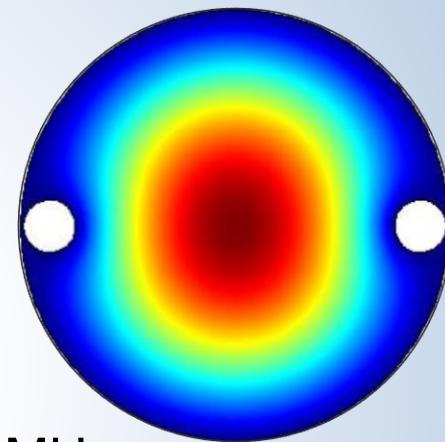
$$(\nabla^2 + \mu\epsilon\omega^2) \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = 0$$

- Assumes mode orthogonality
- Assumes geometric symmetries
 - $x/y (\rho/\Phi)$ symmetry: parity/radial/angular pattern (m,n)
 - z (longitudinal) symmetry: constant cross-section through length of the cavity (p)
- Perturbation theory used when small symmetries are broken

$$\frac{\Delta\omega}{\omega} \approx \frac{\Delta W_m - \Delta W_e}{W}$$

Tuned Cavity Resonators

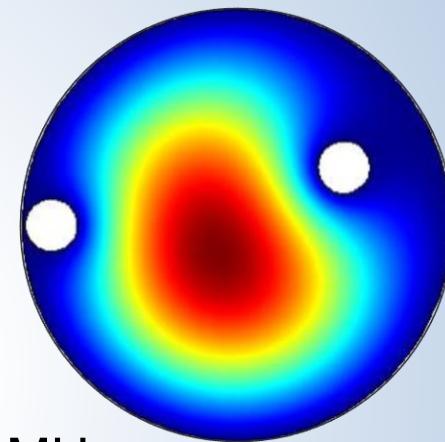
- Breaks $x/y (\rho/\Phi)$ symmetry to change TM frequency by localizing fields
 - First order perturbation theory does not predict $\Delta\omega$ well
 - assumes small changes in field
 - neglects localization effect
 - predicts max frequency of ADMX ~ 700 MHz
 - Requires modeling to accurately predict frequency
 - especially requires modeling to predict form factor, C , and quality factor, Q .



$$C = \frac{\left| \int \mathbf{B}_0 \cdot \mathbf{E} \, dV \right|^2}{B_0^2 V \int \mathbf{E} \cdot \mathbf{E} \, dV}$$

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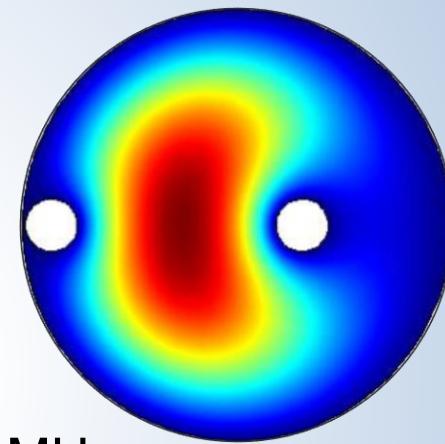
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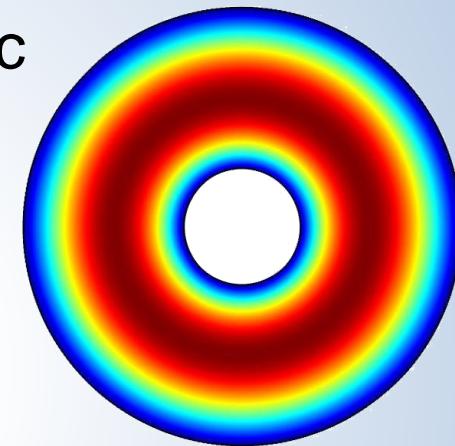
$$C = \frac{\left| \int \mathbf{B}_0 \cdot \mathbf{E} \, dV \right|^2}{B_0^2 V \int \mathbf{E} \cdot \mathbf{E} \, dV}$$

Transverse Symmetry Breaking

- Symmetry breaking changes field
 - Causes localization and asymmetric fields (m, n subscripts have no mathematical meaning)
 - Increase localized field width results in increase wavelength and decreased frequency
- Field solution constant in longitudinal direction

$$\left(\nabla^2 + \mu\epsilon\omega^2 \right) \left\{ \begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right\} = 0$$

- Longitudinal symmetry maintained (“simple” standing waves, ρ subscript meaning maintained)

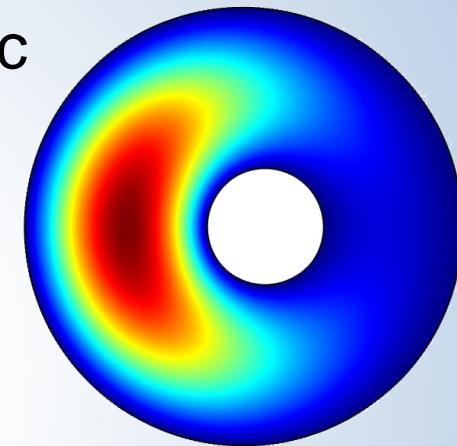


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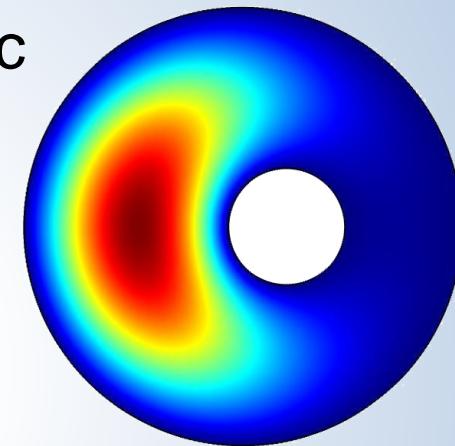


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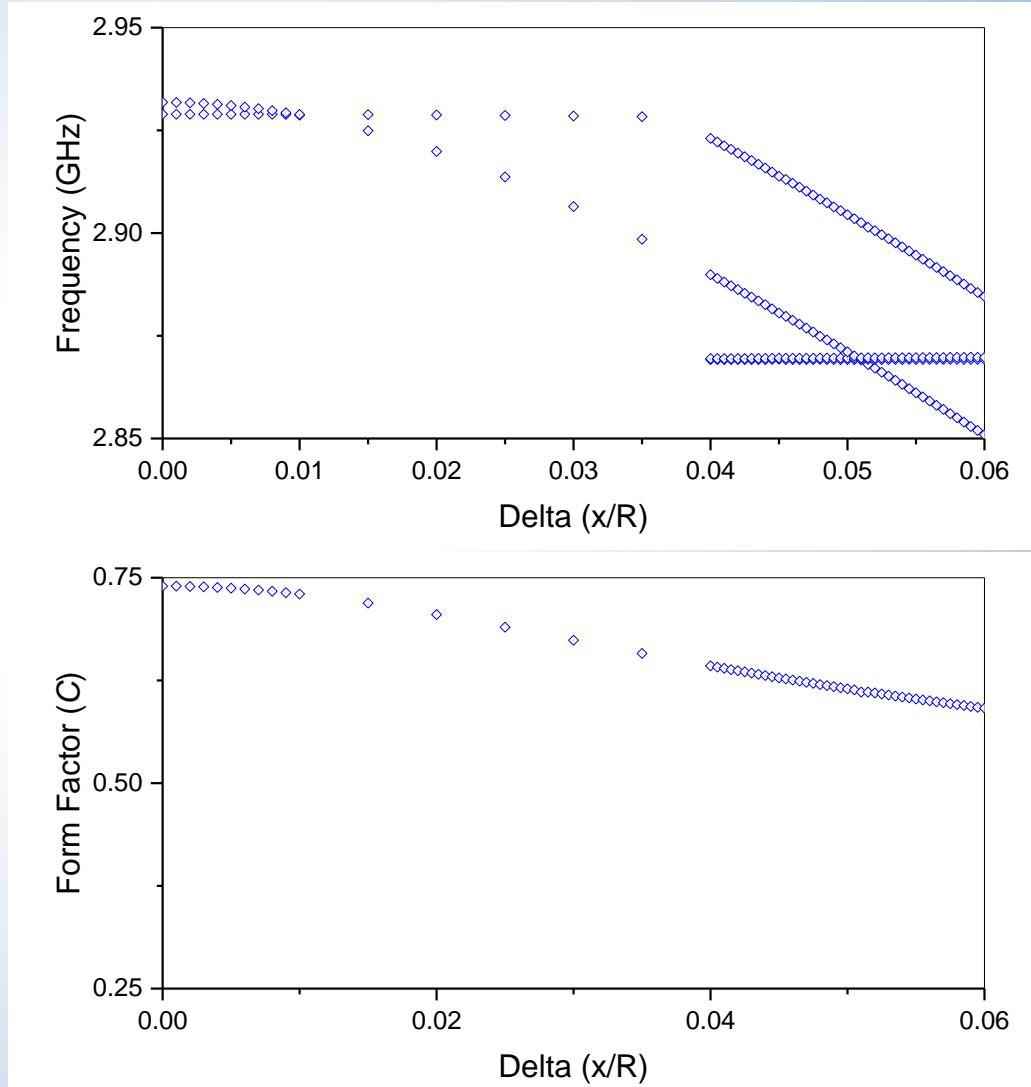
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 - No mode mixing
 - No “loss” in form factor
- Symmetry breaking reduces form factor

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$$\mathbf{E}(x,y,z,t) = \mathbf{E}(x,y)e^{\pm ikz - i\omega t}$$

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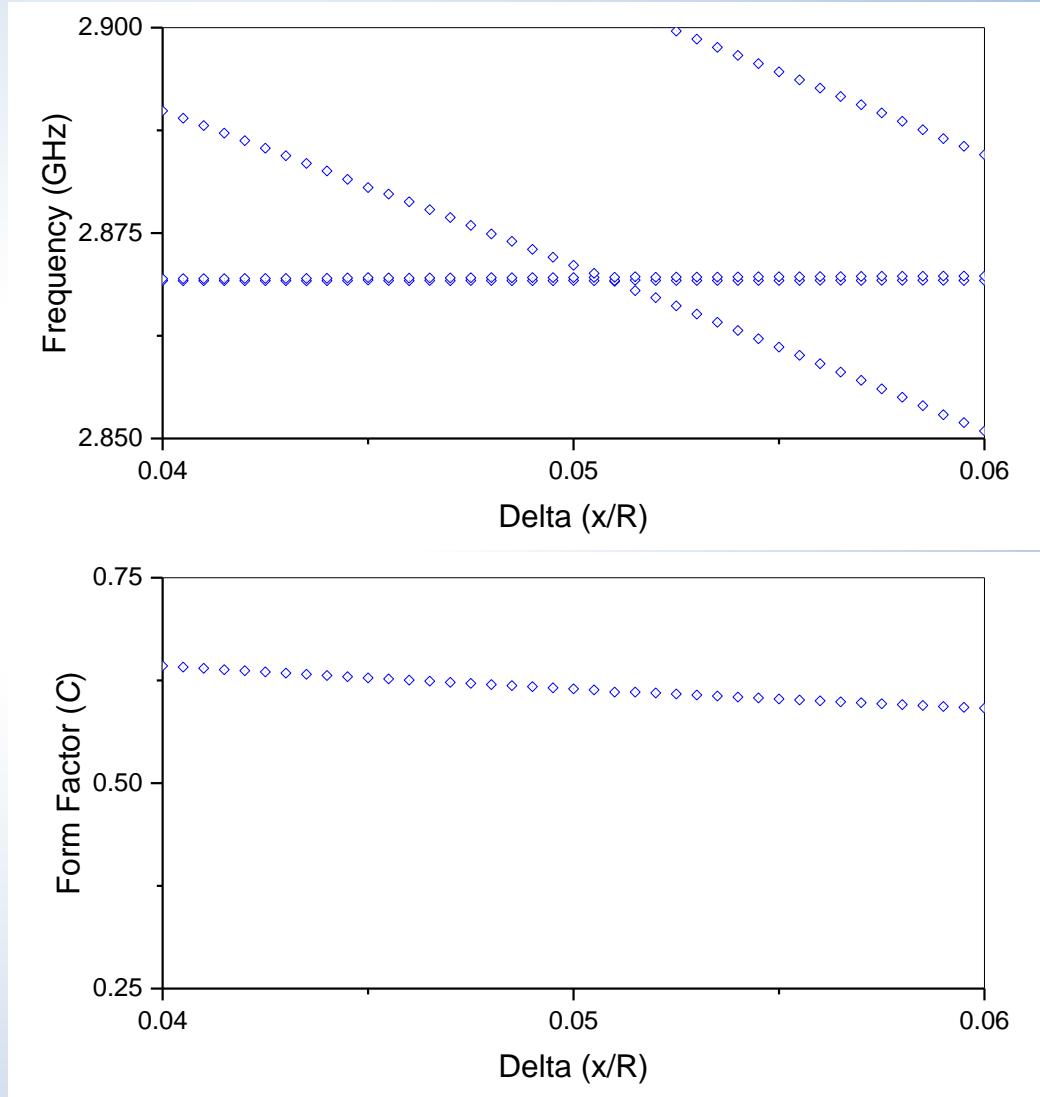
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Transverse Symmetry Breaking

- Inverse relationship between C & Q

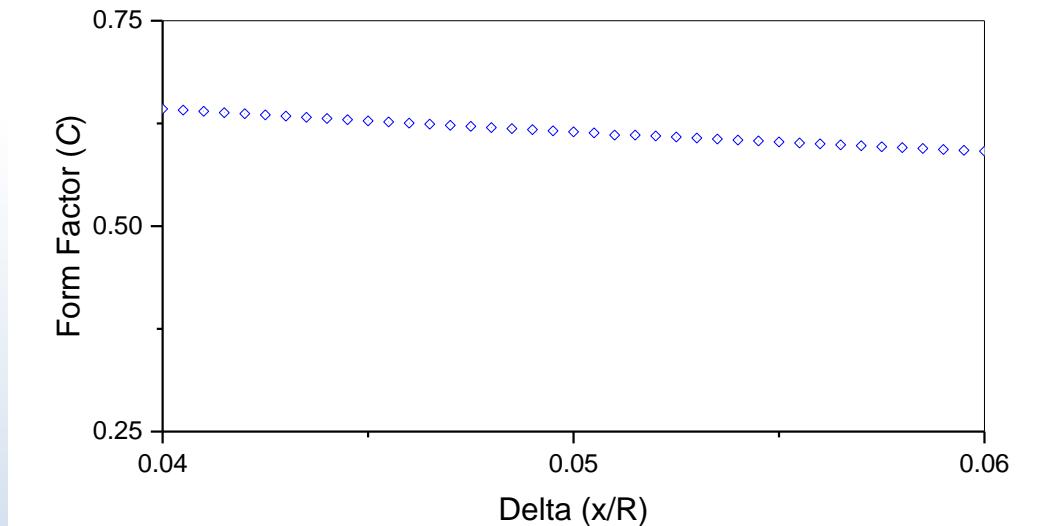
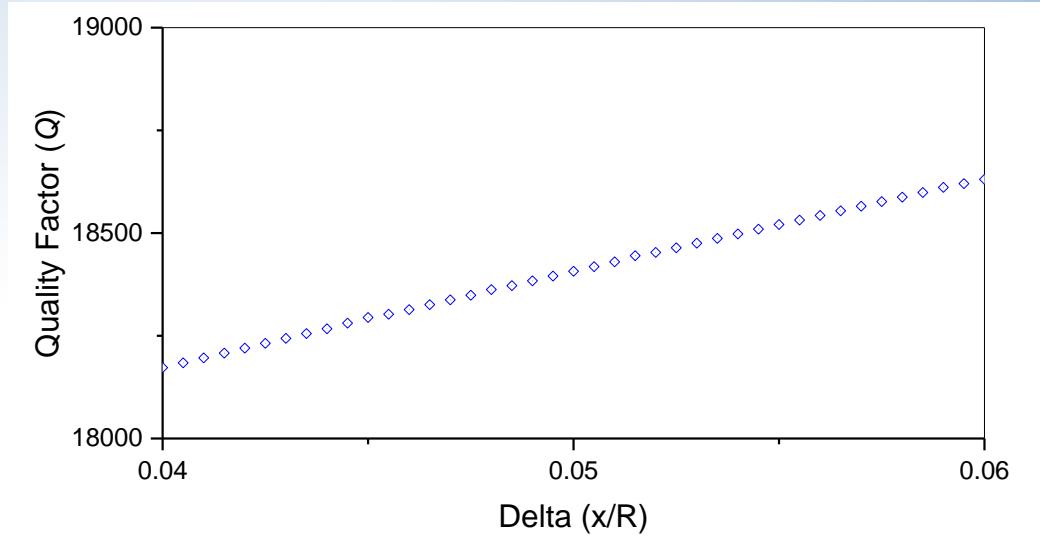
$$C \propto \frac{(\int E_z dV)^2}{\int \mathbf{E} \cdot \mathbf{E} dV}$$

$$Q \propto \frac{1}{1 + \xi \frac{d}{R}}$$

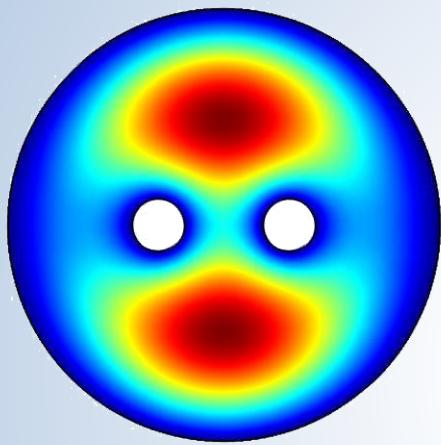
- for TM modes

$$\xi \propto \frac{(\int E_z dV)^2}{\int \mathbf{E} \cdot \mathbf{E} dV}$$

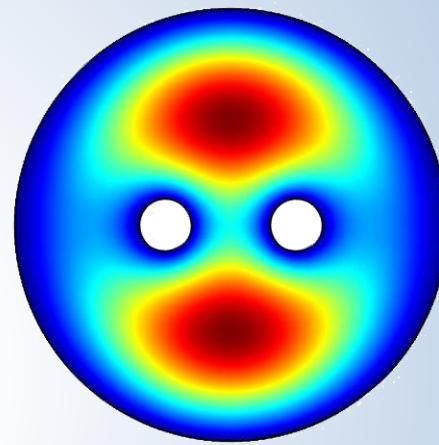
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Parity Symmetry Breaking

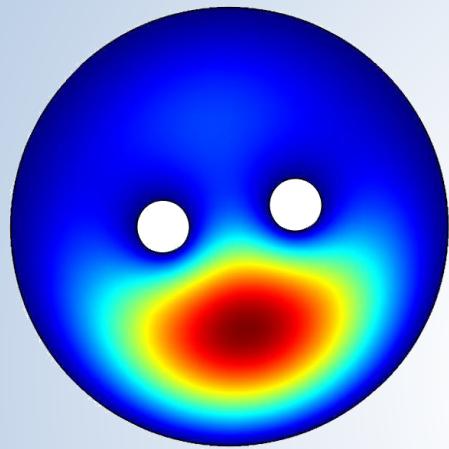


$$\Delta/R = 0.00$$

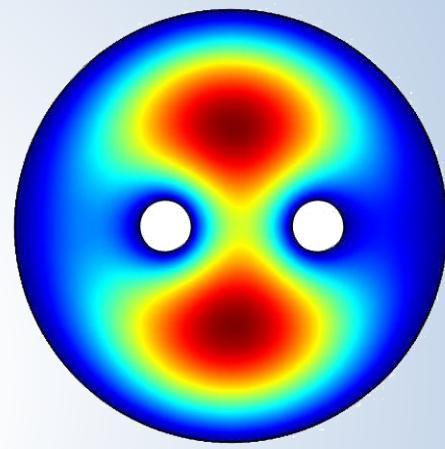


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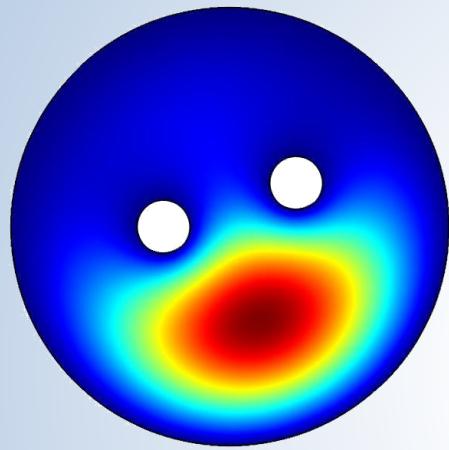


$\Delta/R = 0.10$

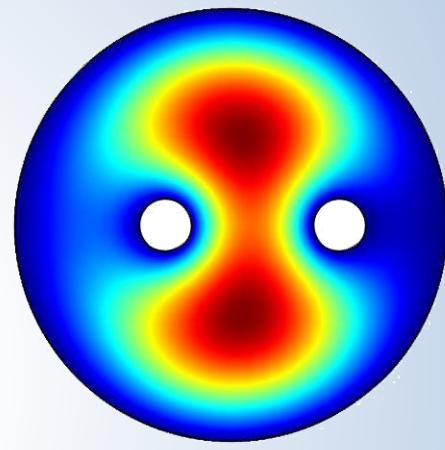


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Parity Symmetry Breaking



$\Delta/R = 0.20$



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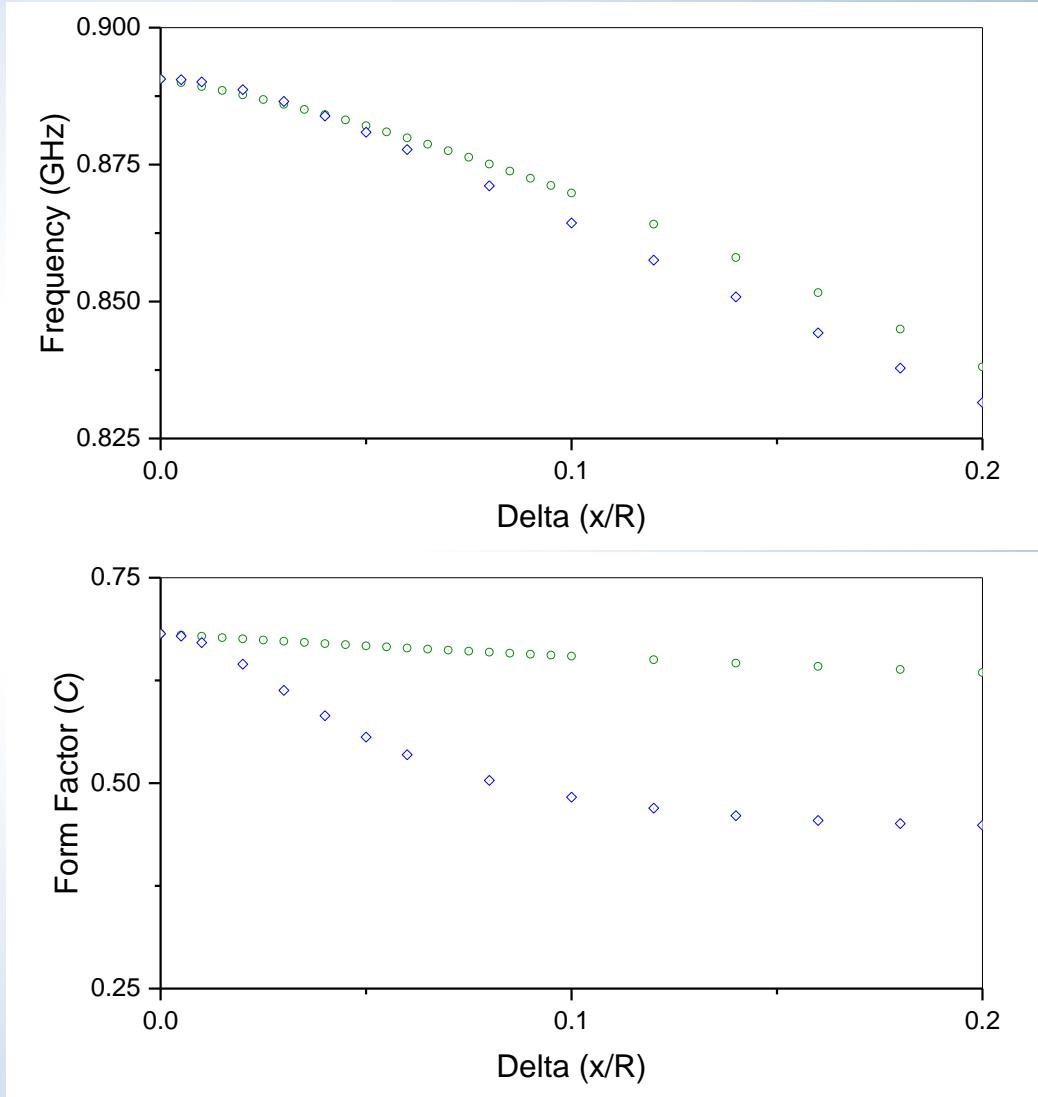
Parity Symmetry Breaking

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- Symmetry breaking reduces form factor
- Breaks some degeneracies

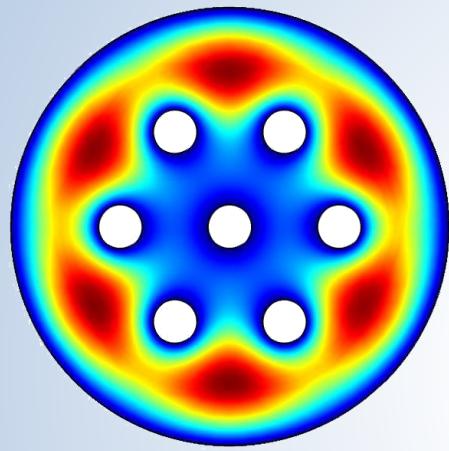
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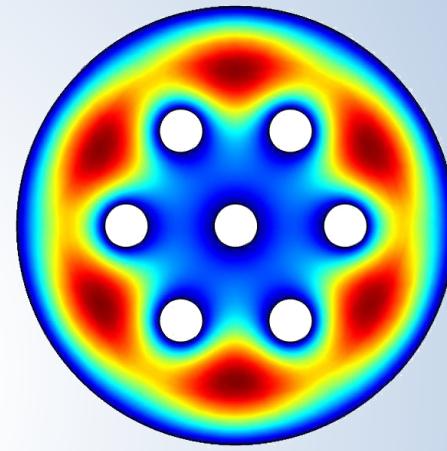
$$\mathbf{H}(x,y,z,t) = \mathbf{H}(x,y)e^{\pm ikz - i\omega t}$$



Angular Symmetry Breaking

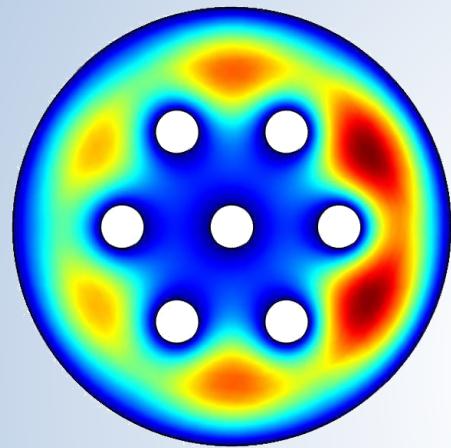


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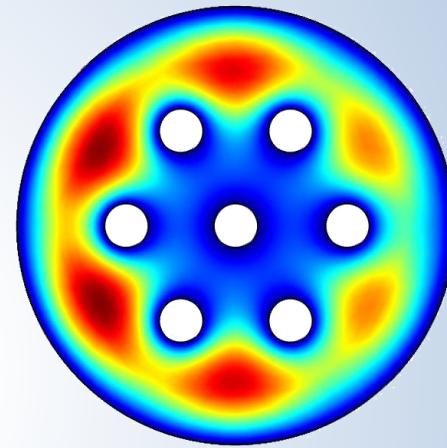


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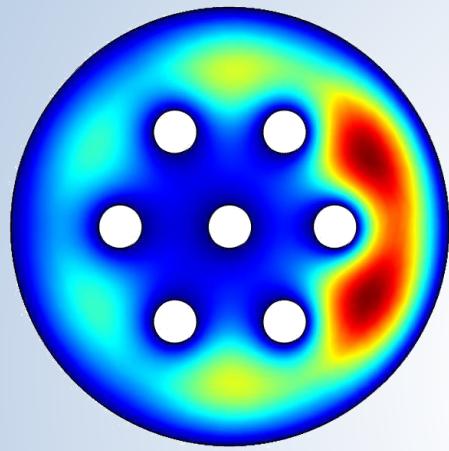


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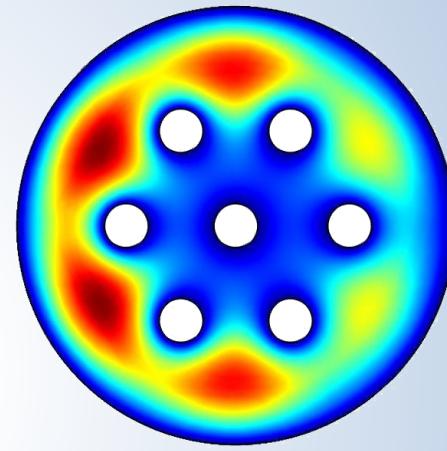


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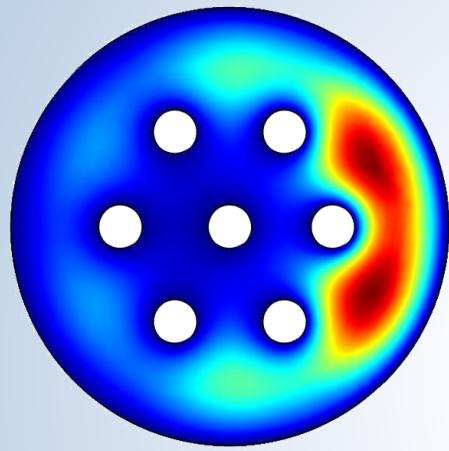


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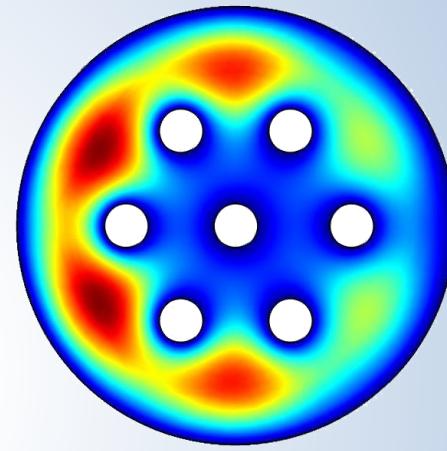


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Angular Symmetry Breaking



$$\Delta/R = 0.30$$



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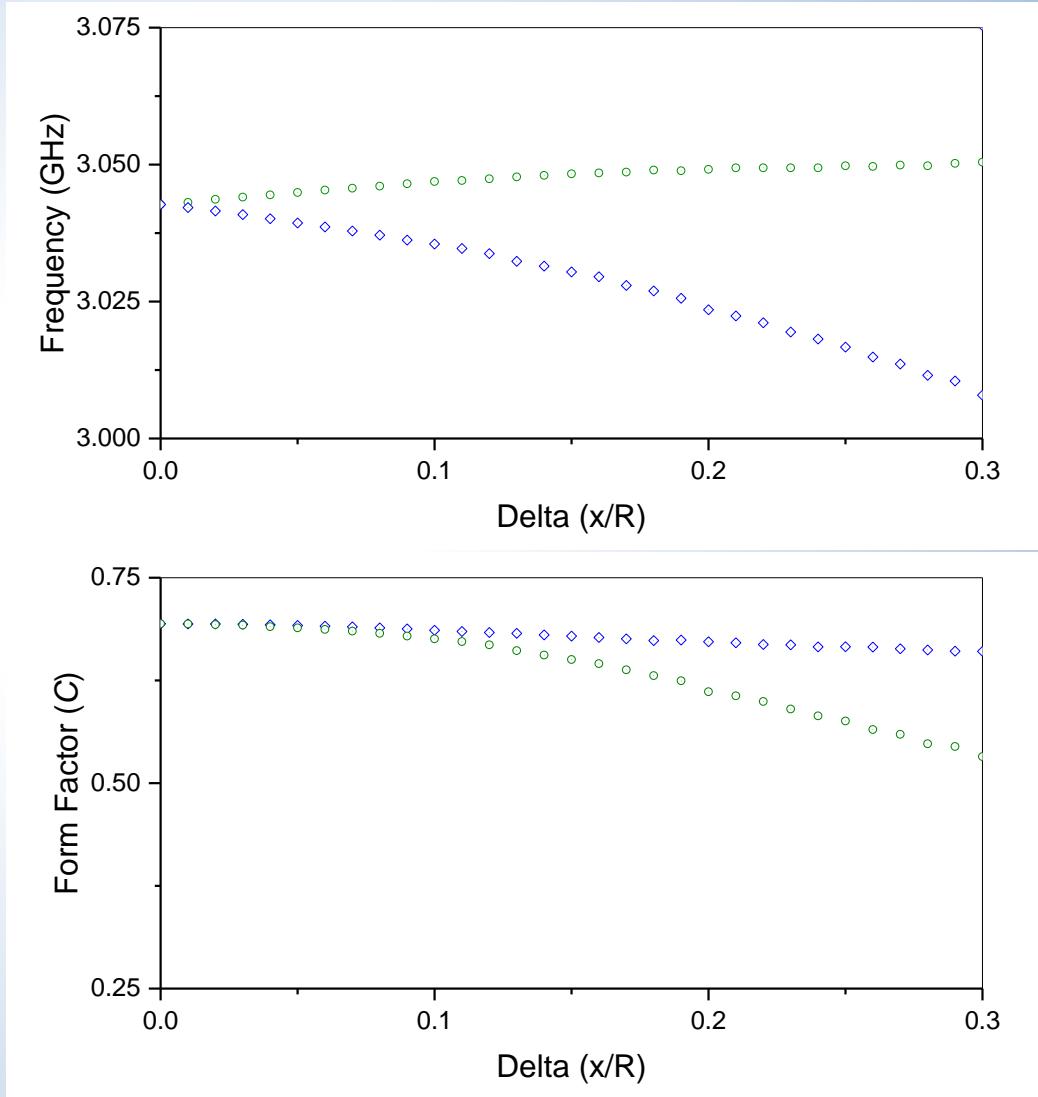
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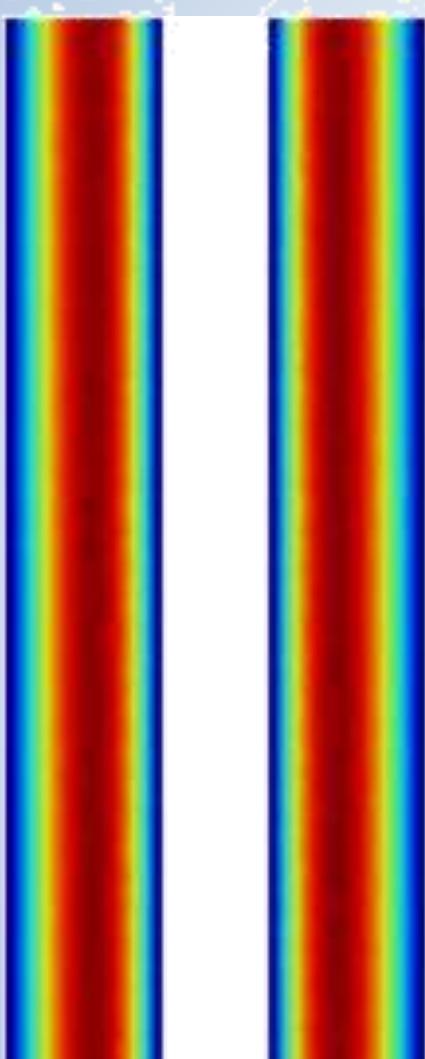
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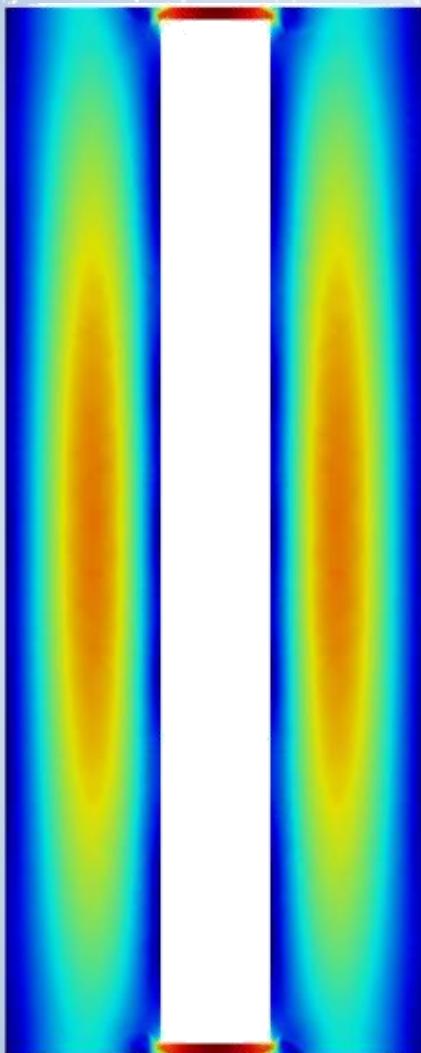
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- “Complex” standing waves

$$\Psi = \Psi(x, y, z) e^{-i\omega t}$$

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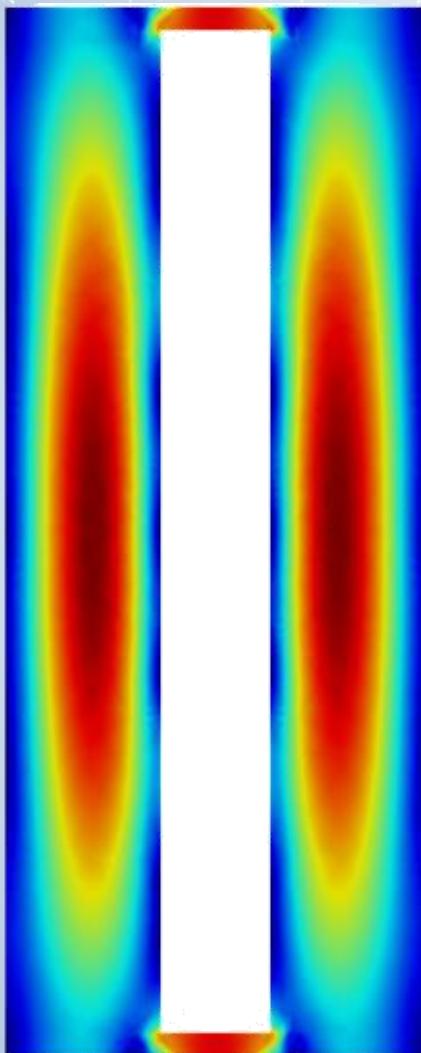
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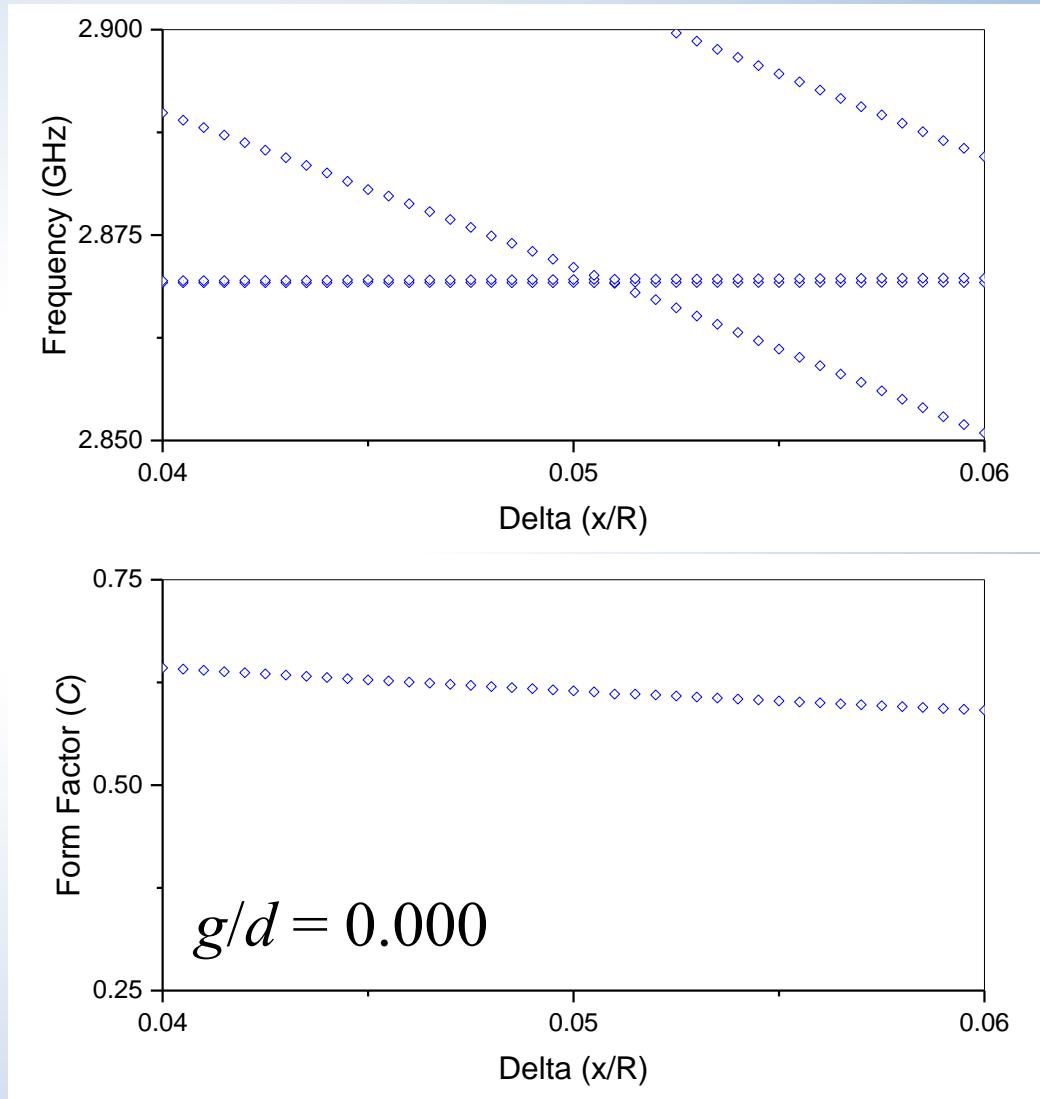
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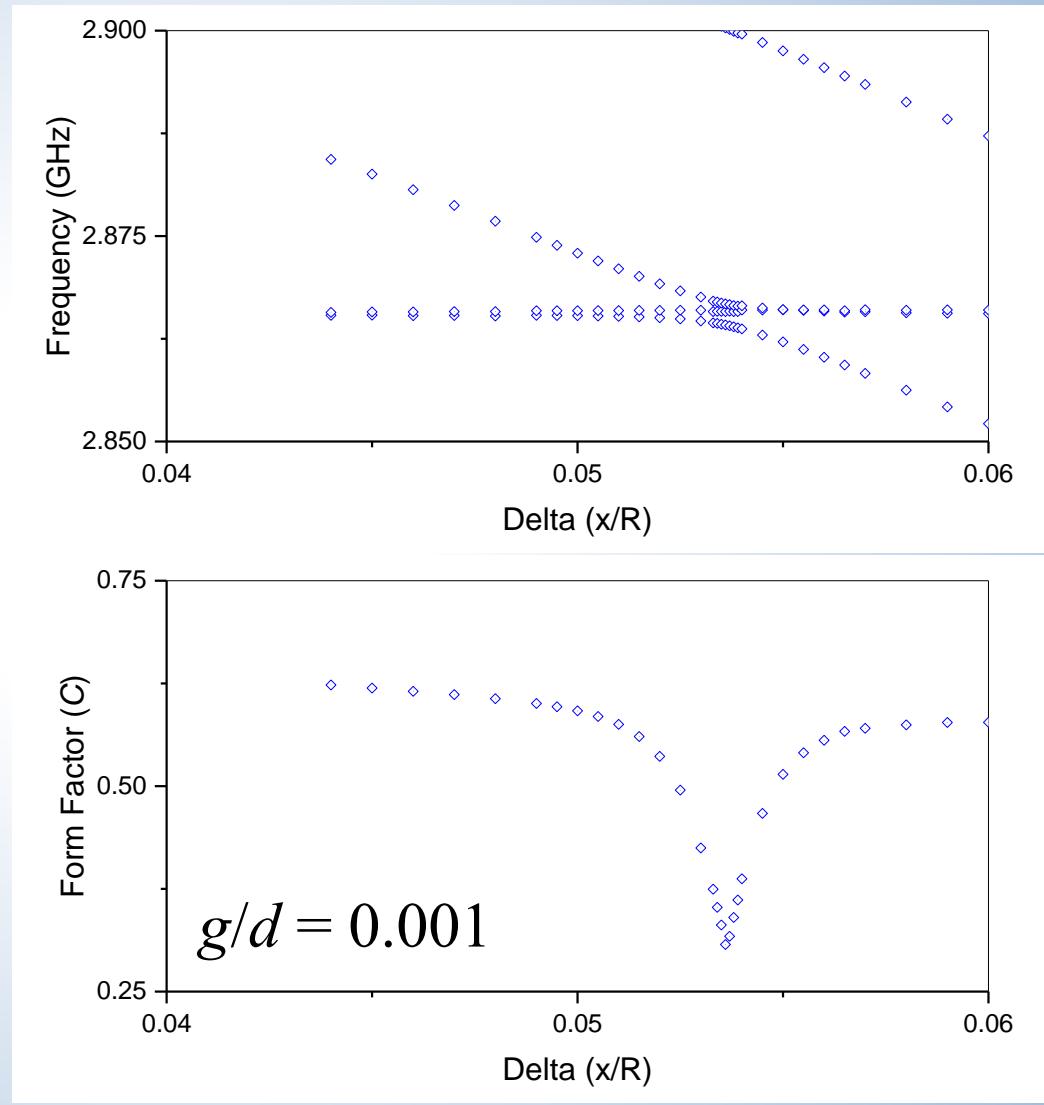
Longitudinal Symmetry Breaking (Gap)

- Mode orthogonality not maintained
 - Mode mixing
 - Gap in frequency
- Symmetry breaking reduces form factor
 - Mode mixing further reduces form factor
- Mode crowding
 - Breaks degeneracies
 - New modes (reentrant modes)



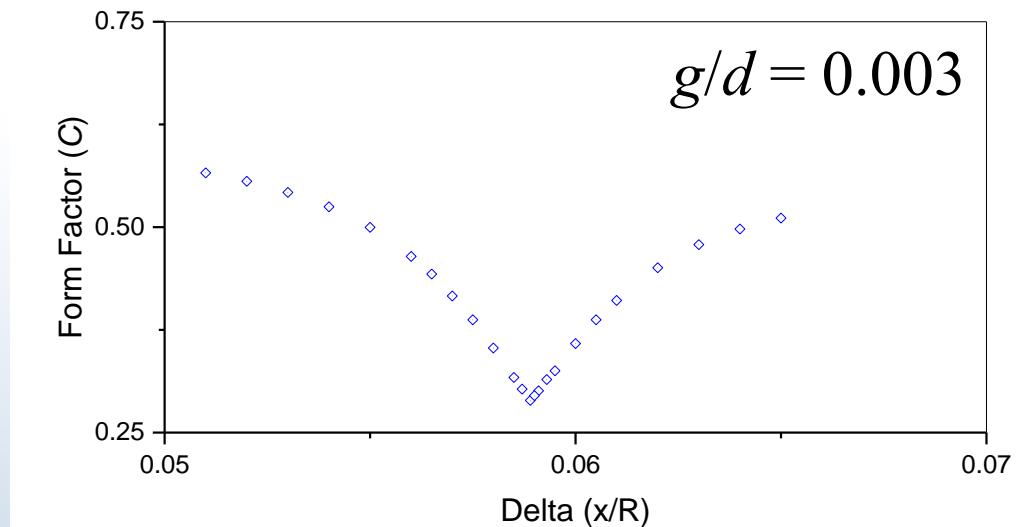
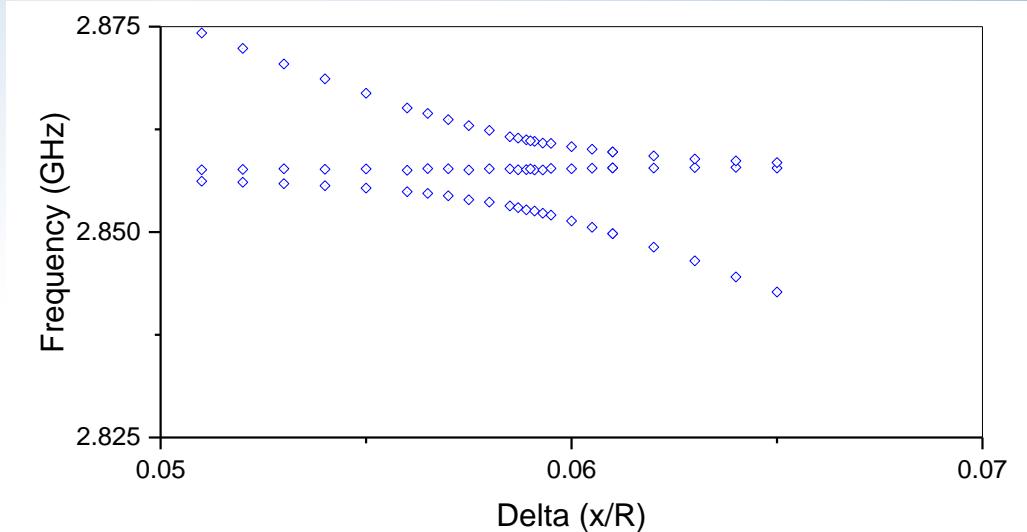
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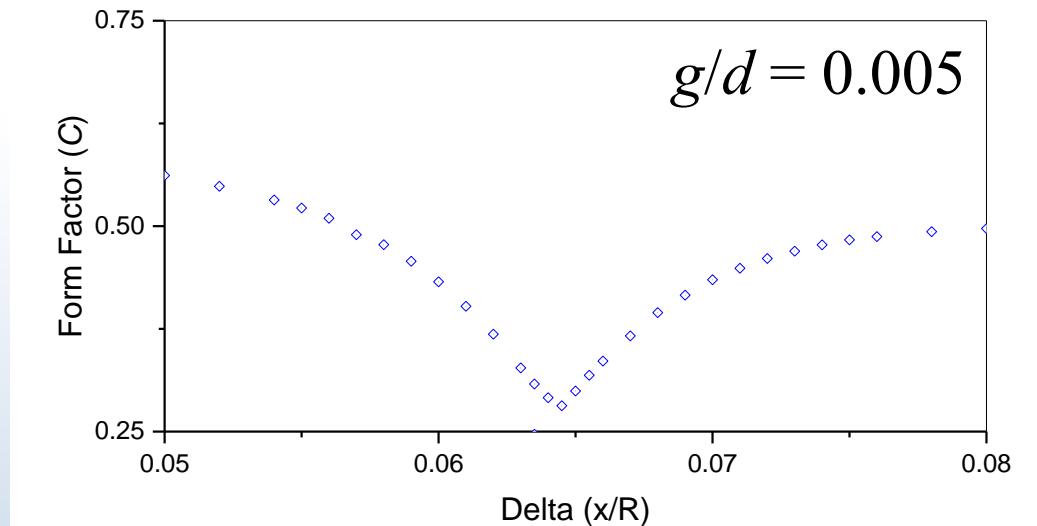
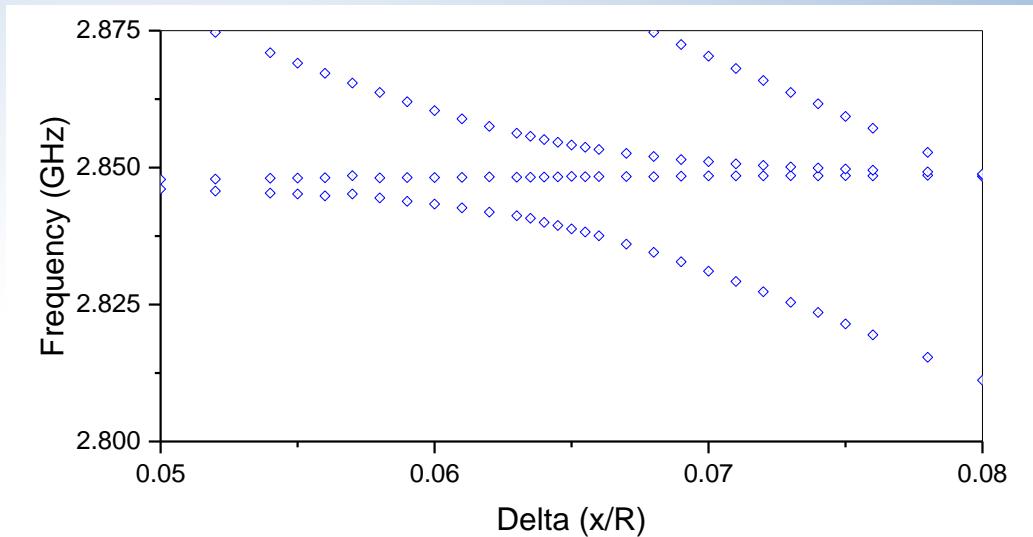
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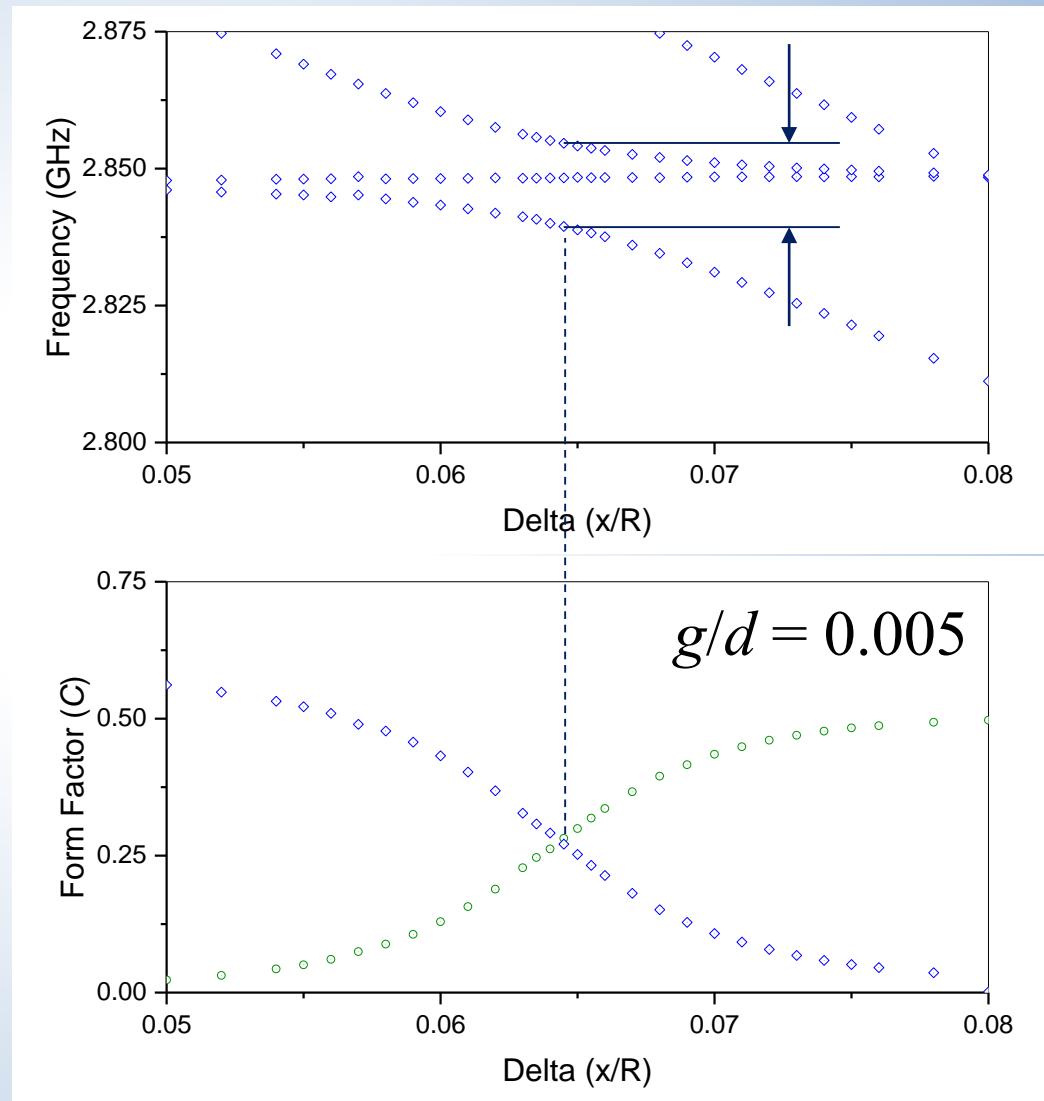
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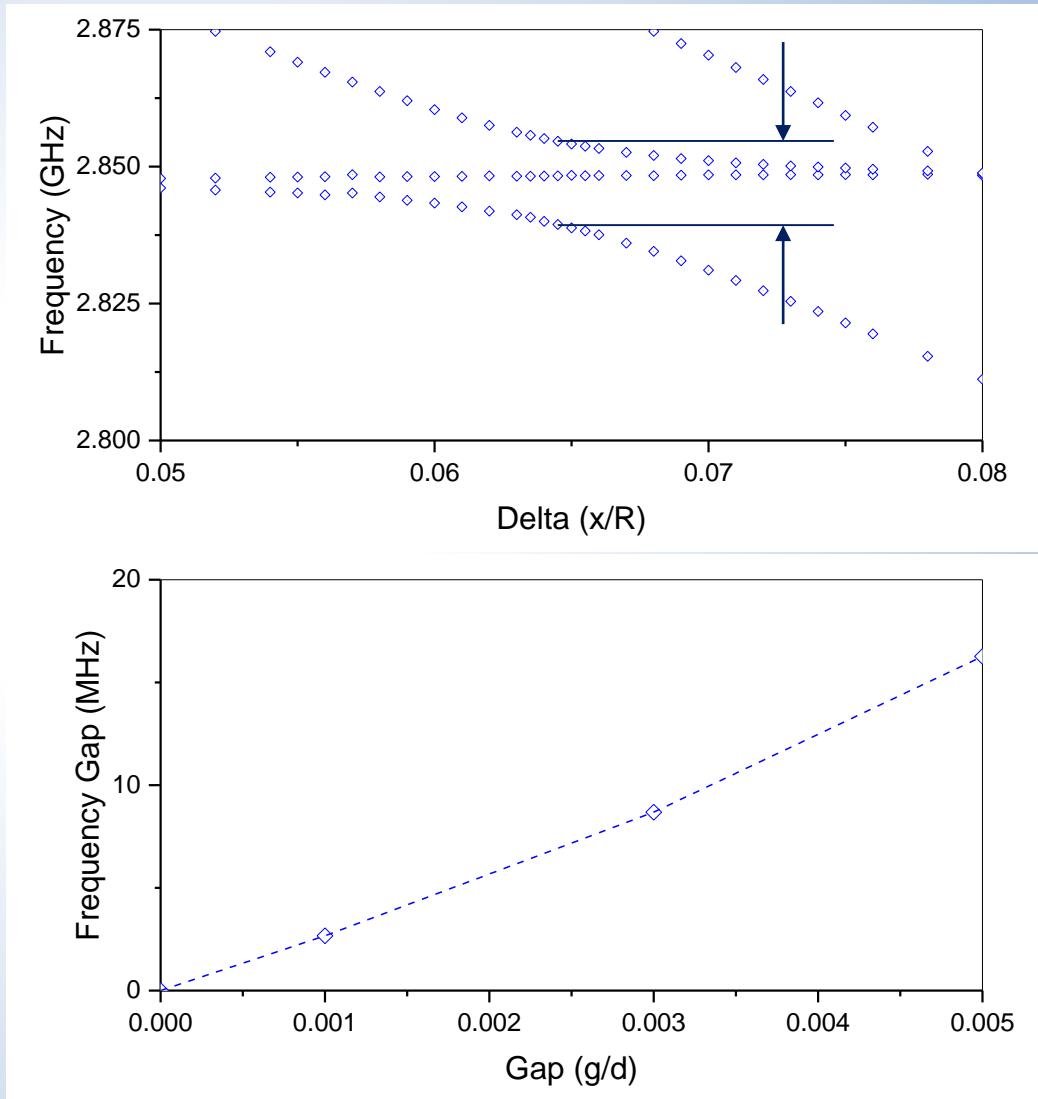
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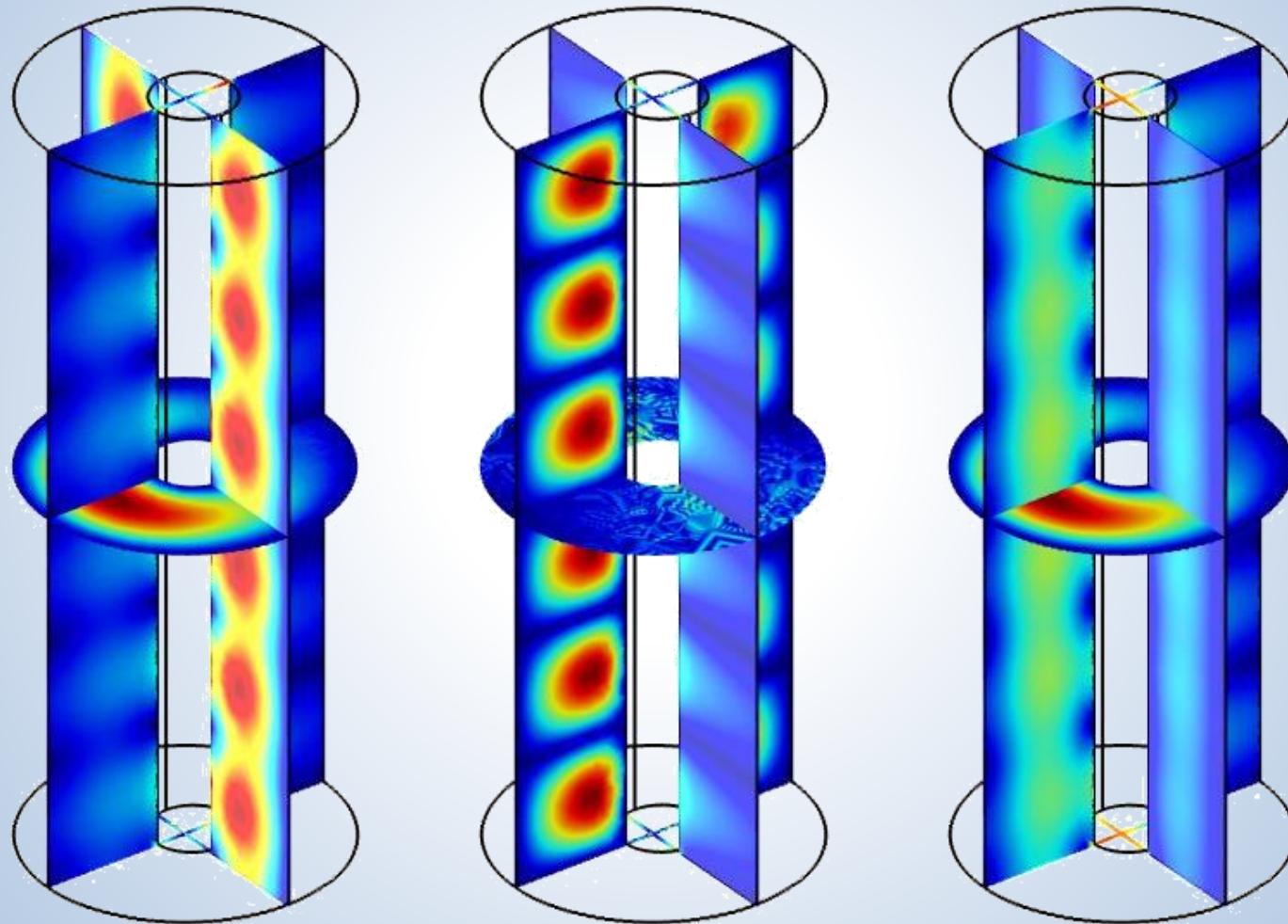


Frequency Gap

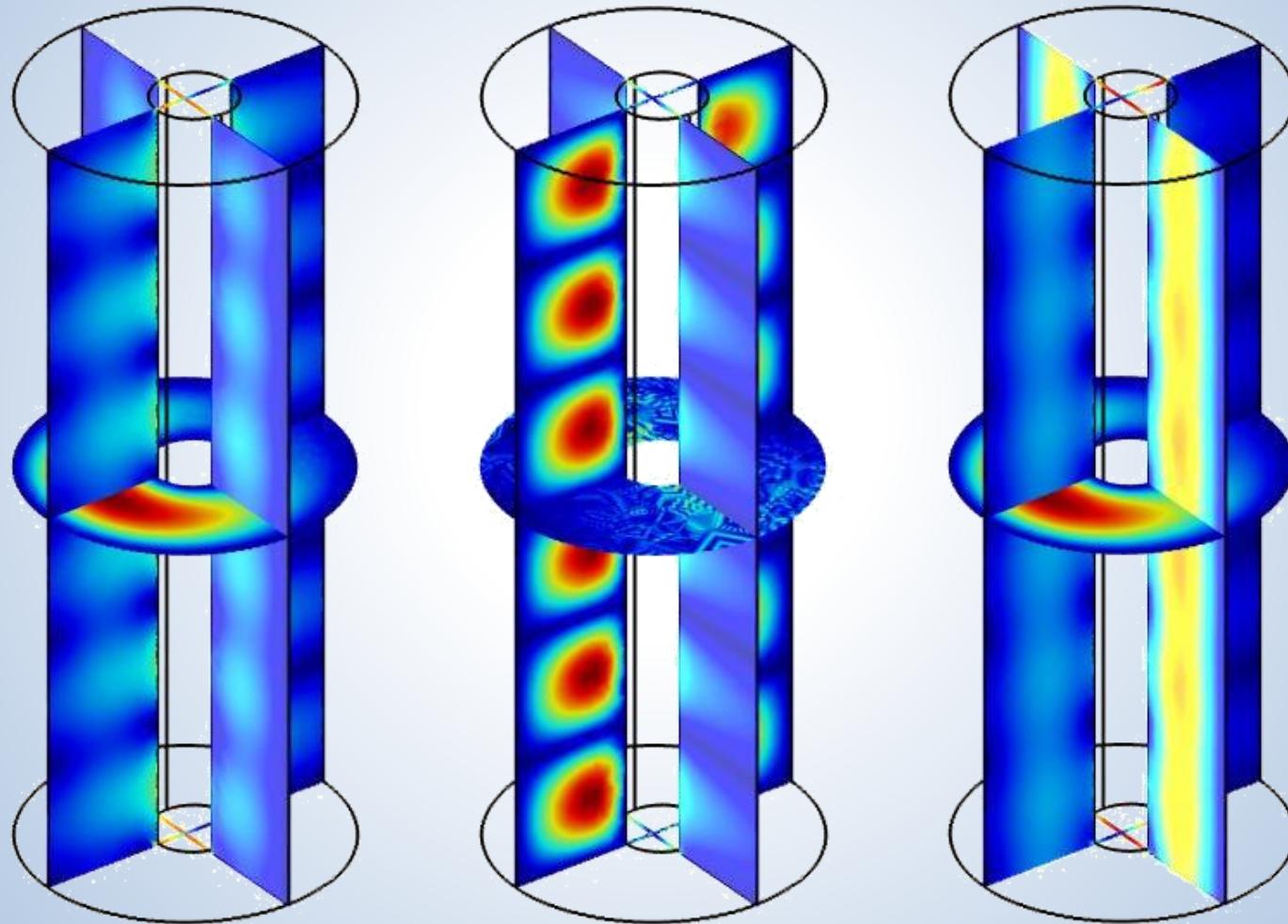
- Mode mixing causes gap in scan frequency range
- Frequency gap is proportional to rod gap



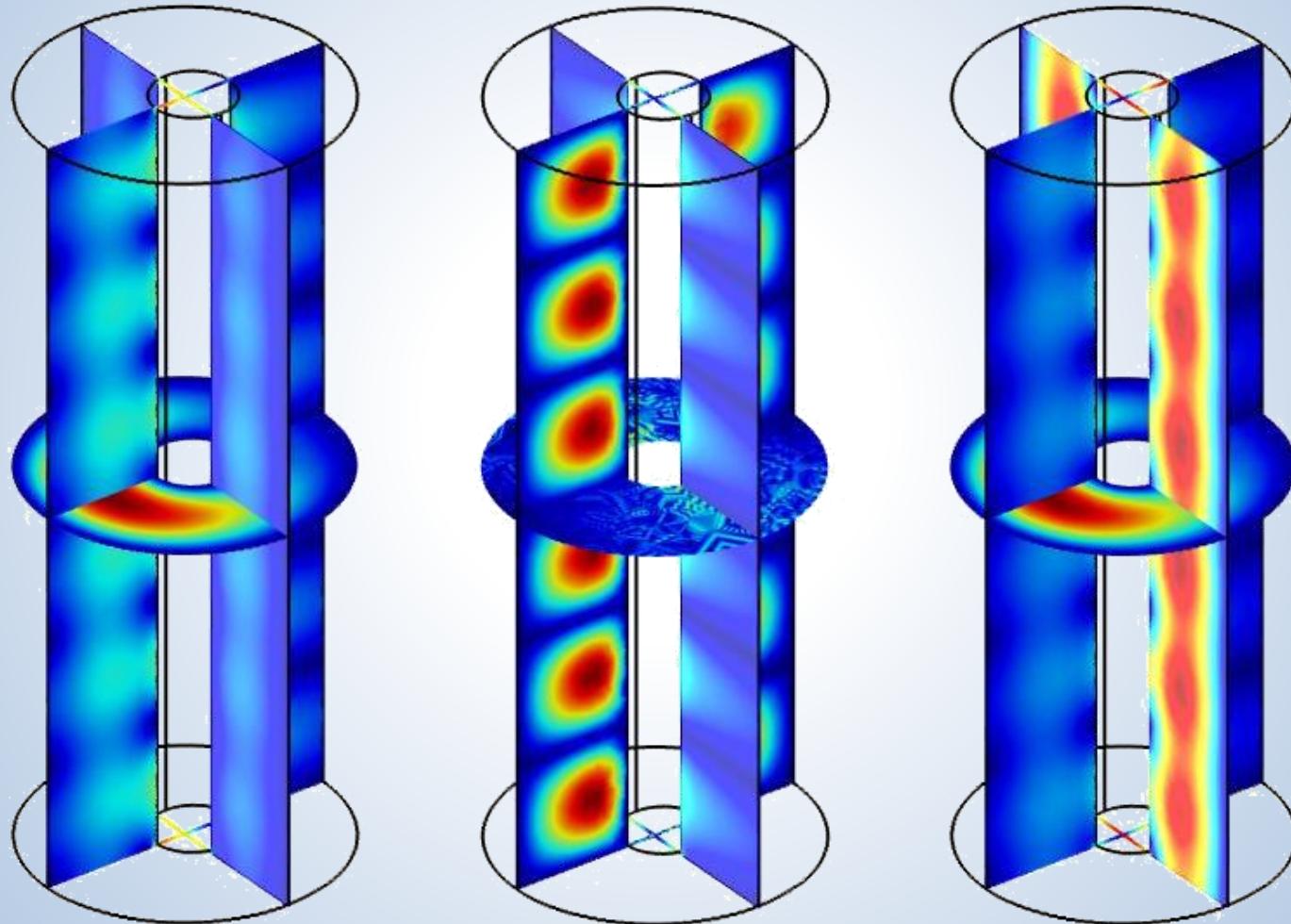
Mode Mixing



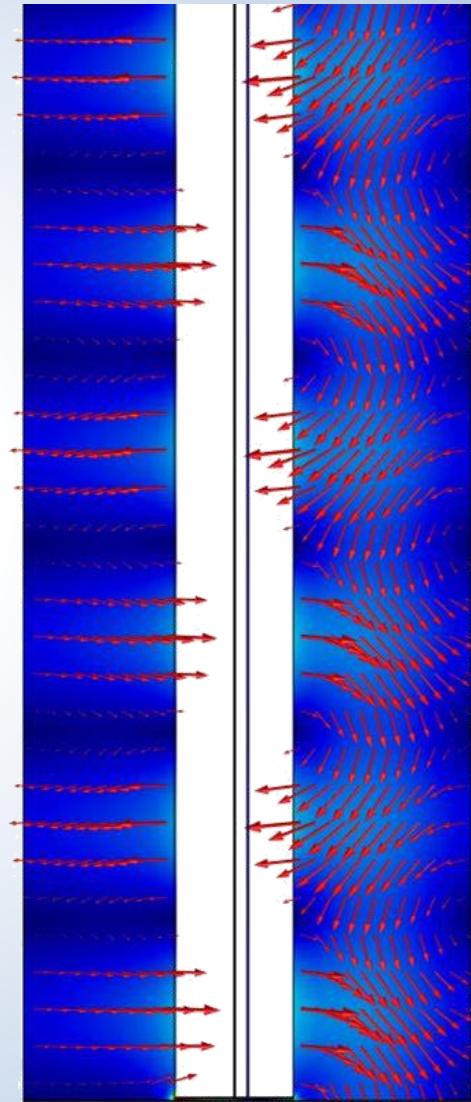
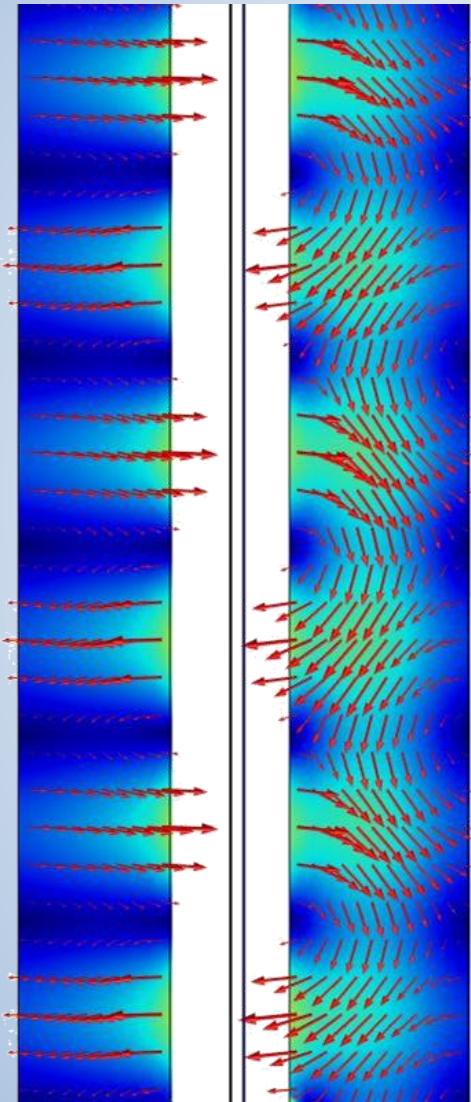
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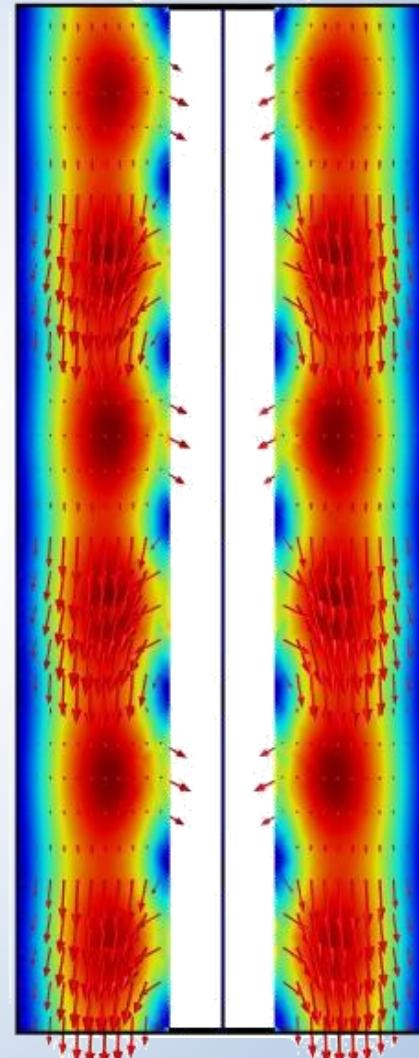
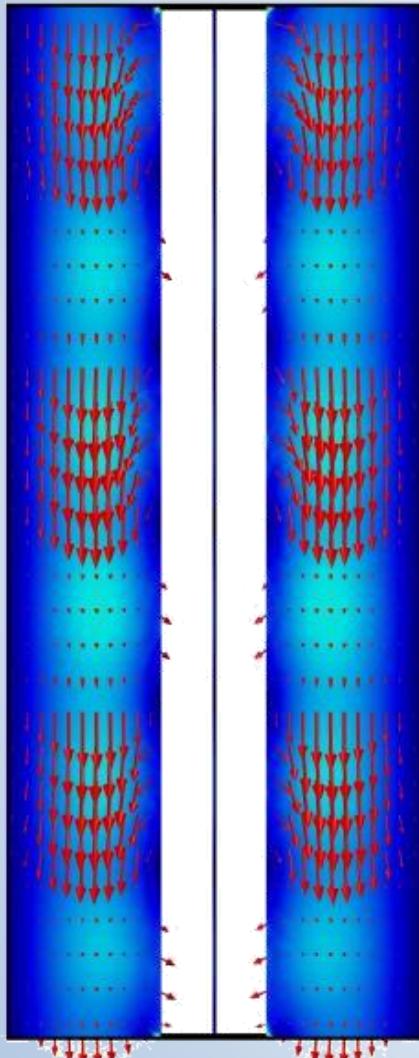
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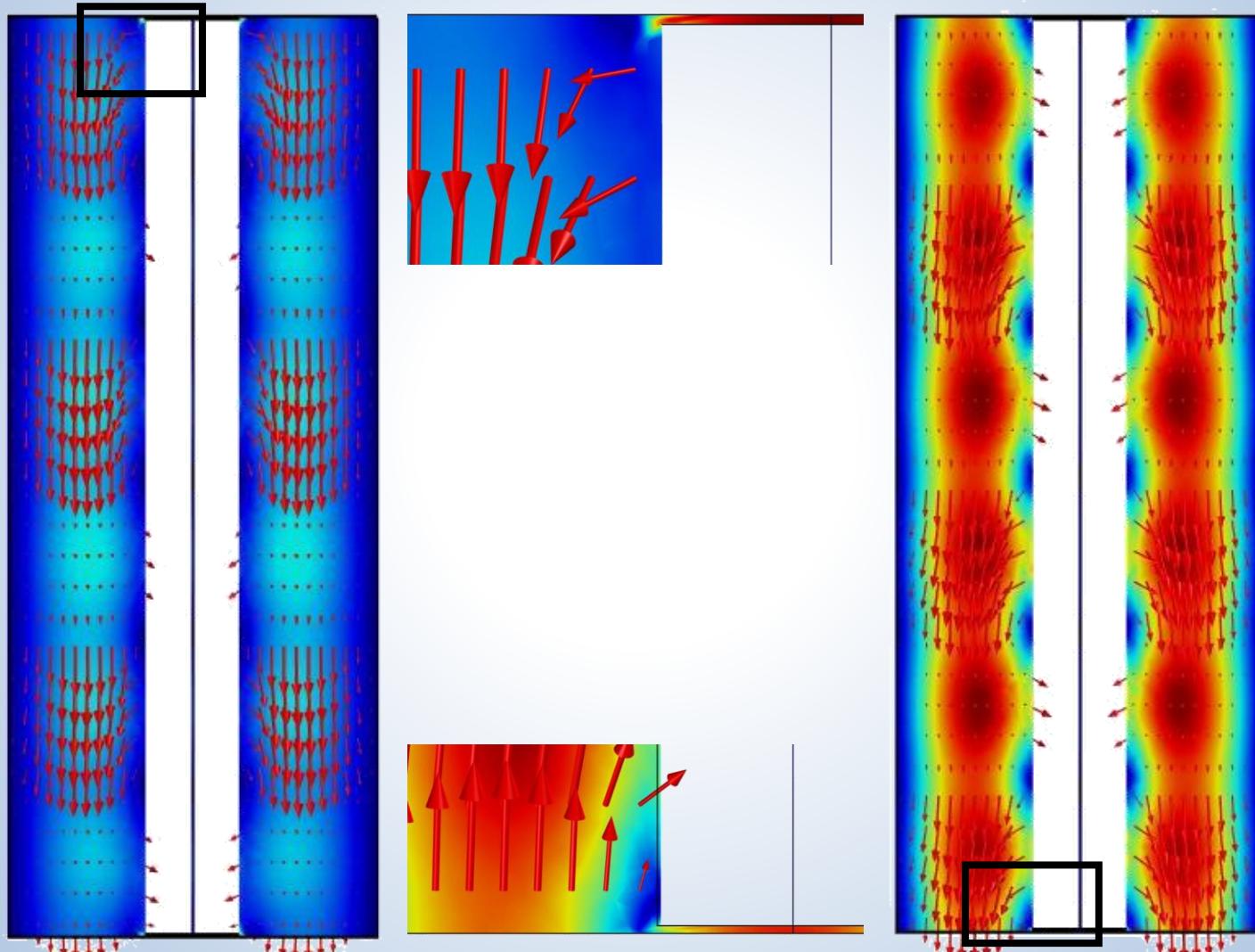
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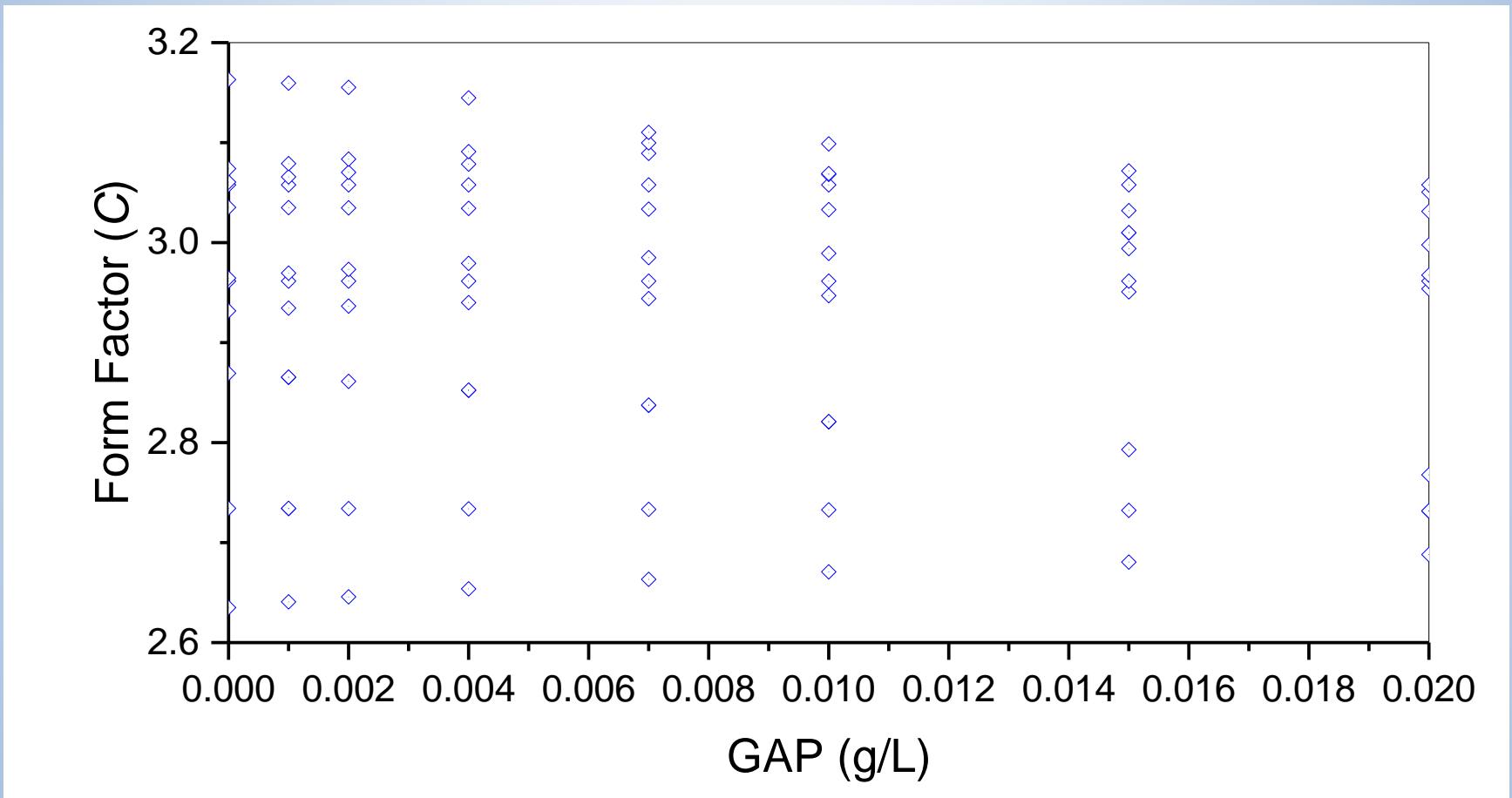
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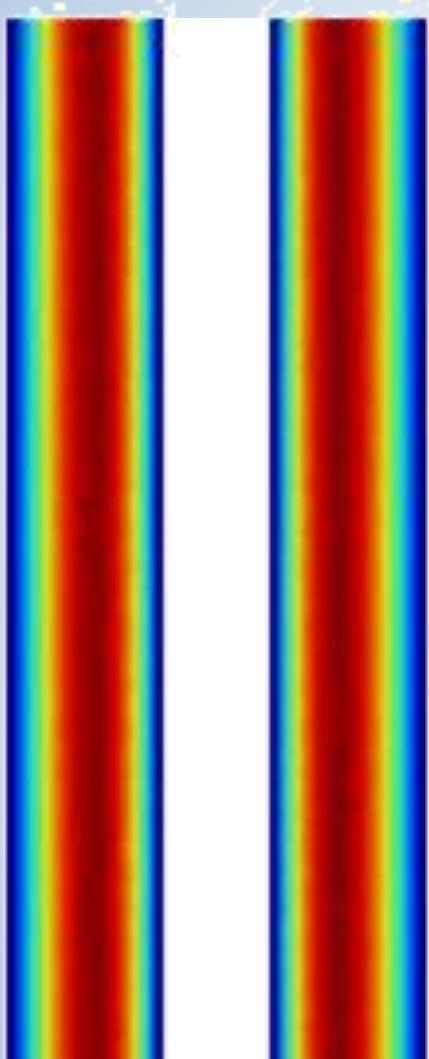
Mode Mixing



Mode Crowding



Longitudinal Symmetry Breaking (Tilt)



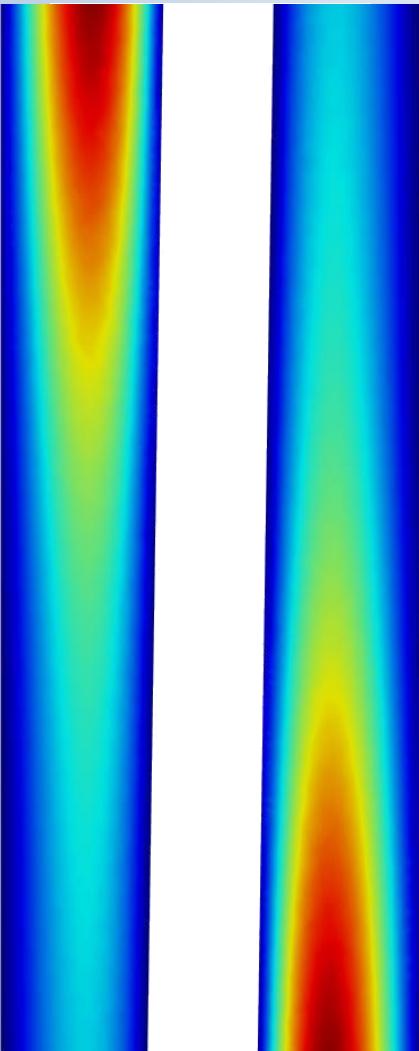
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$$\Psi = \Psi(x, y, z) e^{-i\omega t}$$

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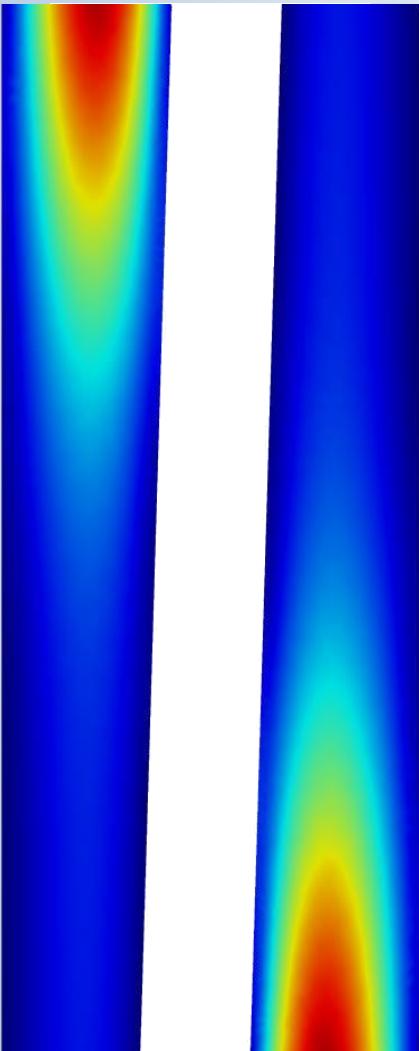
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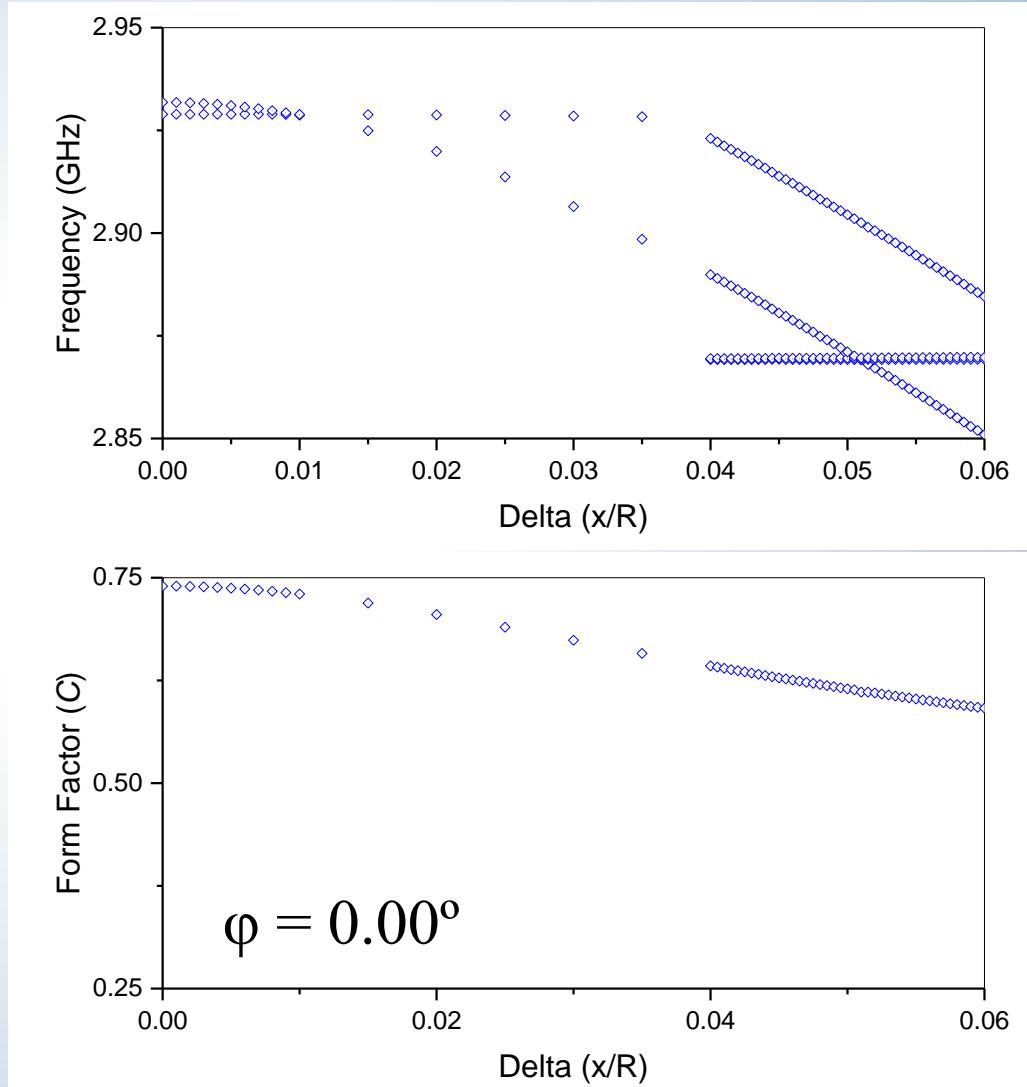
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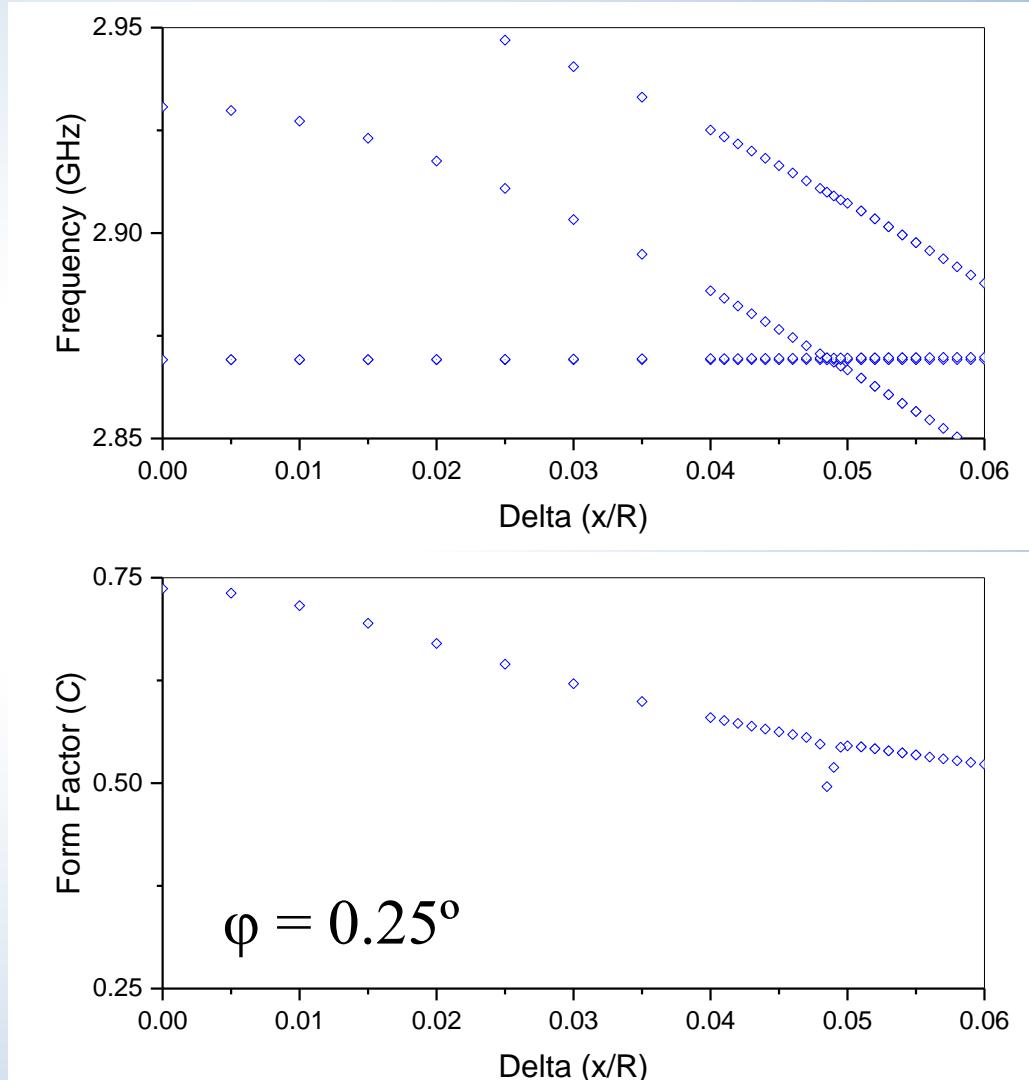
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 - New modes (splitting of mode from new parity symmetry)



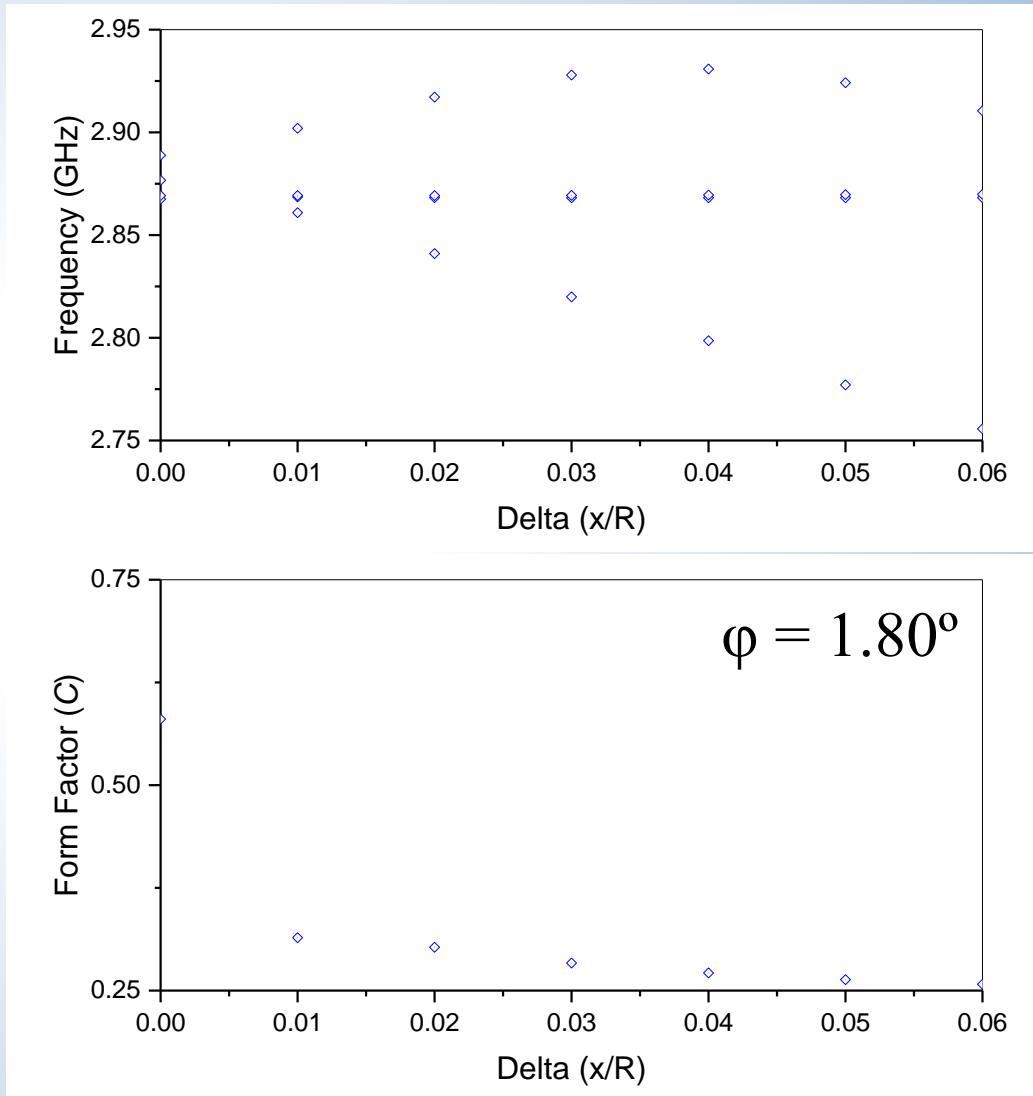
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 - New modes (splitting of mode from new parity symmetry)

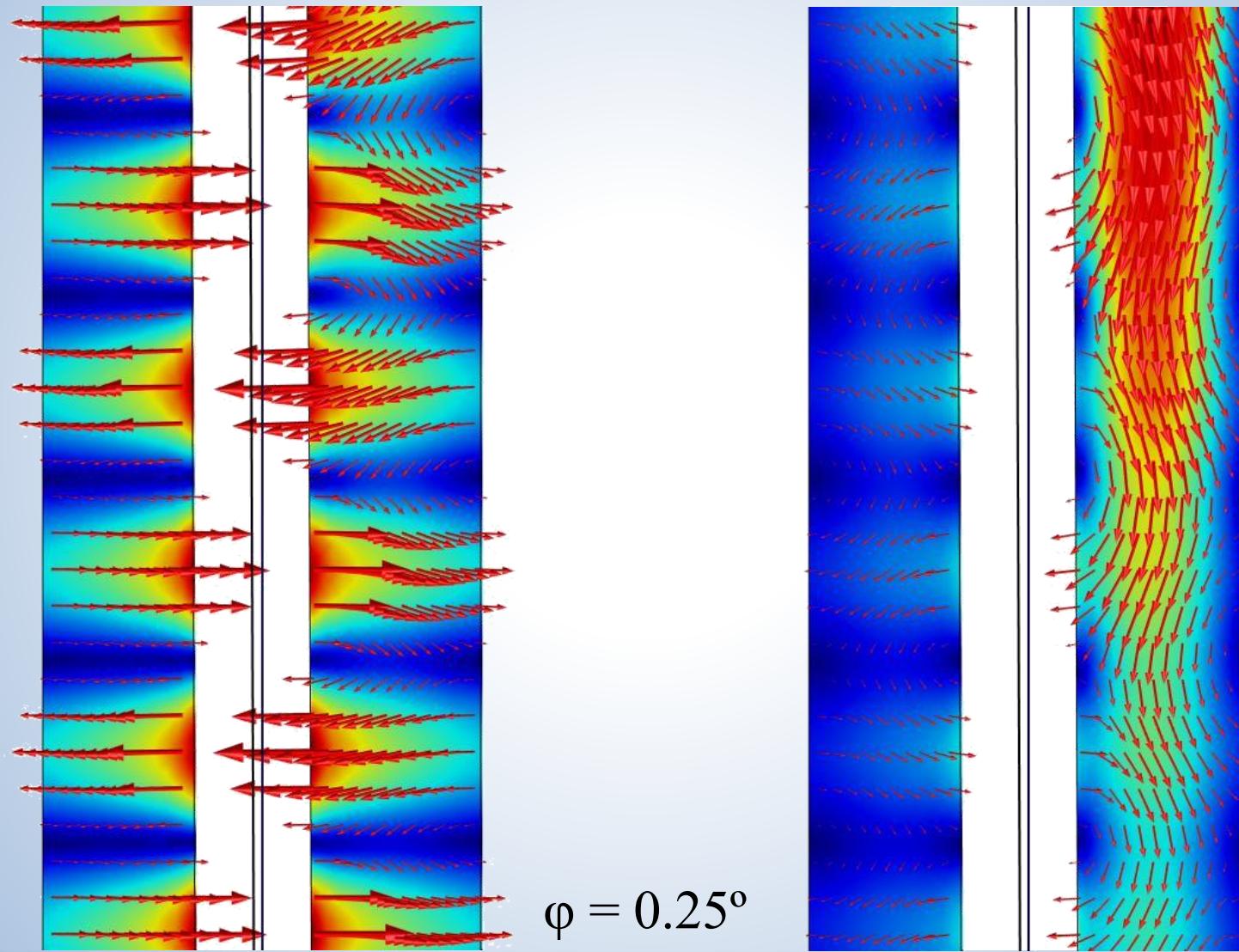


Longitudinal Symmetry Breaking (Tilt)

- Mode orthogonality not maintained
 - Mode mixing
- Symmetry breaking reduces form factor
 - Mode mixing further reduces form factor
- Mode crowding
 - Breaks degeneracies
 - New modes (splitting of mode from new parity symmetry)



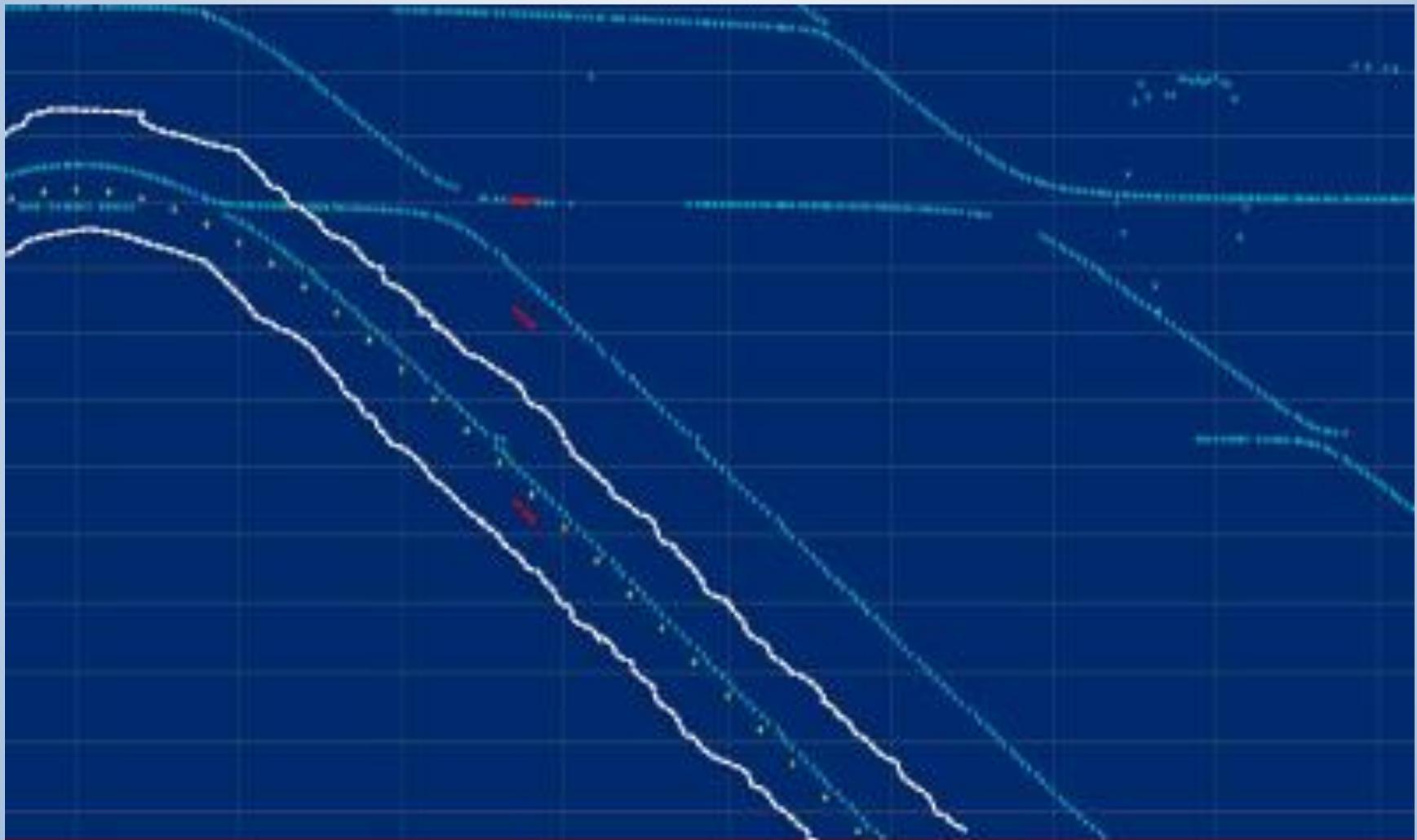
Mode Mixing



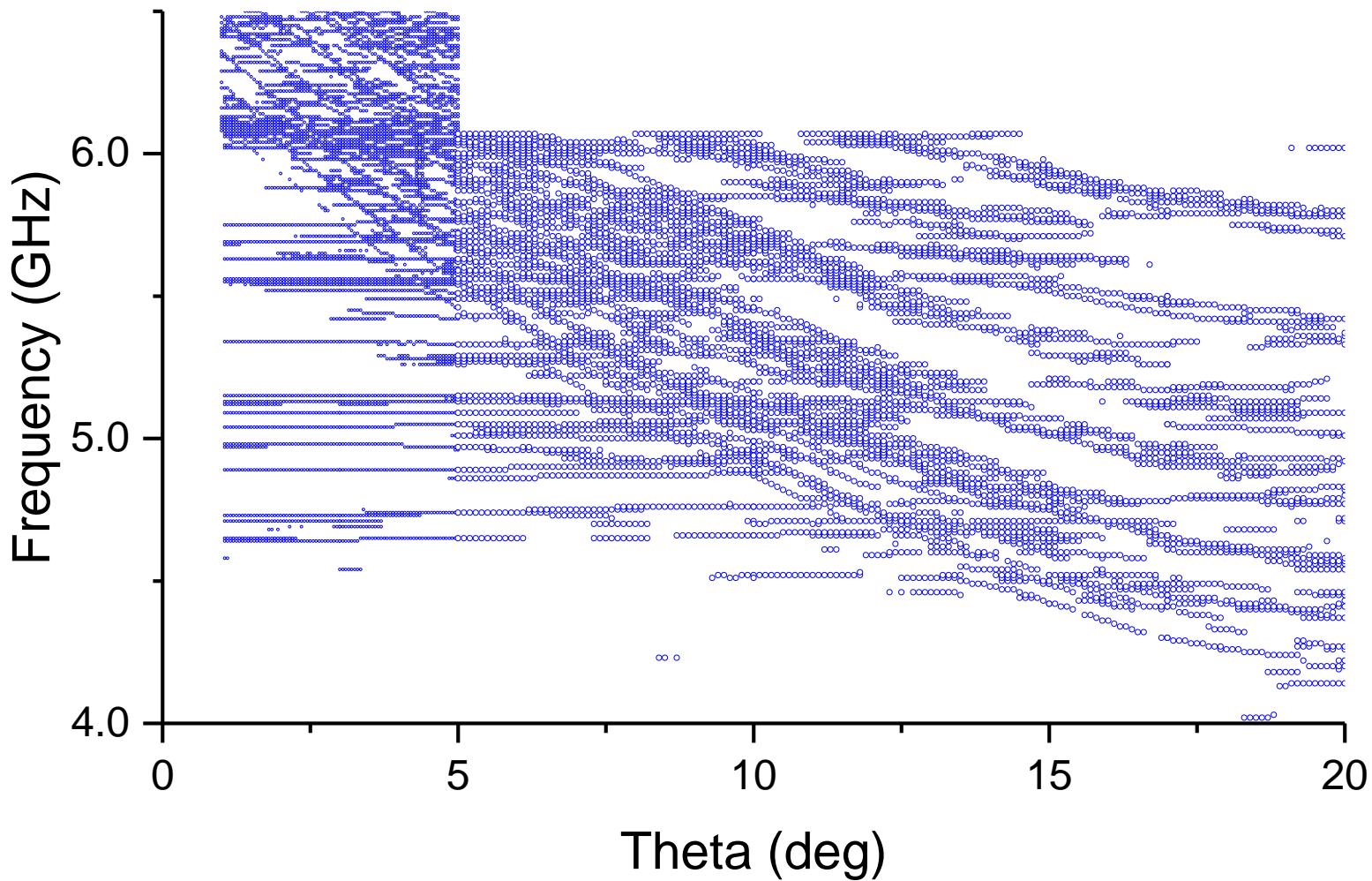
Mode Identification Techniques

- Mode map
- Field perturbation methods
 - Rod insertion
 - Bead pull
- Power/phase test

Mode Map



Mode Map



Rod Insertion

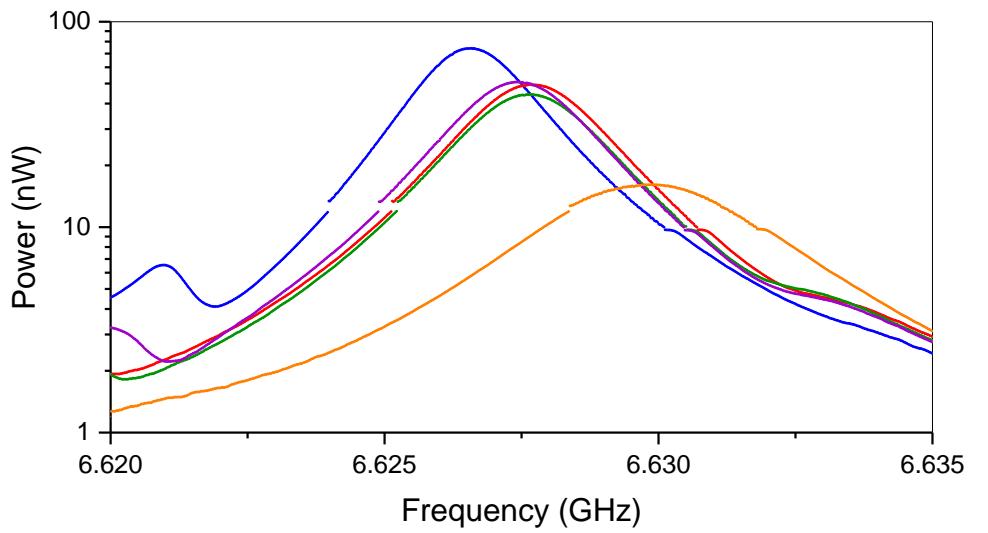
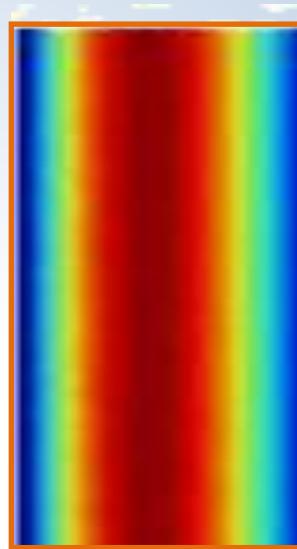
- Discerns TM modes
 - Conductive rod increases TM

$$(\nabla^2 + \mu\epsilon\omega^2) \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = \begin{Bmatrix} \nabla(\nabla \cdot \mathbf{E}) \\ (\nabla \times \mathbf{H}) \times \frac{\nabla\epsilon}{\epsilon} \end{Bmatrix}$$

- Perturbation theory can estimate frequency shift

$$\frac{\Delta\omega}{\omega} \approx \frac{\Delta W_m - \Delta W_e}{W}$$

- Increase $\frac{b}{\omega} = \frac{1}{Q}$



Rod Insertion

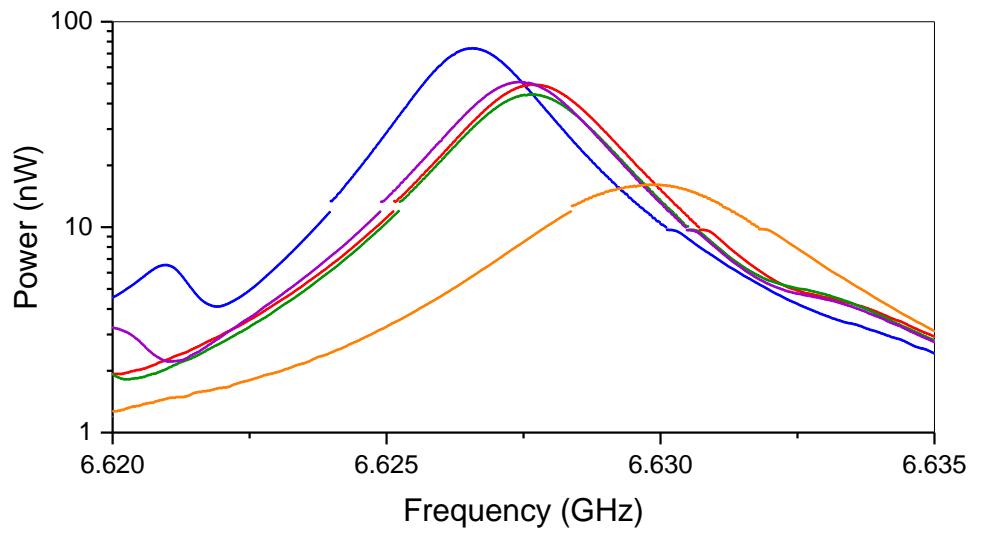
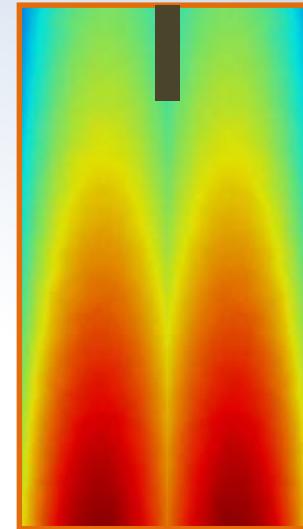
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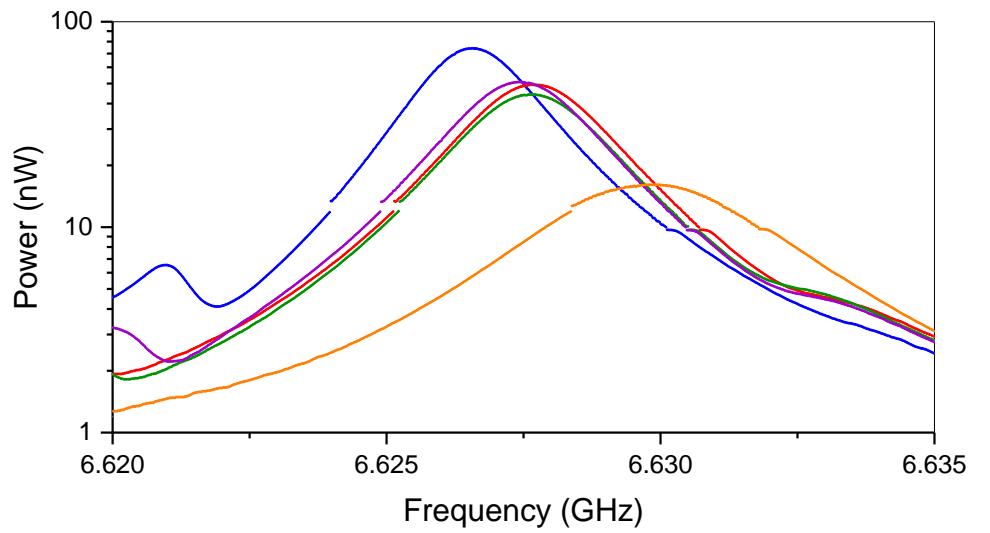
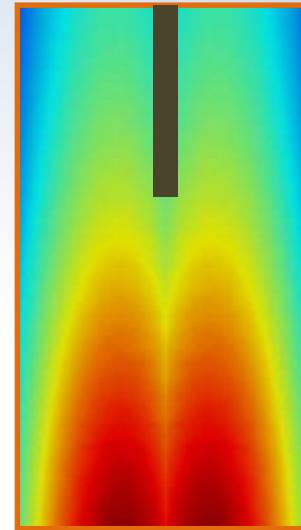
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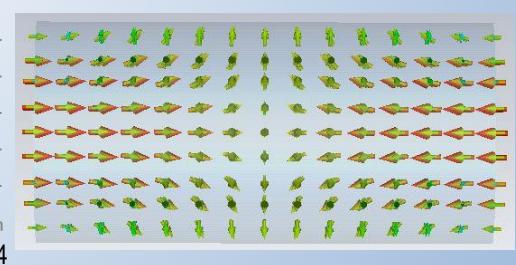
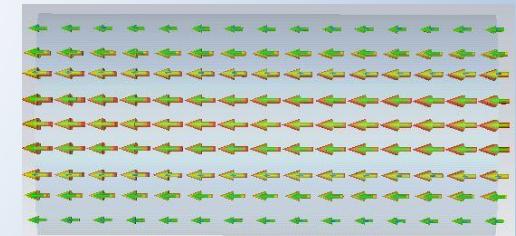
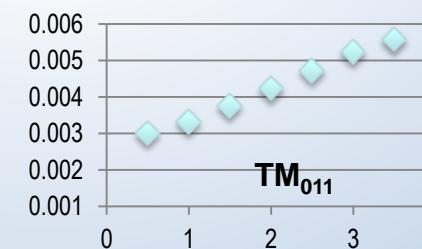
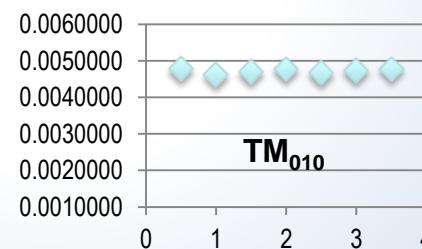
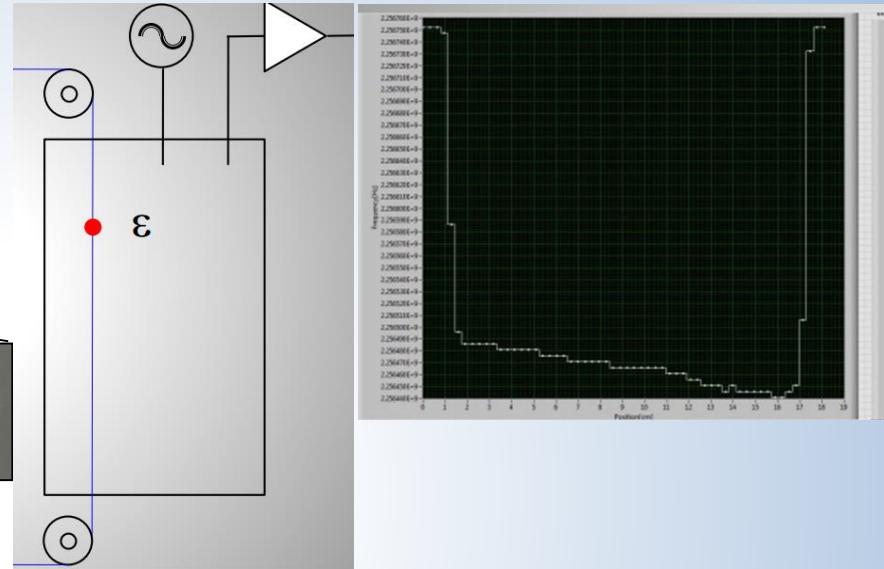
- Increase $\frac{b}{\omega} = \frac{1}{Q}$



Bead Pull



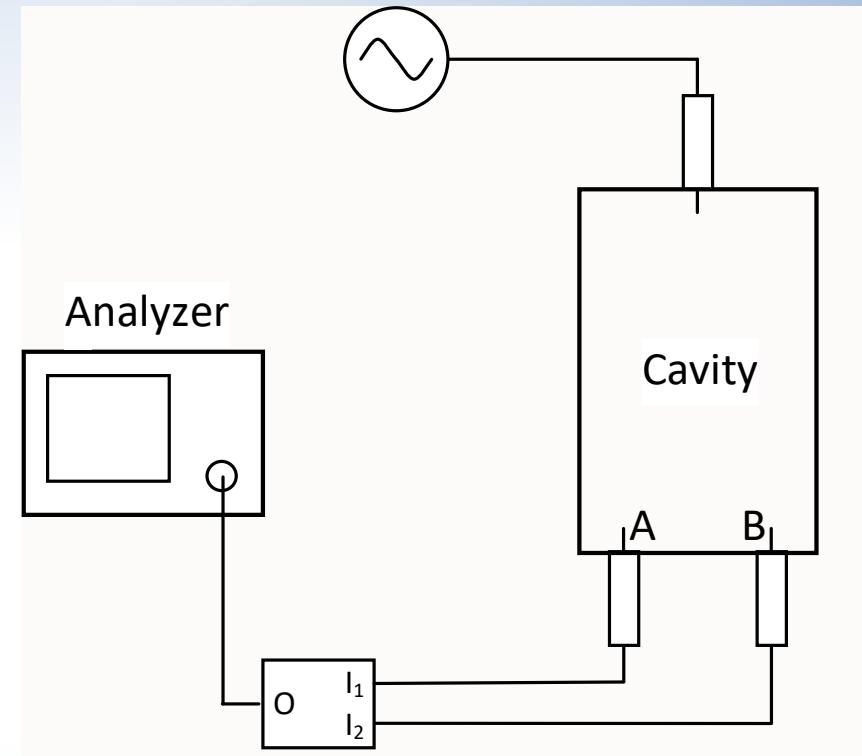
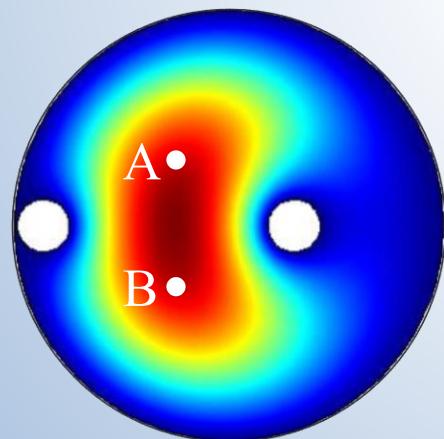
$$\frac{\omega^2 - \omega_0^2}{\omega_0^2} = k \int_{\Delta r} \frac{\mu H^2 - \epsilon E^2}{2U} dv$$



- Discerns E-field direction and strength along a vertical line

Power/Phase Test

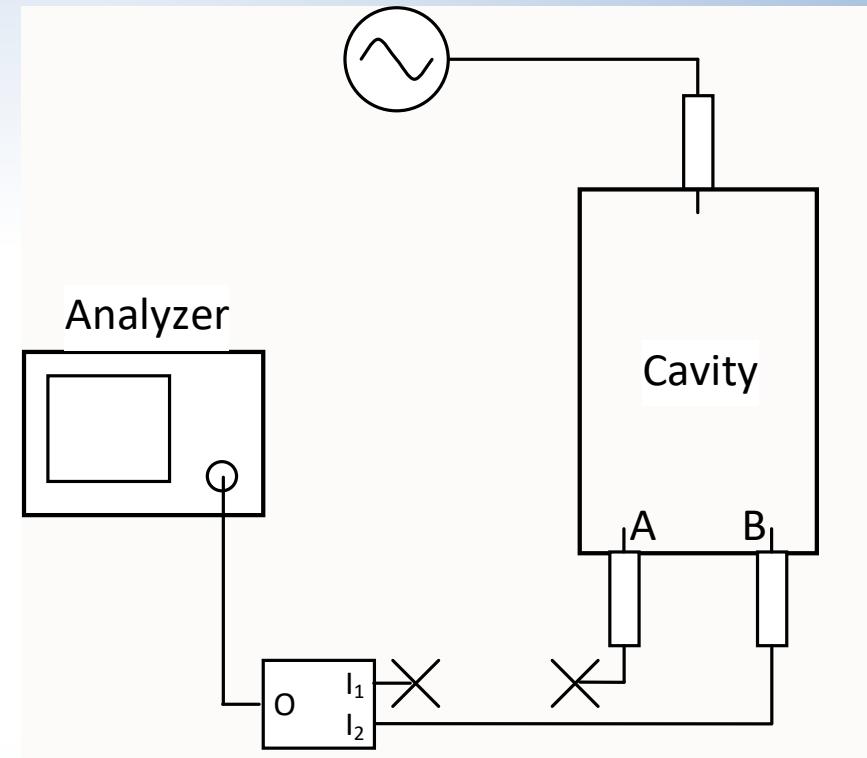
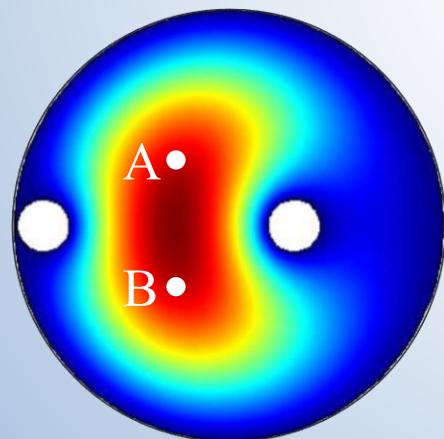
- Discerns in-phase/out-of-phase fields at known locations
 - Antennas at same or opposite endplate
 - Identifies transverse and longitudinal errors



$$P_{AB} \approx \frac{1}{2} \left(P_A + P_B + 2\sqrt{P_A P_B} \cos \left(\frac{\Delta\omega}{\omega_A} + \phi \right) \right)$$

Power/Phase Test

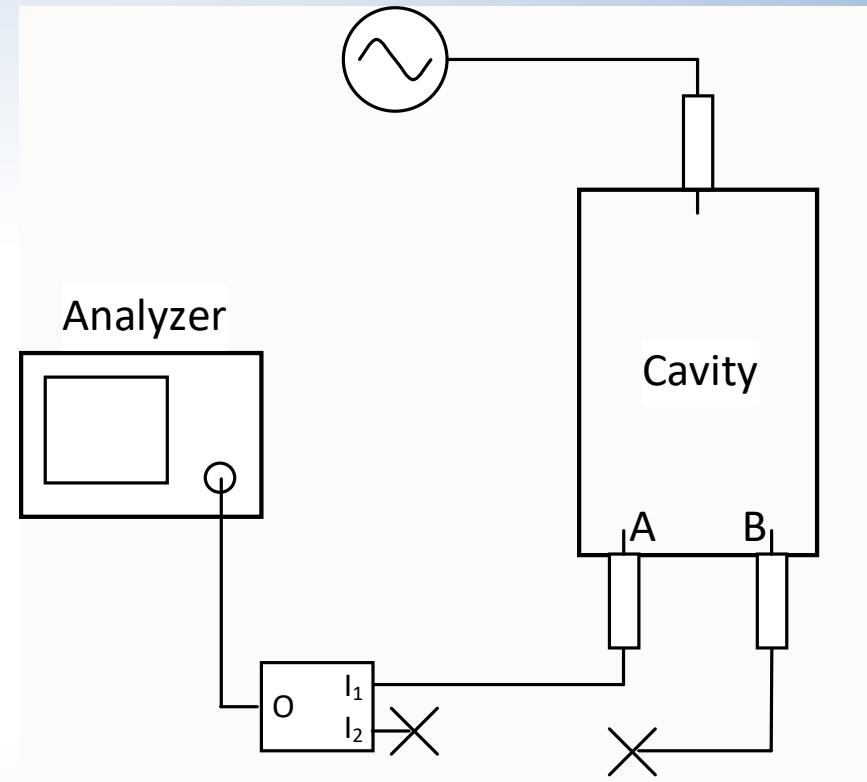
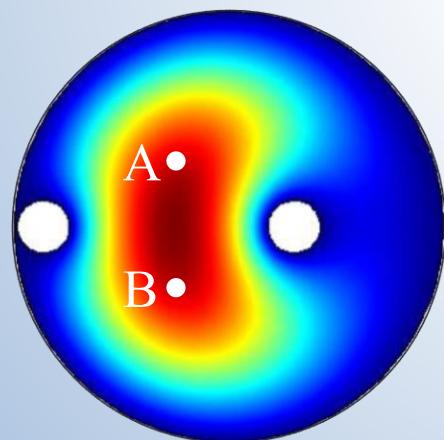
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Power/Phase Test

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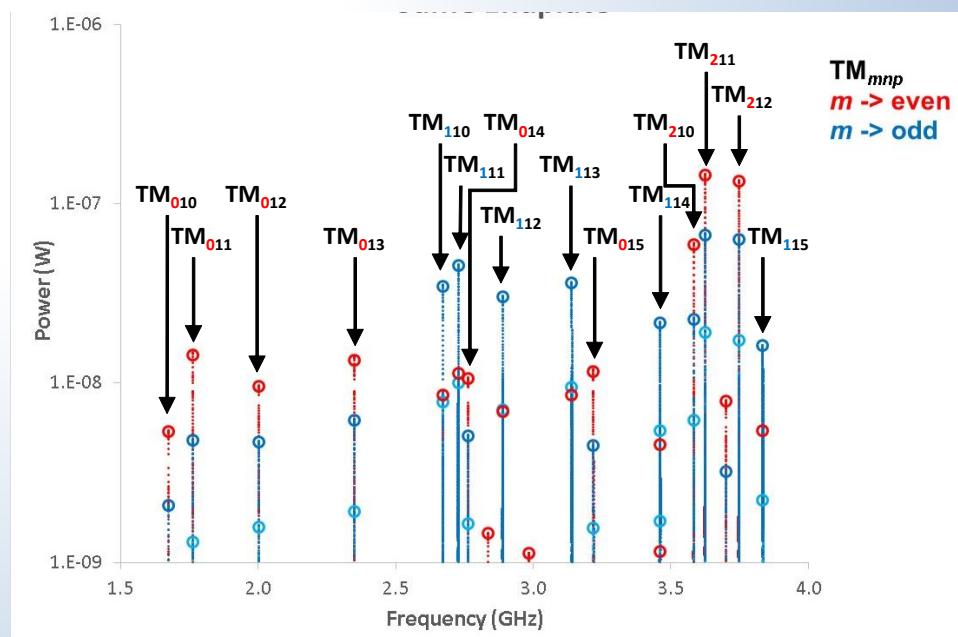
Power/Phase Test

- For TM modes (theoretically)

$$P_{AB} = 2P_A \cos(\phi)$$

- Antennas on same endplate test transverse nodes
- Antennas on opposite endplates check longitudinal nodes

- ΔP identifies transverse errors
- $\frac{\Delta\omega}{\omega_A}$ and ϕ identify longitudinal errors



Conclusion

- Transverse symmetry breaking changes TM frequency and reduces C .
- Longitudinal symmetry breaking causes orthogonality breaking, mode mixing, mode crowding, and reduces C .
- Significant symmetry breaking requires additional mode identification techniques.