Lies, Damn Lies, and your Analysis: Practical Statistics for Neutrino Physics

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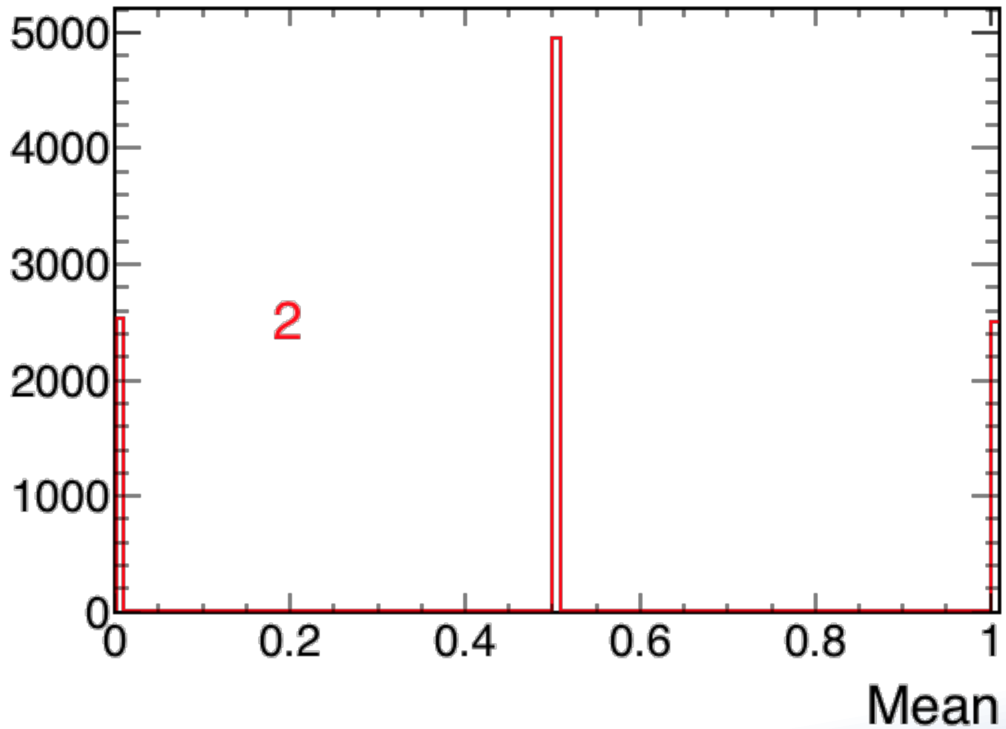
WIKIPEDIA
The Free Encyclopedia

Statistics is a branch of mathematics dealing with the collection, analysis, interpretation, presentation, and organization of data.[1][2]

- A collection of methods to extract meaning from data.
  - There are many, many methods.
  - The question you need to answer – is the method I'm using appropriate to my situation?
  - Make sure you're clear about what you did, so others can interpret your results.
- **You are making an argument using data.**
- The answers are never simply "yes" or "no"
  - There is always a degree of uncertainty or level of agreement.

Central Limit Theorem

- The sum of a sufficiently large number of **independent random variables**.
  - It does not matter what distribution the underlying random variables come from.
- Example: coin flips. Heads = 0, tails = 1
  - Clearly not normally distributed.
- However, if we look at the distribution of the means:



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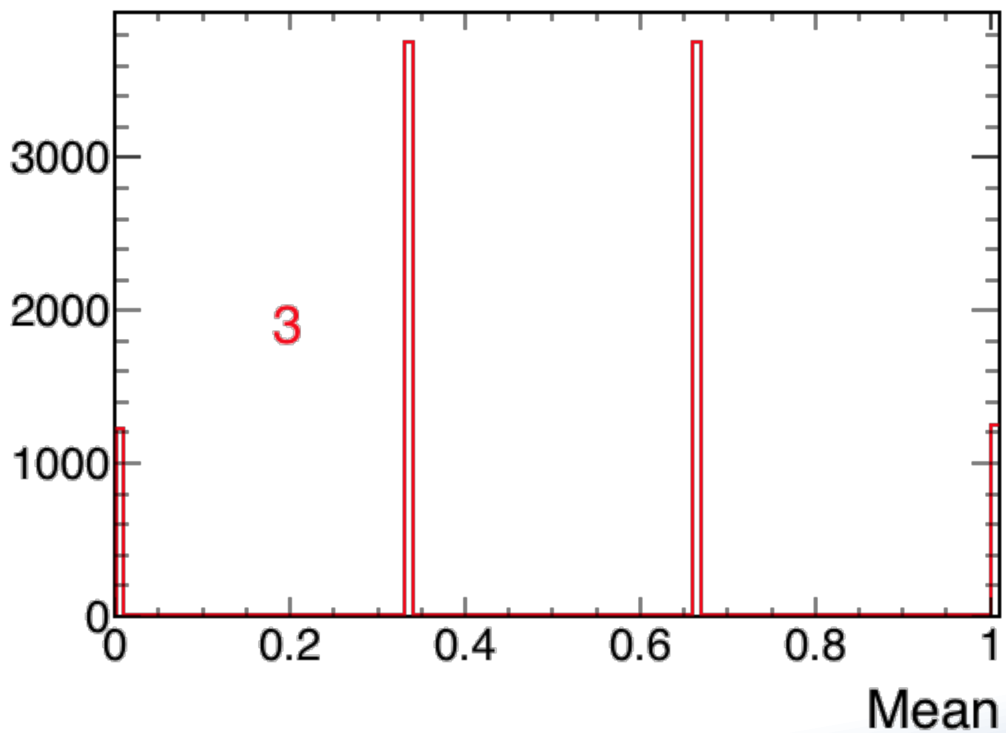

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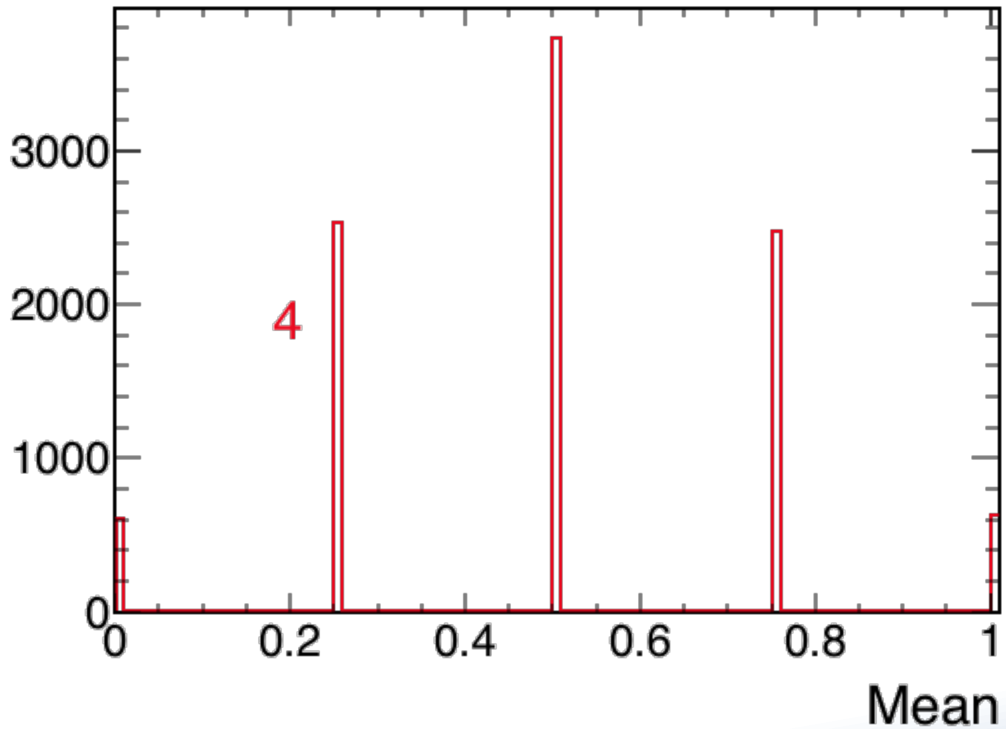

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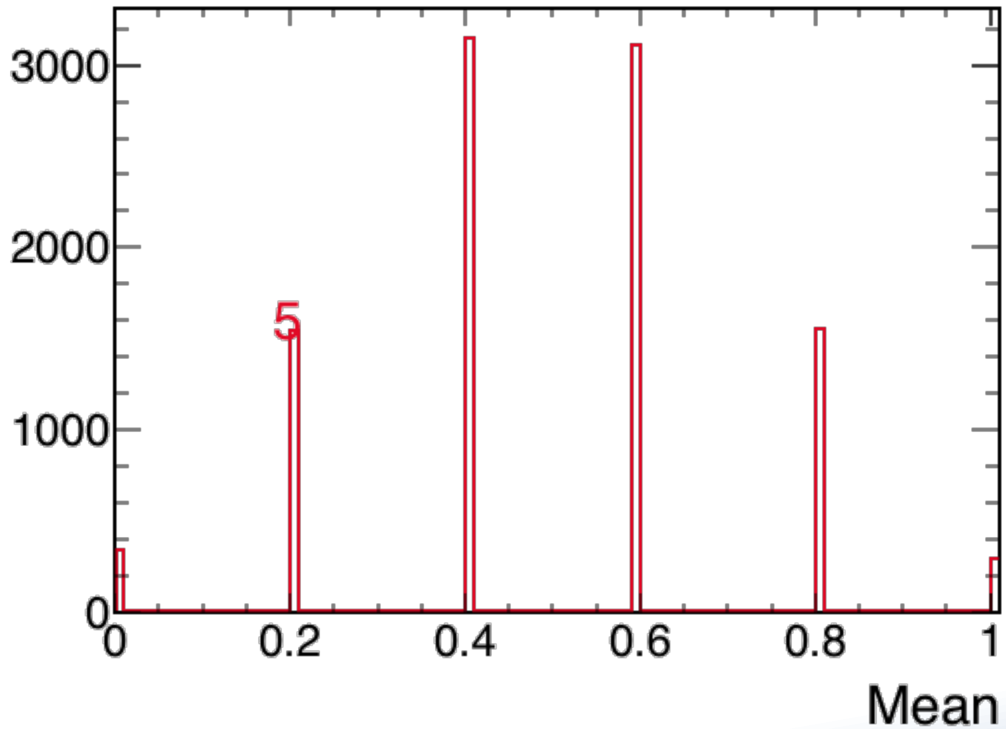

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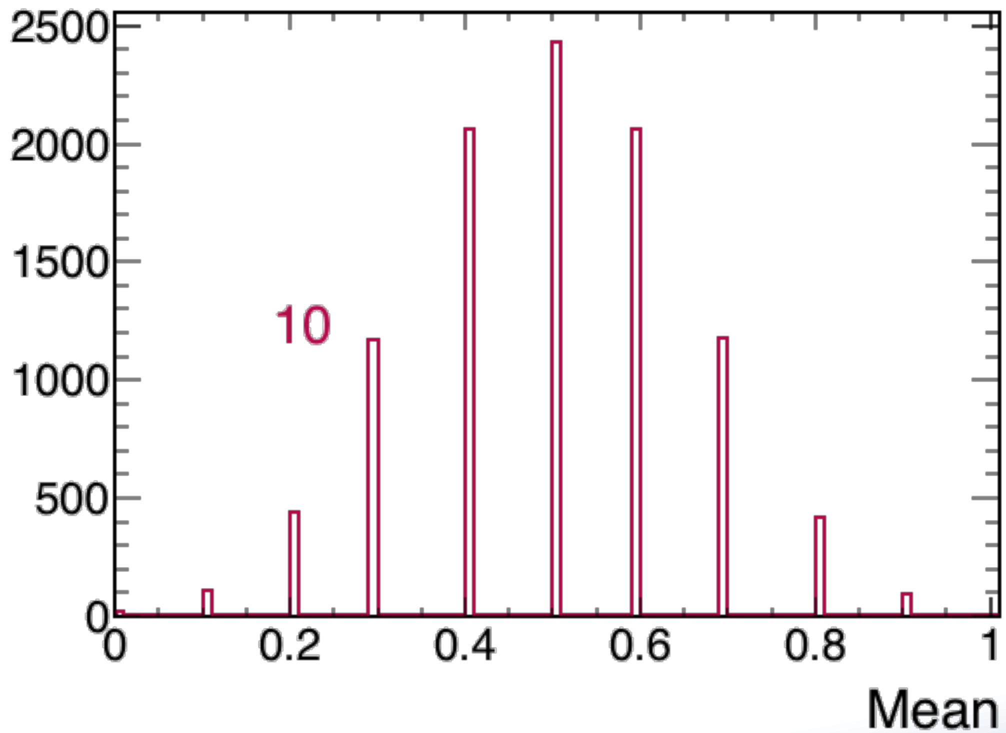

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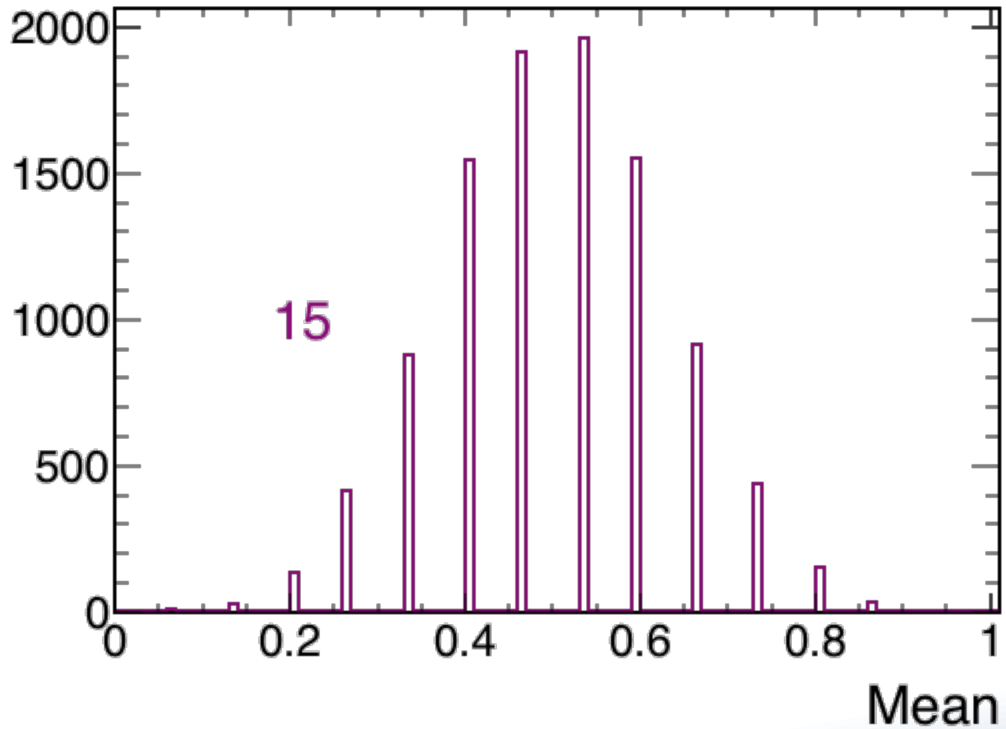

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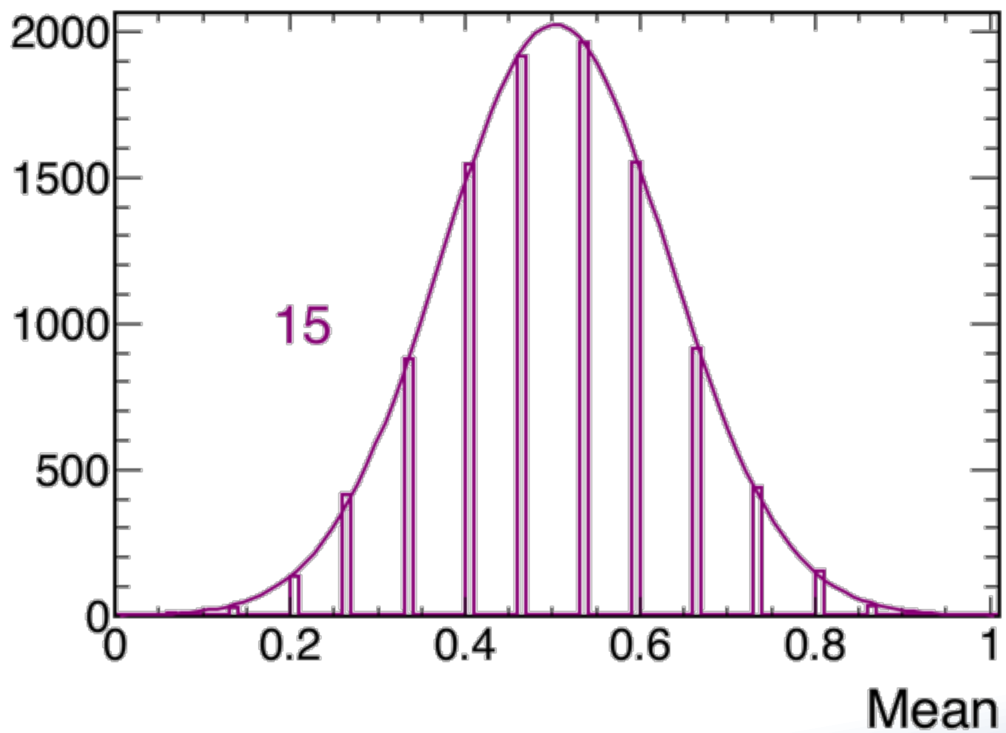
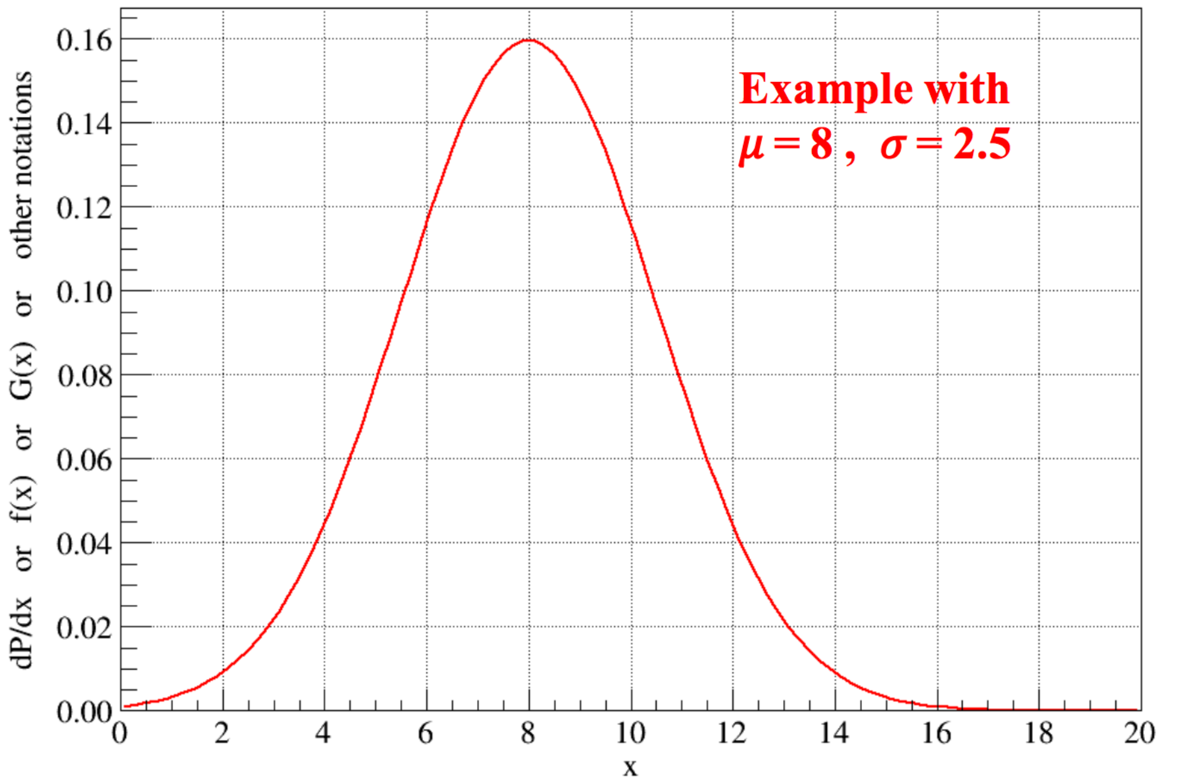
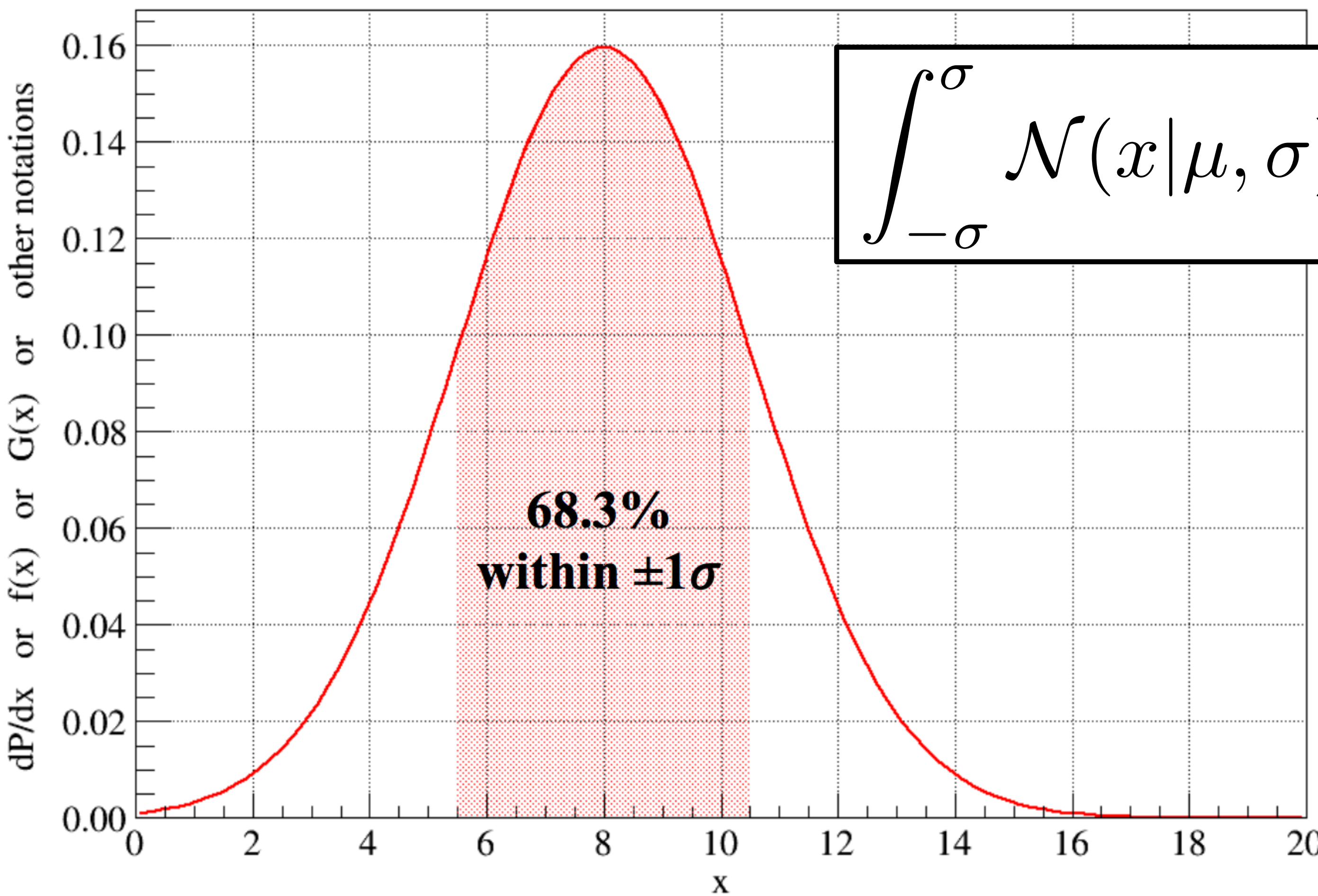


- This is why, under most circumstances, we treat errors as "Gaussian"…because most of the time it works.
- When doesn't it work?
  - Mostly when the stats are too low, plus a few other edge cases.

$$\mathcal{N}(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

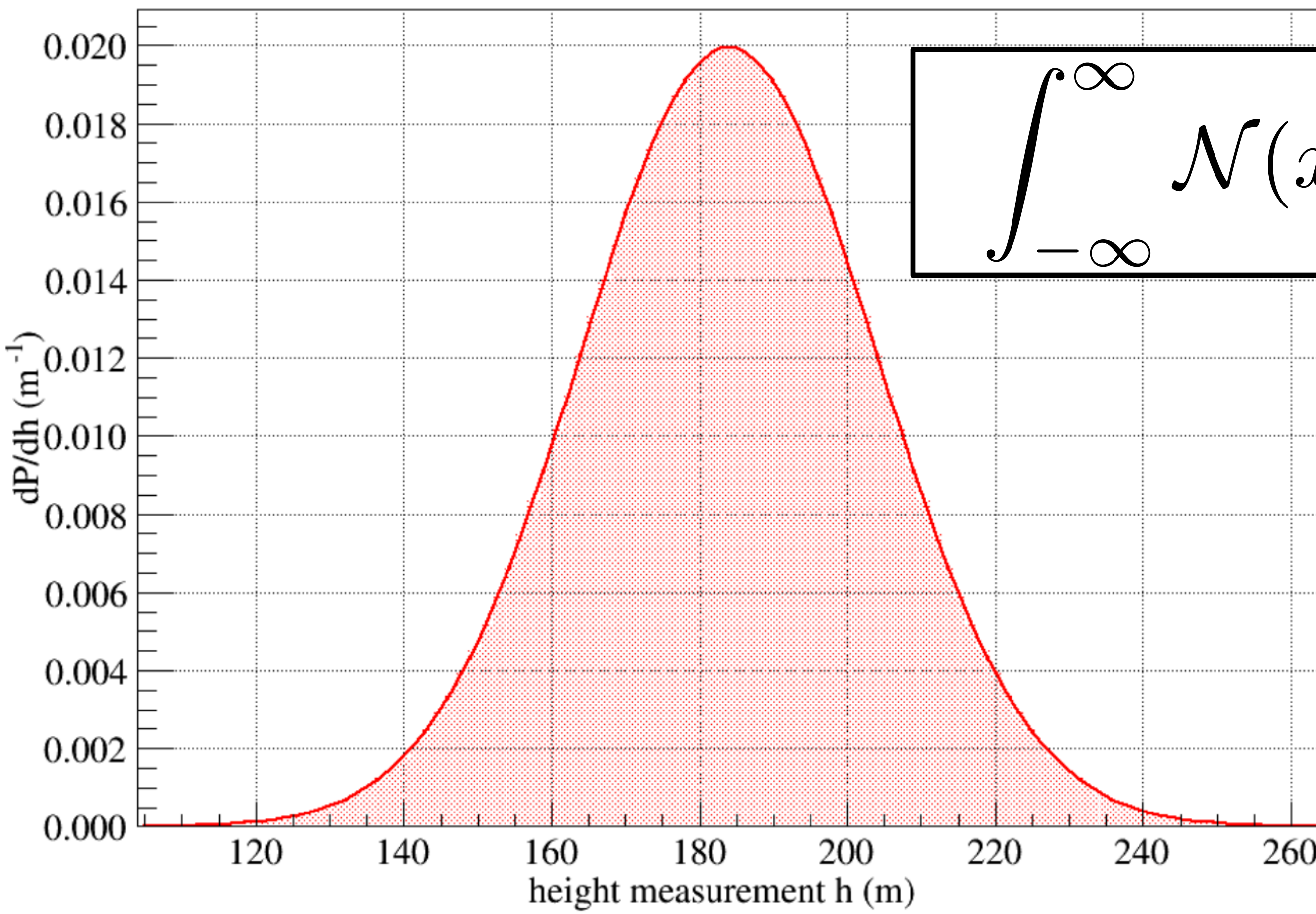


$$\int_a^b \mathcal{N}(x|\mu,\sigma)dx =$$ probability that *x* is between *a* and *b*



$$\boxed{\int_{-\sigma}^{\sigma} \mathcal{N}(x|\mu,\sigma)dx = 0.683}$$

$$\int_a^b \mathcal{N}(x|\mu,\sigma)dx = \text{probability that } x \text{ is between } a \text{ and } b$$



$$\boxed{\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma)dx = 1}$$

How to Ask a Statistical Question

- The term for this is a "hypothesis test."
- H_0: Null hypothesis
  - The specific case, such as A and B are the same
- H_1: Alternative hypothesis
  - The alternative to the null – A and B are different
- Significance level
  - How high a rate of false positives (rejecting the null, even if it is true) can you tolerate.
  - $\boxed{\alpha = 0.05}$ is common, but often not sufficient for physics.

Are two means the same?

$$\mu_1 = 2.5 \pm 0.1 \quad \mu_2 = 3.1 \pm 0.3$$

- H_0: The difference between the means is 0
  - $\mu_2 - \mu_1 = 0$
- H_1: The means are different
  - $\mu_2 - \mu_1 \neq 0$
- I can tolerate a 5% chance of saying they are different, even if they really are the same.
- Now, let's do the test.

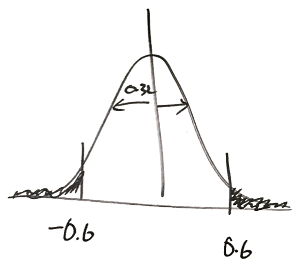
$$X = \mu_2 - \mu_1 = 0.6$$

Are these different at $p = 0.05$?

$$\sigma_{\mu_2 - \mu_1} = \sqrt{\sigma_{\mu_1}^2 + \sigma_{\mu_2}^2} = 0.32$$

How often do we get 0.6 or more extreme assuming

$$X \sim N(0, 0.32)?$$



$$\int_{-\infty}^{-z} N(x,1)\,dx + \int_{z}^{\infty} N(x,1)\,dx \stackrel{?}{<} p$$

$$1 - \underbrace{\int_{-z}^{z} N(x,1)\,dx}_{\mathrm{Erf}(x/\sqrt{2})} \stackrel{?}{<} p$$

$$X \rightarrow Z = \frac{X - \mu}{\sigma} = \frac{0.6 - 0}{0.32} = 1.88$$

$$Z \sim N(0,1)$$

$$0.06 > 0.05$$

"Fail to reject H_0"

$$p = 1 - \int_{-Z}^{Z} \mathcal{N}(x, 1) dx$$

- This integral doesn't have an analytical solution, but we need it all the time, so it's results are readily available as the "error function"

```
// Z-score (sigmas) -> p-value

root [4] 1 - TMath::Erf(1.88 / TMath::Sqrt(2))
(Double_t) 0.0601081
```

A Little Vocabulary

- Z is our **test statistic**
  - A single number we calculate as a "summary" of our data.
- You want to know how the test statistic is supposed to be distributed under the null hypothesis.
  - You need to know the distribution to calculate a *p*-value.
- Generally, there are **assumptions that must be met** for this to be true.
- If the conditions are not met, or there is no simple test statistic, all is not lost.
  - There are "non-parametric" techniques.

Is there signal above the background?

- Let's say we're members of a neutrino experiment called SOνA
  - The Statistical Off-axis νe Appearance Experiment
- Thanks to our powerful off-axis design we expect only **1 background event**.
  - And since this is SOνA we have no systematic errors!
- We open the box and **observe 6 events**.
- Did we observe ν_e appearance?

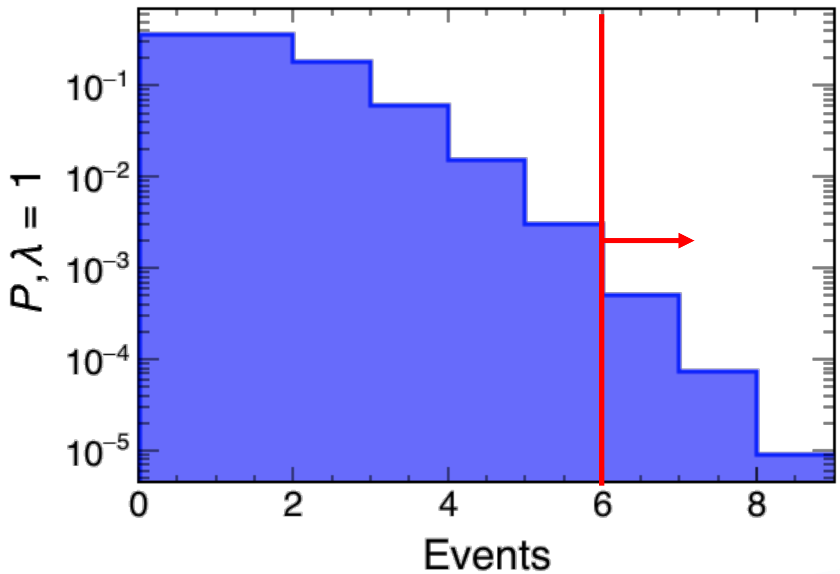
- Let’s translate into a hypothesis test:
- H_0: Our observation is consistent with the background.
  - X = B
- H_1: There is a signal above our background estimate.
  - X > B
- We are making an important claim, so we require $\alpha = 0.0027$ (3 σ)

```
// p-value -> sigmas
root [6] TMath::NormQuantile(1 - 0.0027/2)
(Double_t)3.0
```

- How is this different from the mean test?
  - The numbers involved are *small*.
  - This test is 1-sided instead of 2-sided
  - The distribution is not Gaussian, it is Poisson.
- How do we know it's Poisson?
  - This distribution describes the number of **independent events** (neutrinos in the FD)
  - occurring within a **fixed time interval** (periods 1&2).
  - This almost always describes neutrino physics data.
- But, if you have many events, then the Poisson just becomes…

- First – what is our test statistic?
  - Just the number of observed events.
  - We know, under the null hypothesis, how that should be distributed – Poisson, rate 1
- We need to calculate a *p*-value to compare to our α.
  - To do that, we again need to integrate a distribution.

$$p = \sum_{X}^{\infty} \mathcal{P}(x, 1)$$



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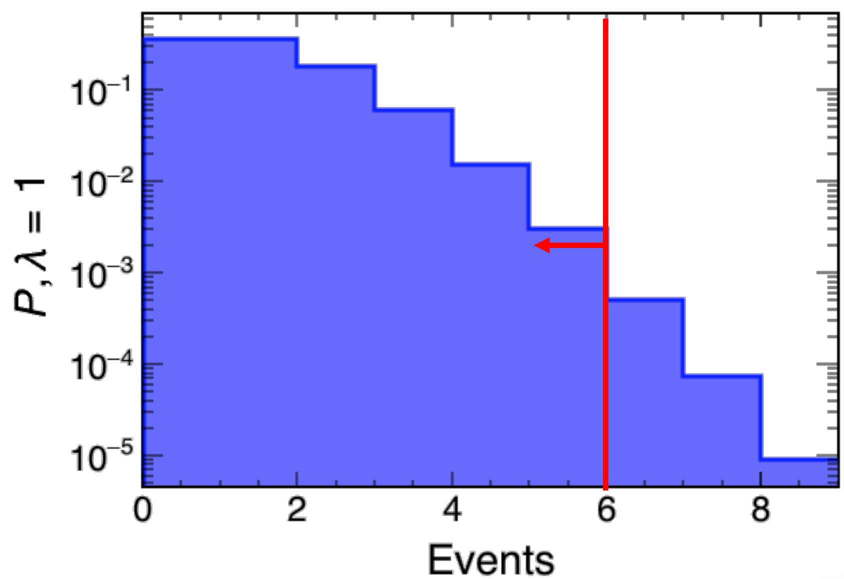
$$p = \sum_{X}^{\infty} \mathcal{P}(x,1) = 1 - \sum_{0}^{X-1} \mathcal{P}(x,1)$$



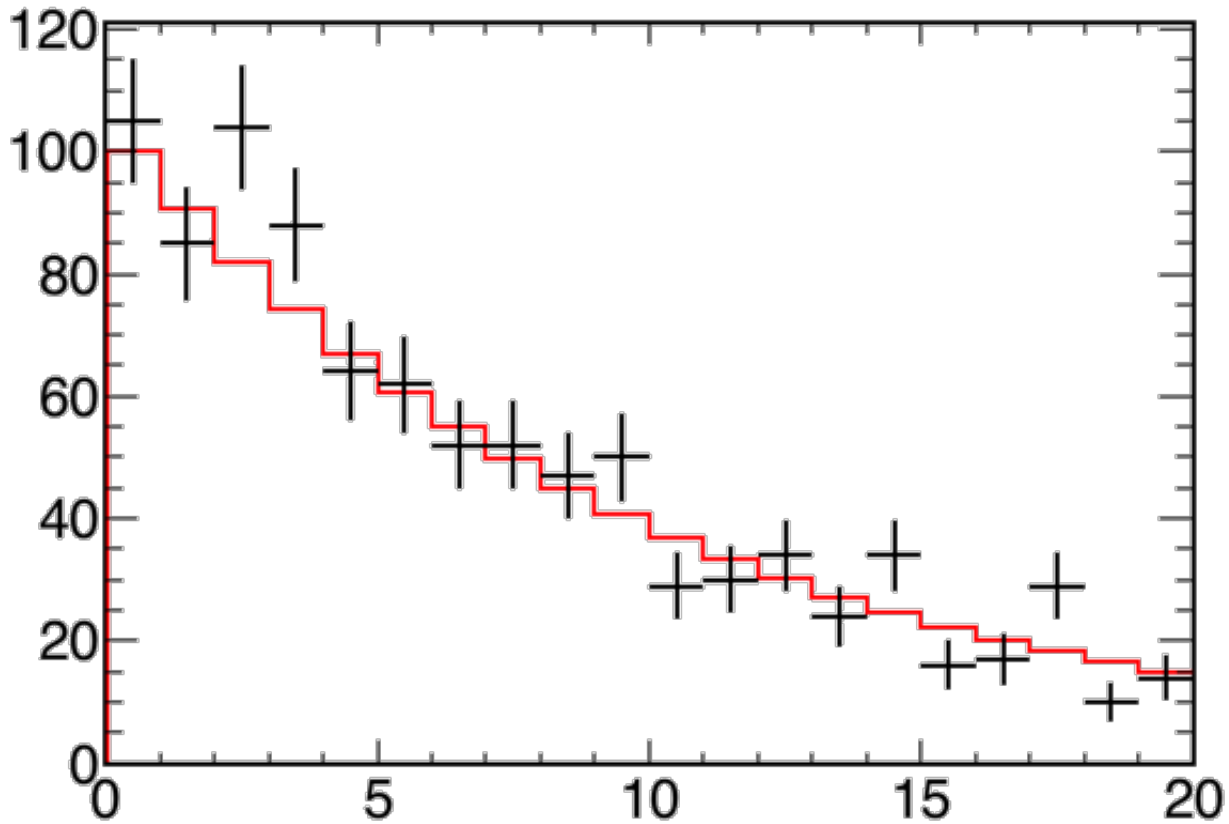
- Again, let's take advantage of built-in functions which already have the integral of the Poisson distribution.

```
root [14] 1 - ROOT::Math::poisson_cdf(5,1)
(double) 0.00059418
```

- p (0.000594) < α (0.0027)
  - We reject the null hypothesis.
  - We have evidence of something other than background at the 3σ-level.

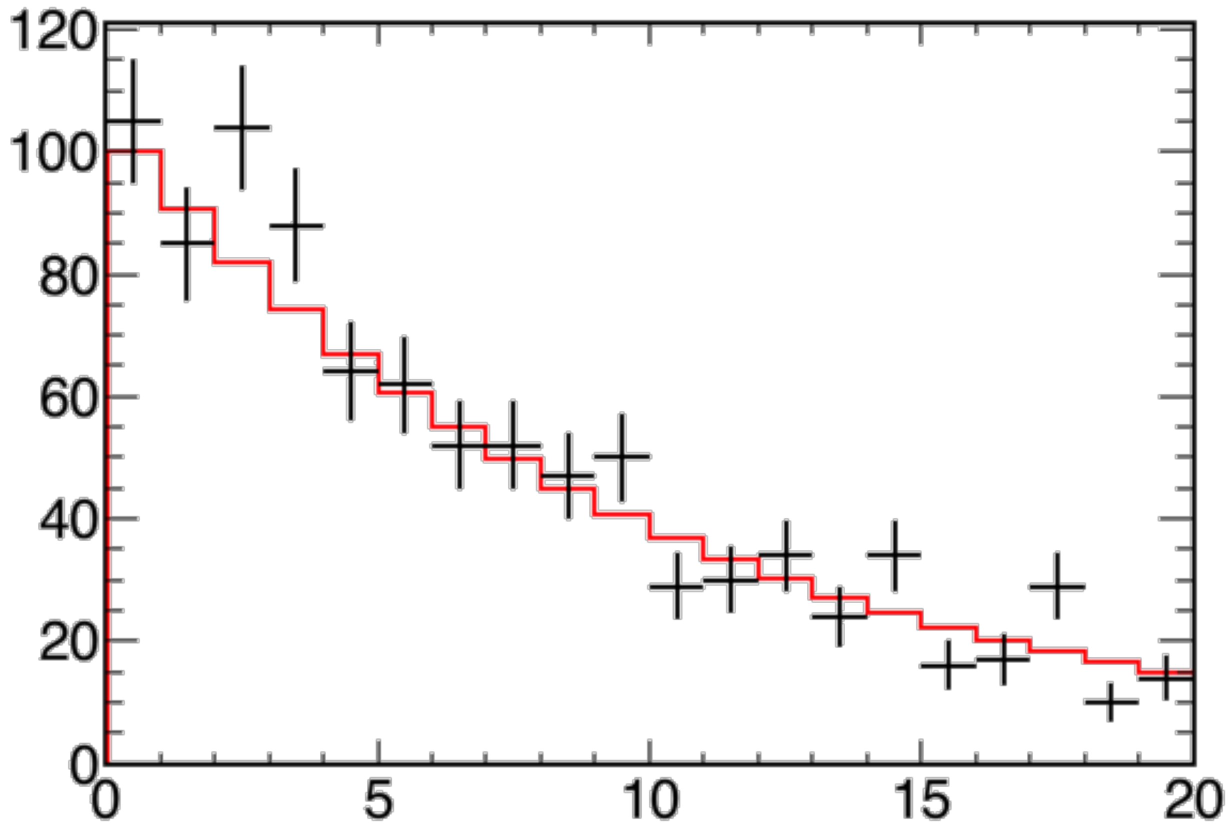
Data/MC Agreement

- Does the model (red) describe the data (black)?



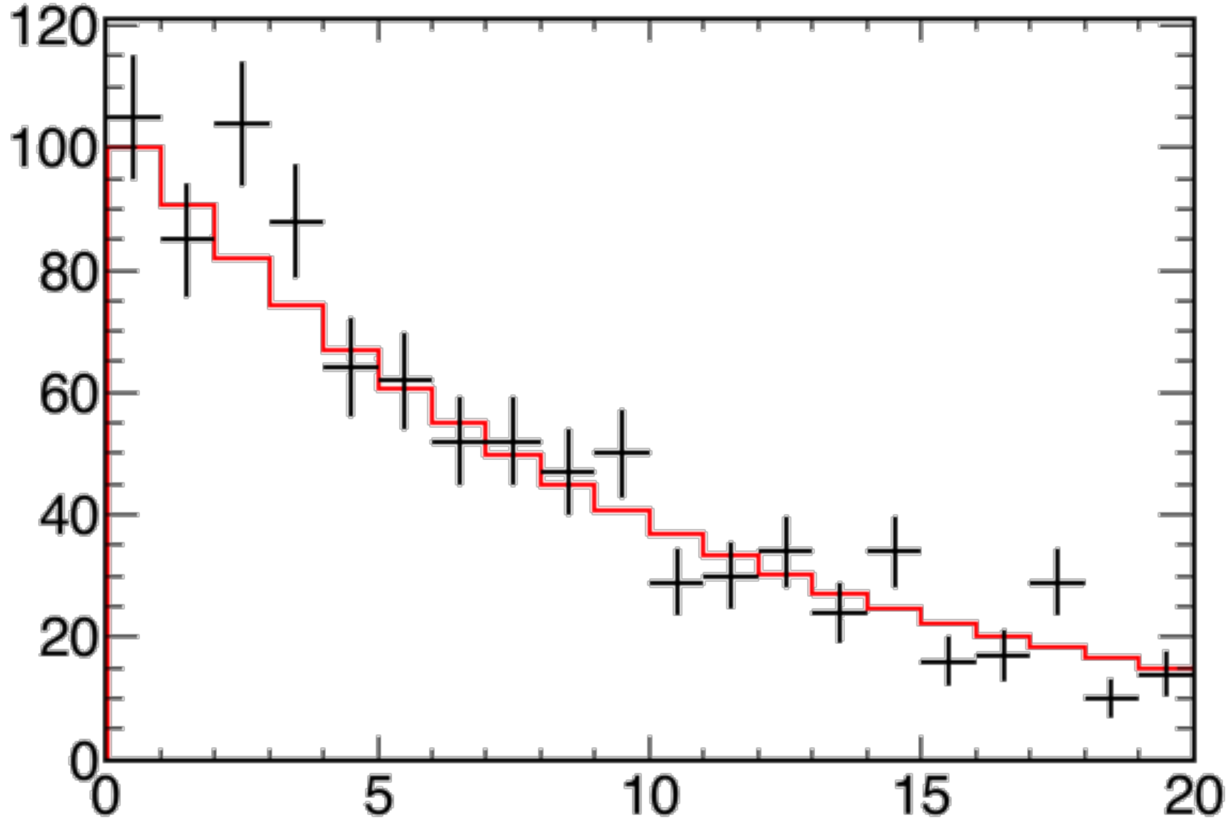
Data/MC Agreement

- Is the data consistent with having been drawn from the model, given its uncertainties?



Data/MC Agreement

- Hypothesis test:
  - H_0: The data was drawn from the model in red.
  - H_1: The data is not consistent with the model.
  - $\alpha = 0.05$



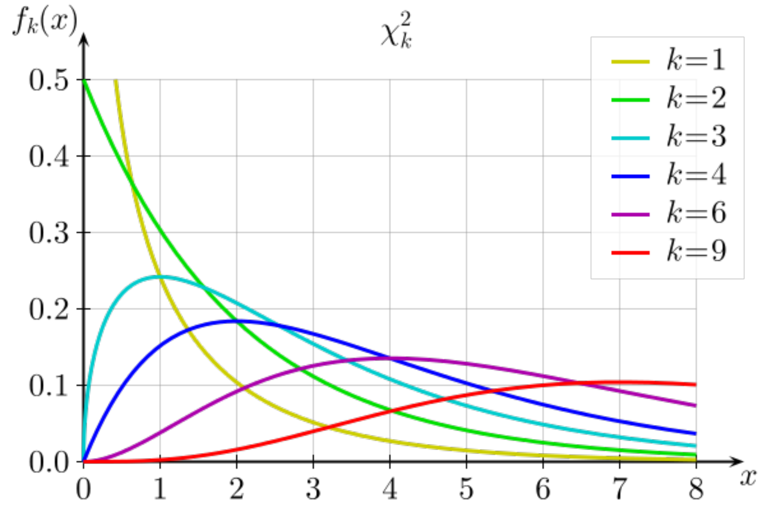
$$T = \sum_{i}^{N} \frac{(O_i - E_i)^2}{\sigma^2} \longrightarrow \sum^{N} Z^2 \sim \chi^2(N)$$

$$\underset{x}{O_i} \sim N(\underset{\mu}{E_i}, \sigma)$$

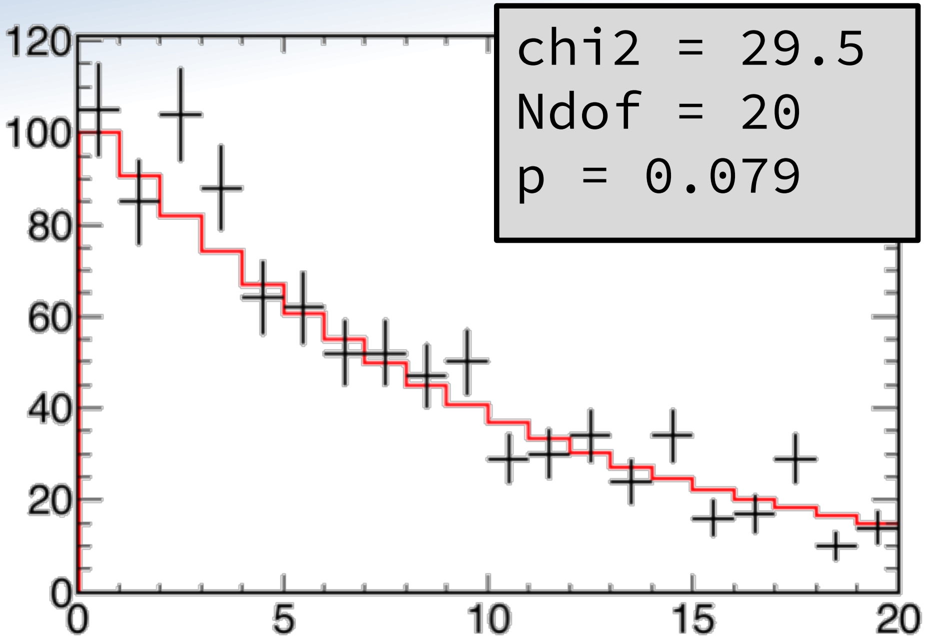
$$\frac{x - \mu}{\sigma} \longrightarrow Z$$

- This means that, assuming the null is true, we know what *T*'s distribution should be: the chisquared.

$$T = \sum^{N} Z^2 \sim \chi^2(N)$$



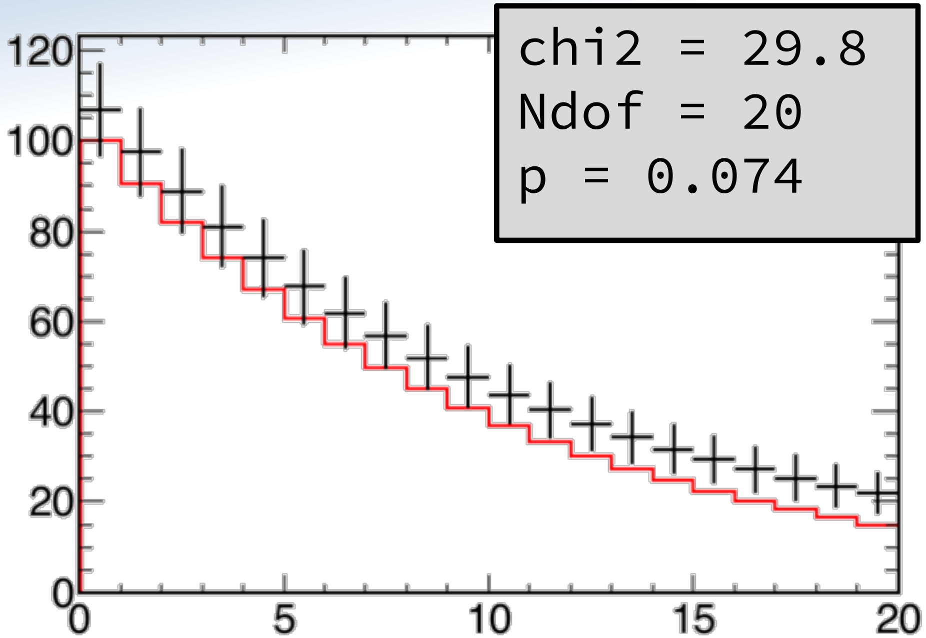
- This means that we can calculate *T* for our histograms, and then look up that value in this distribution to get a *p*-value.
  - Note that χ^2 depends on the number of "degrees of freedom"
  - For a histogram, Ndof = number of bins.




```python
def CalcChi2(hmc, hdata):
    chi2 = 0
    for i in range(1, hmc.GetNbinsX()+1):
        ei = hmc.GetBinContent(i)
        oi = hdata.GetBinContent(i)
        sigma = sqrt(ei)
        chi2 += (ei - oi)**2 / sigma**2
    return chi2

chi2 = CalcChi2(hpred, hrand)
Ndof = hpred.GetNbinsX()
p = TMath.Prob(chi2, Ndof)
```

- With p of 0.08, we fail to reject the H_0.

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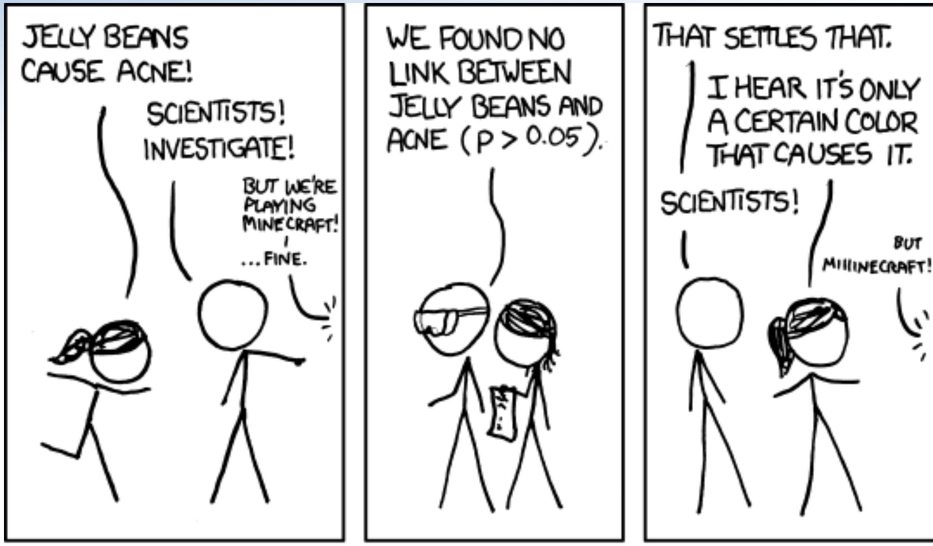
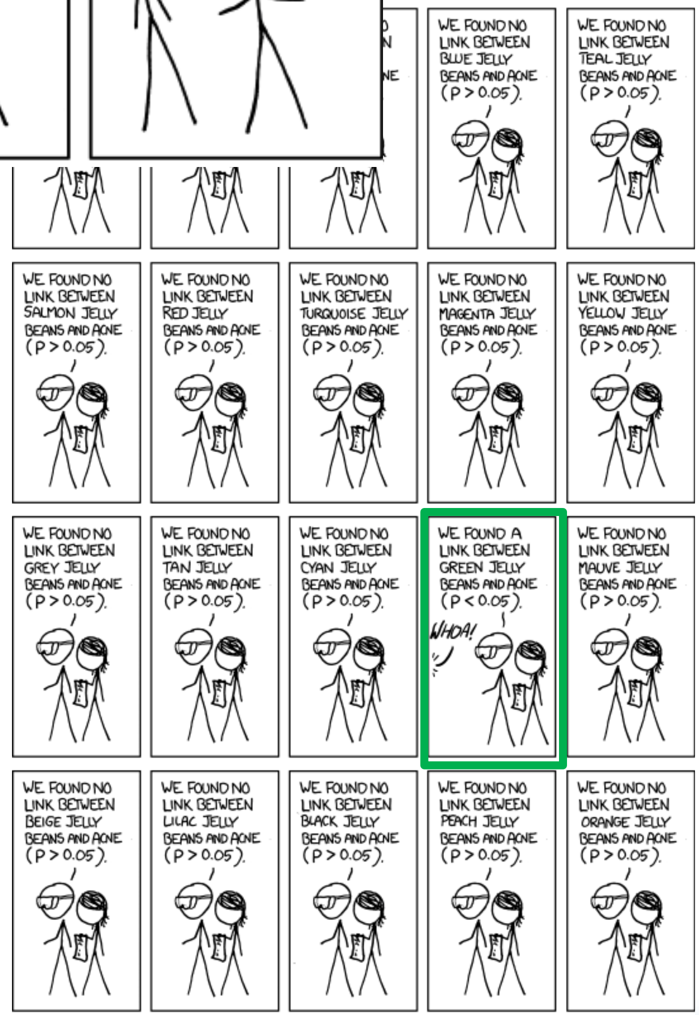
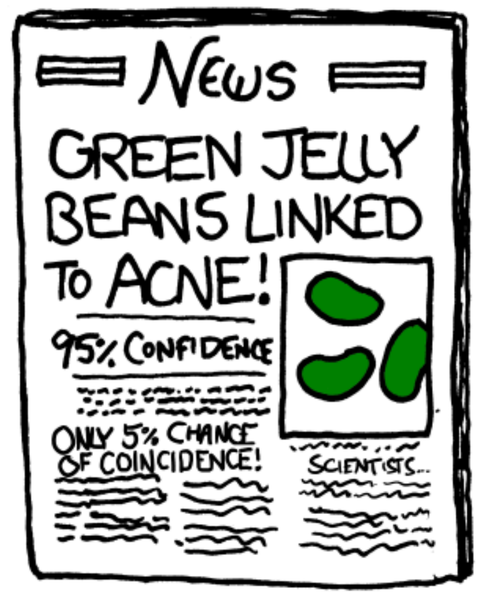
```

- Statistical tests are not a substitute for looking at the data!
- The results from a test are piece of the argument – they are not an answer themselves.

- Also – be warry of the “look elsewhere” effect.
  - More often a problem for us when looking at data-MC comparisons.







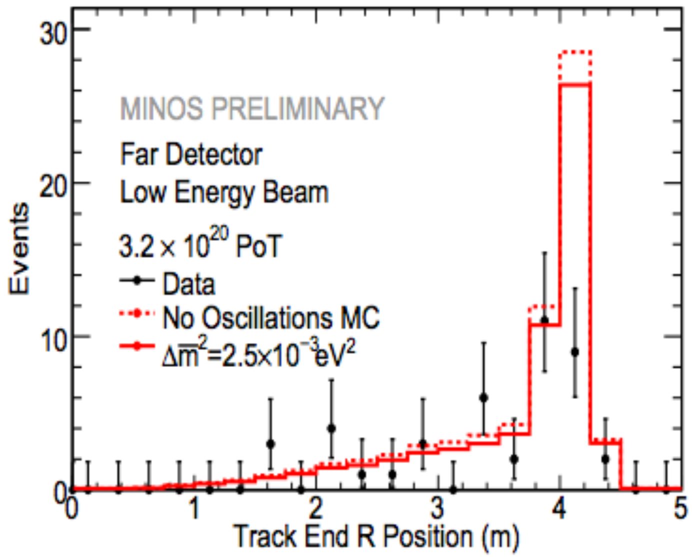
https://xkcd.com/882/

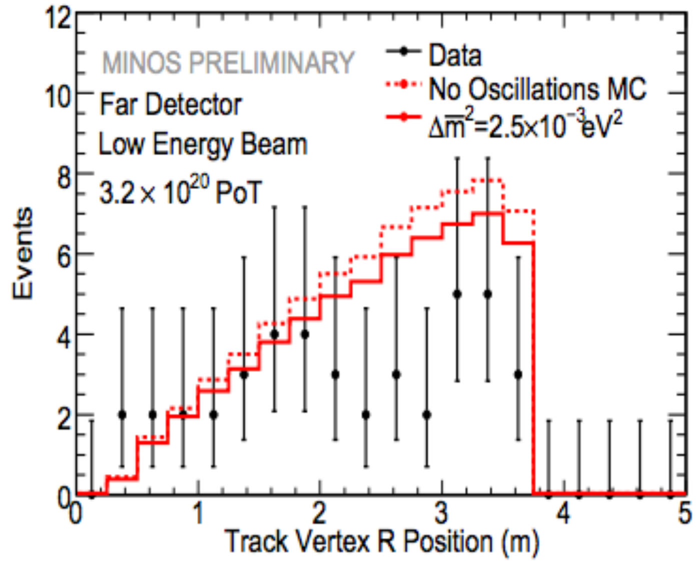
Real Life Examples

MINOS 2008

Anomalous FD Distributions

Z. Isvan



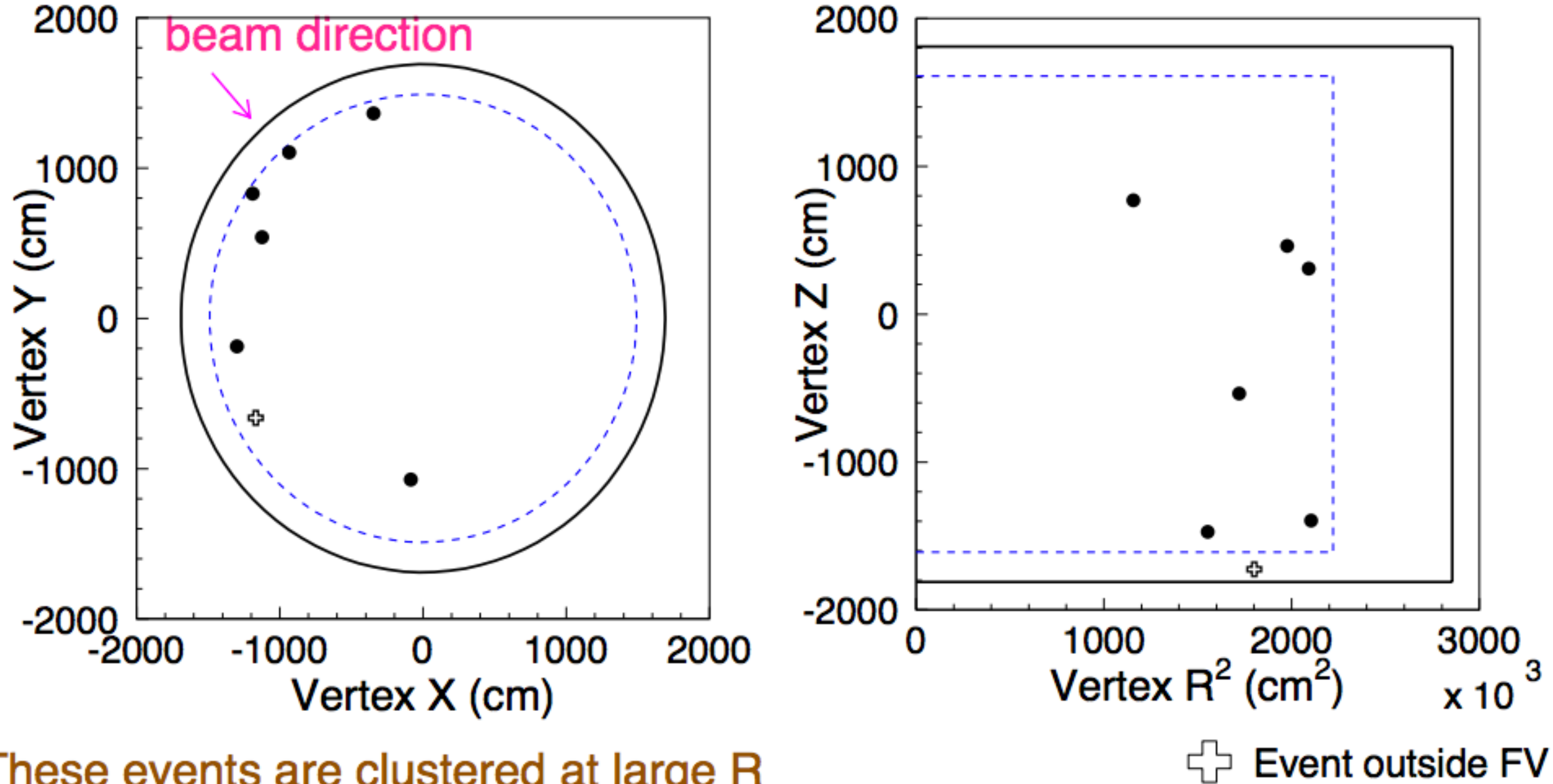


- Track End R has $\sim 3.3\sigma$ discrepancy at 4.1m (26 events expected 9 events seen discrepancy of $(26\text{-}9)/\sqrt{26} = 3.3\sigma$)!
  - Essentially all the missing events are in a single Track End R bin.
- Vertex R distribution also shows discrepancy in region r>2m.

Real Life Examples

T2K 2011

Vertex distribution of ν_e candidate events



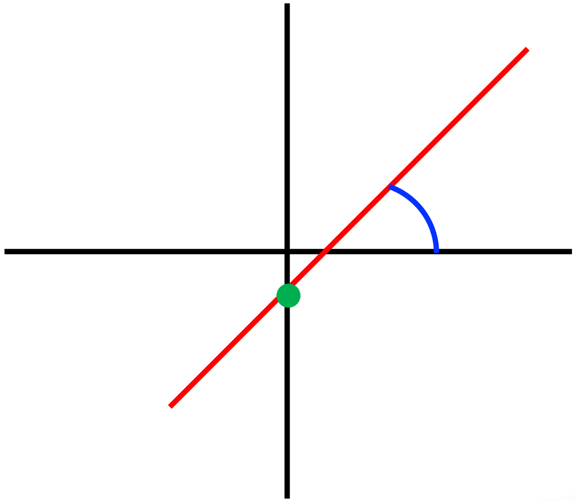
These events are clustered at large R

→ Perform several checks. for example

* Check distribution of events outside FV → no indication of BG contamination
* Check distribution of OD events → no indication of BG contamination
* K.S. test on the R^2 distribution yields a p-value of 0.03

Parameter Estimation

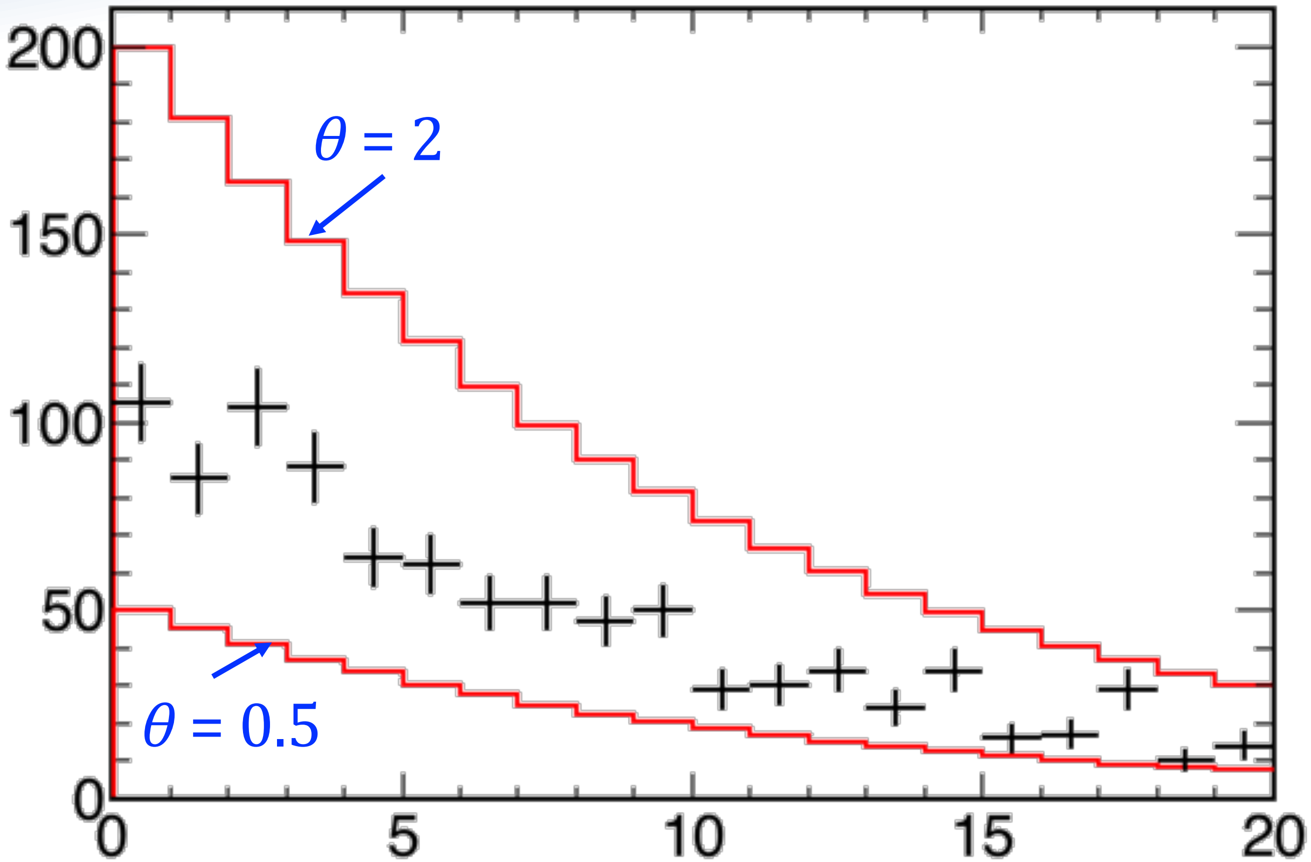
- Up until now, we've been asking yes-or-no questions.
- Often, what we want is to measure a value – this is **parameter estimation**.
  - In addition to data, this requires a model.
  - The parameters are the values which describe that model.
  - For example, a line is described by it's **slope** and y-intercept.



- So, how do we estimate parameters given a model and data?
- We use a method called **maximum likelihood**
  - The key to which is the likelihood function:

$$\mathcal{L}(\vec{\theta}) = P(\text{your data}|\vec{\theta})$$

  - The probability of your data assuming these parameters are true.



- Let's extend a familiar example.
- Now, we have a model, with a single parameter θ.

- Now, we need a likelihood function.
- To start, let's assume Gaussian errors.

$$\mathcal{L} = P(\vec{O}|\theta) = \prod^{N} e^{(O_i - E_i(\theta))^2/\sigma^2}$$

P of each bin, assuming each is a normal distribution.

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- In practice, instead of maximizing likelihood, we minimize -2 ln *L*
  - Because addition is easier than multiplication.

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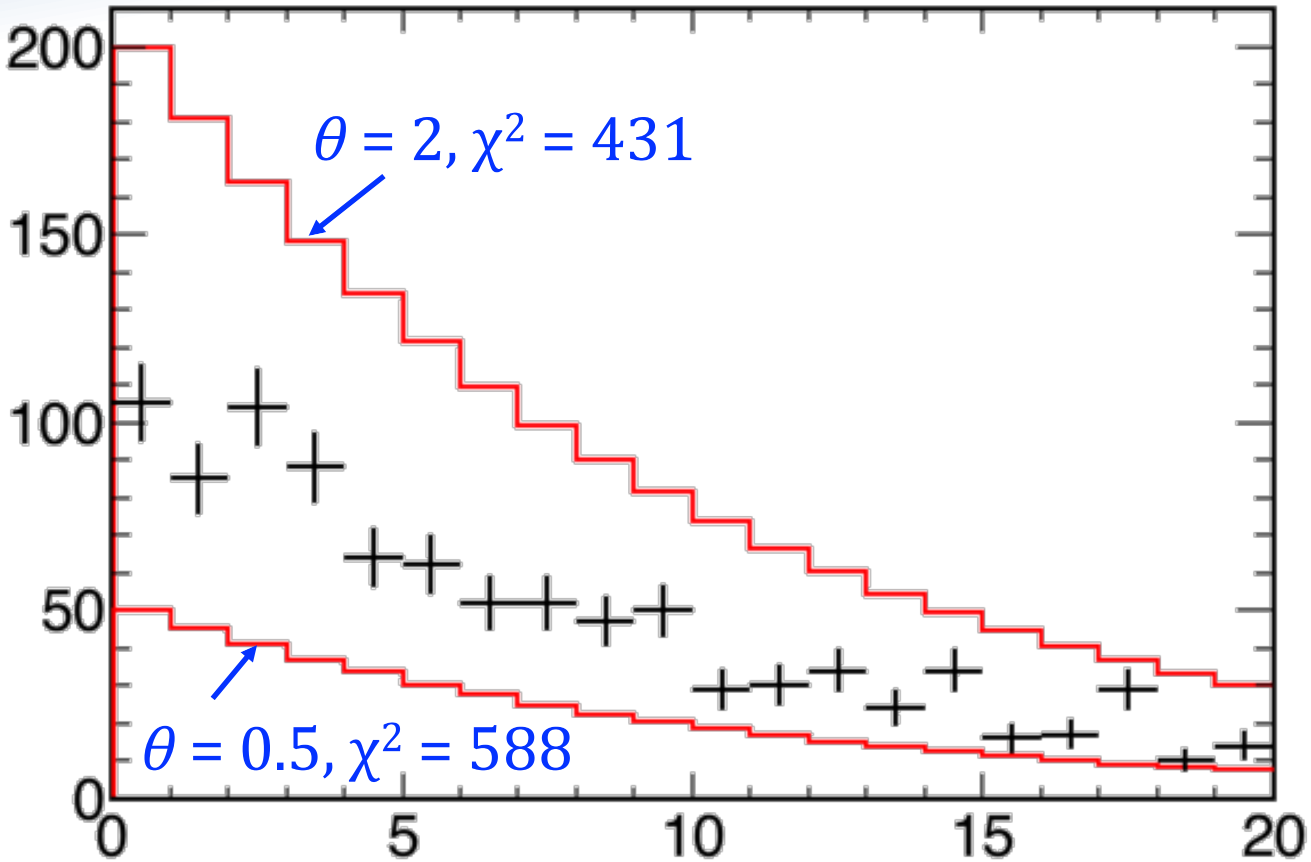
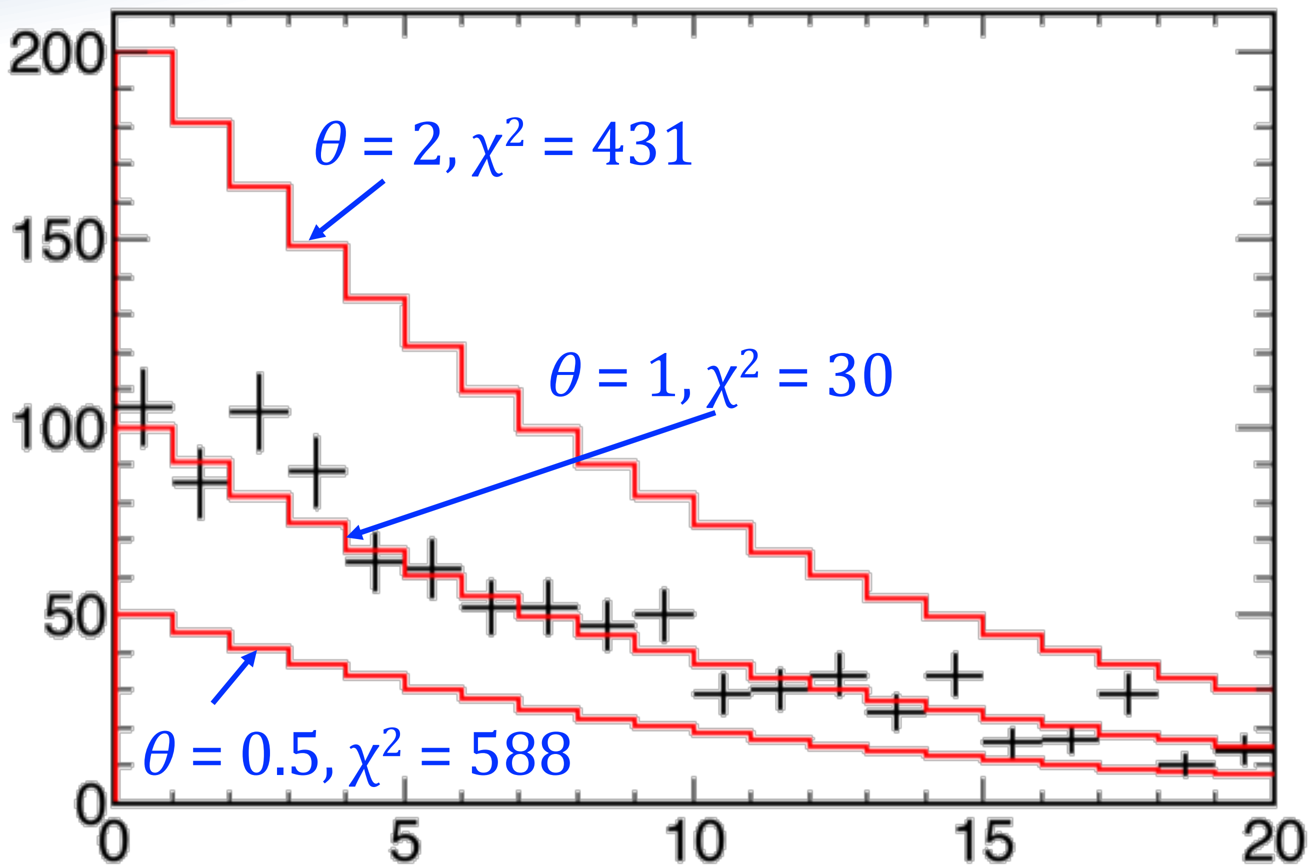
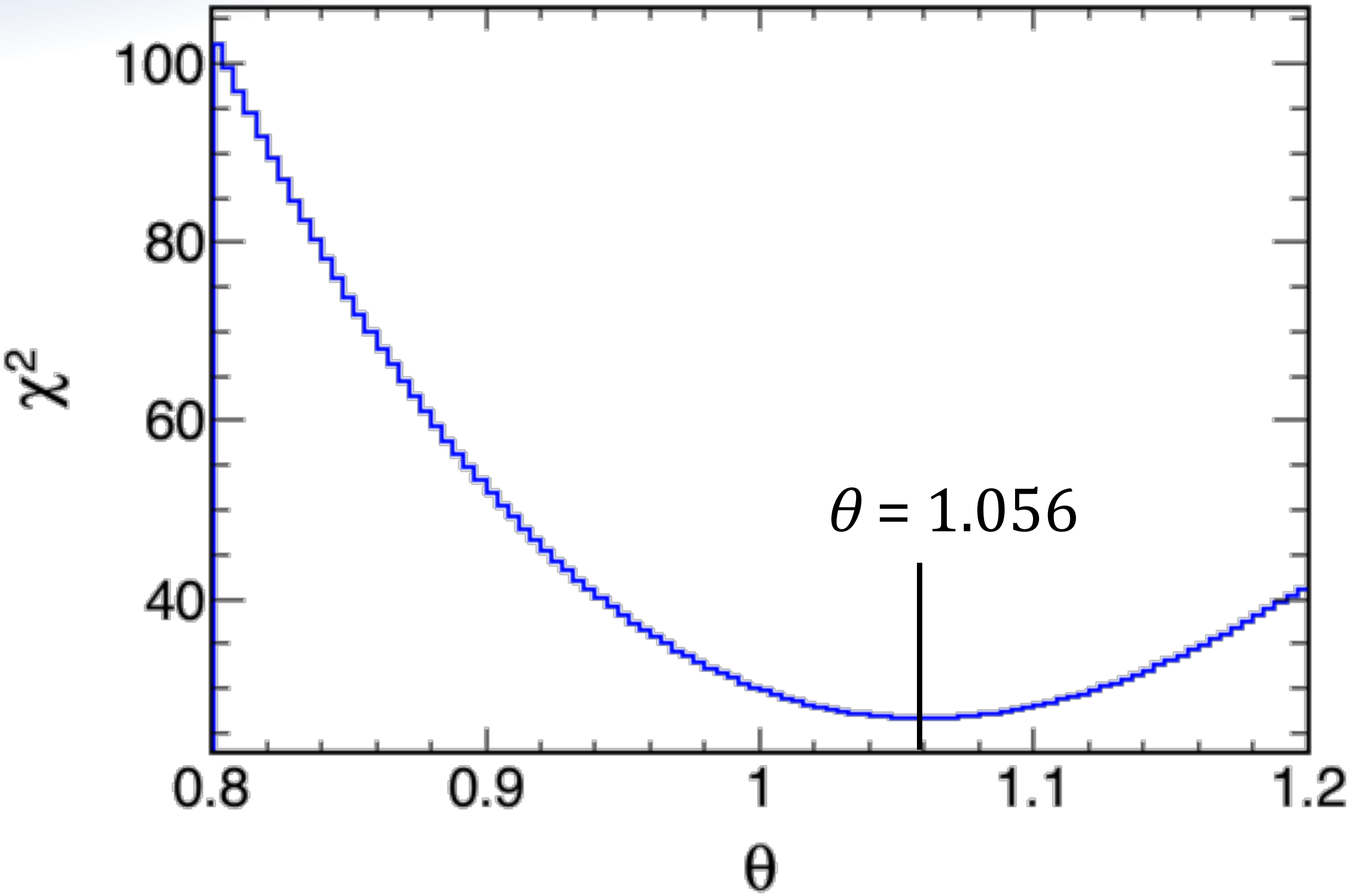
$$-2\ln\mathcal{L}(\theta) = \sum^{N} \frac{(O_i - E_i(\theta))^2}{\sigma^2} = \chi^2$$



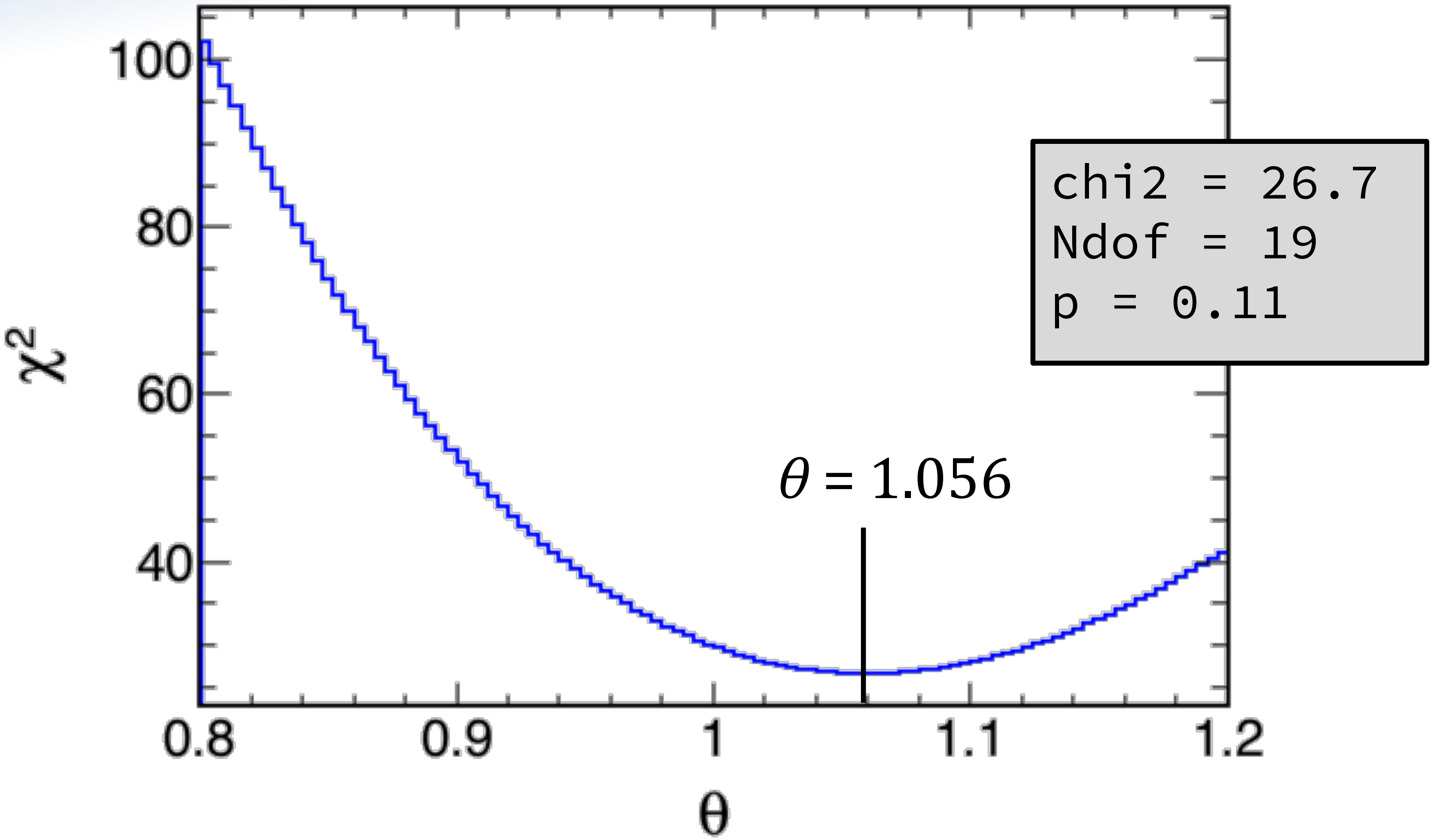
- We can then calculate χ^2 for each possible value of θ.
  - Both of these are pretty bad.



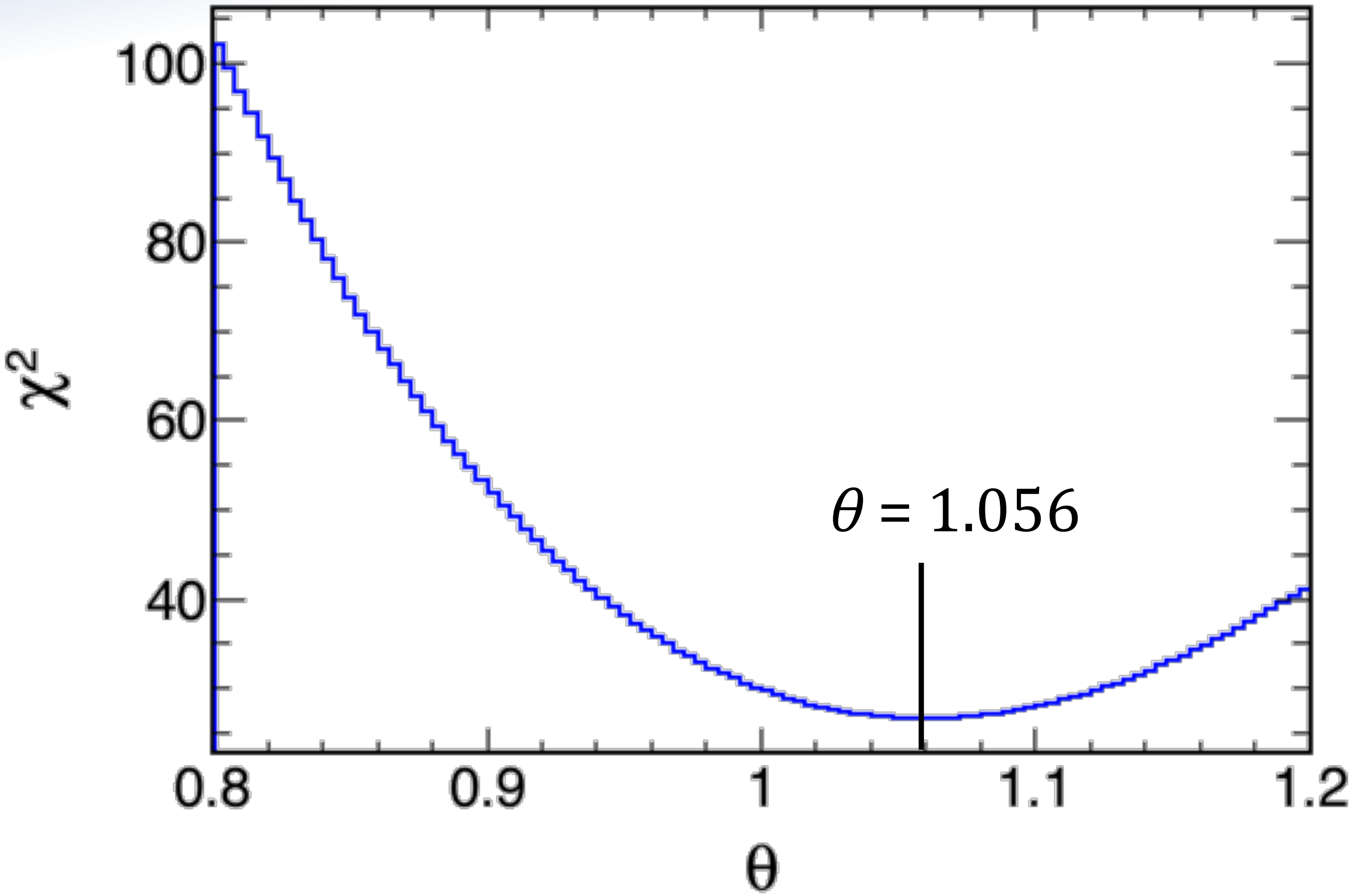
- We can then calculate χ^2 for each possible value of θ.
  - But 30 is pretty good.



- We find the minimum χ^2 (maximum L) when $\theta = 1.054$
- This is our *maximum likelihood estimate*, or “best fit”



- We can also ask, "how good a fit is this?"
  - Is this a reasonable model of this data?
- That is just the hypothesis test we did before.
  - But – you need to subtract 1 for each free parameter in the fit.



- An even better question – what is our uncertainty on our estimate?

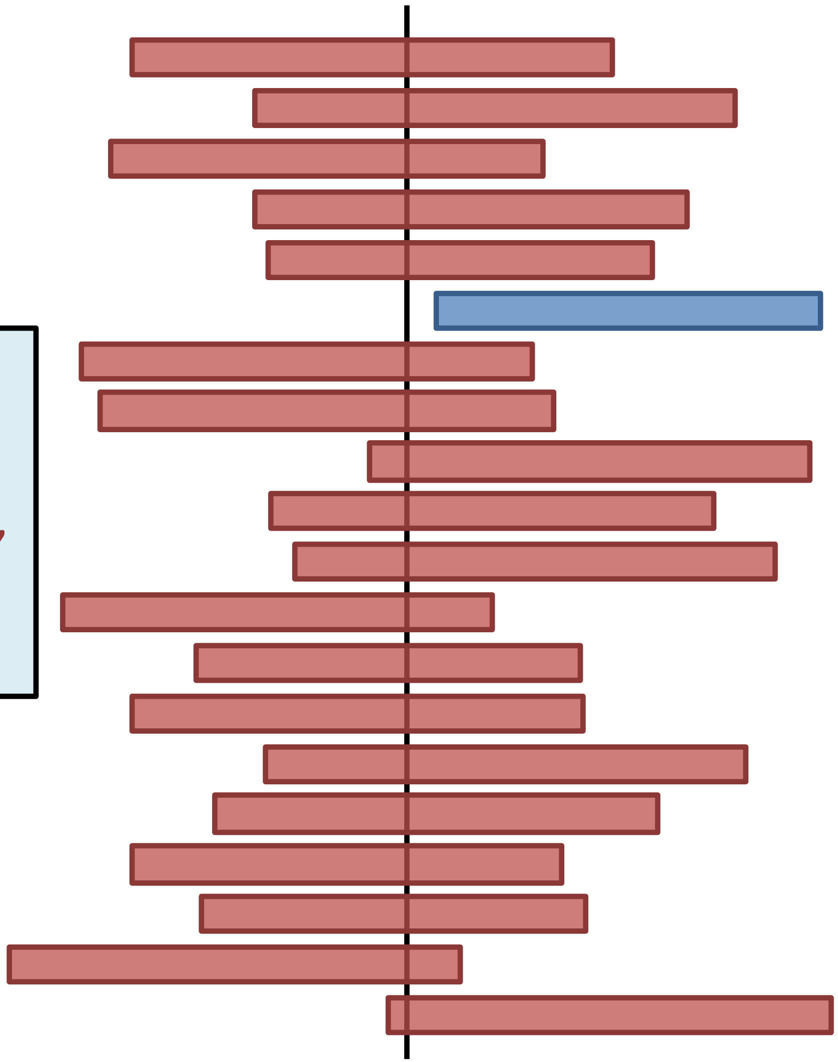
Building Confidence Intervals

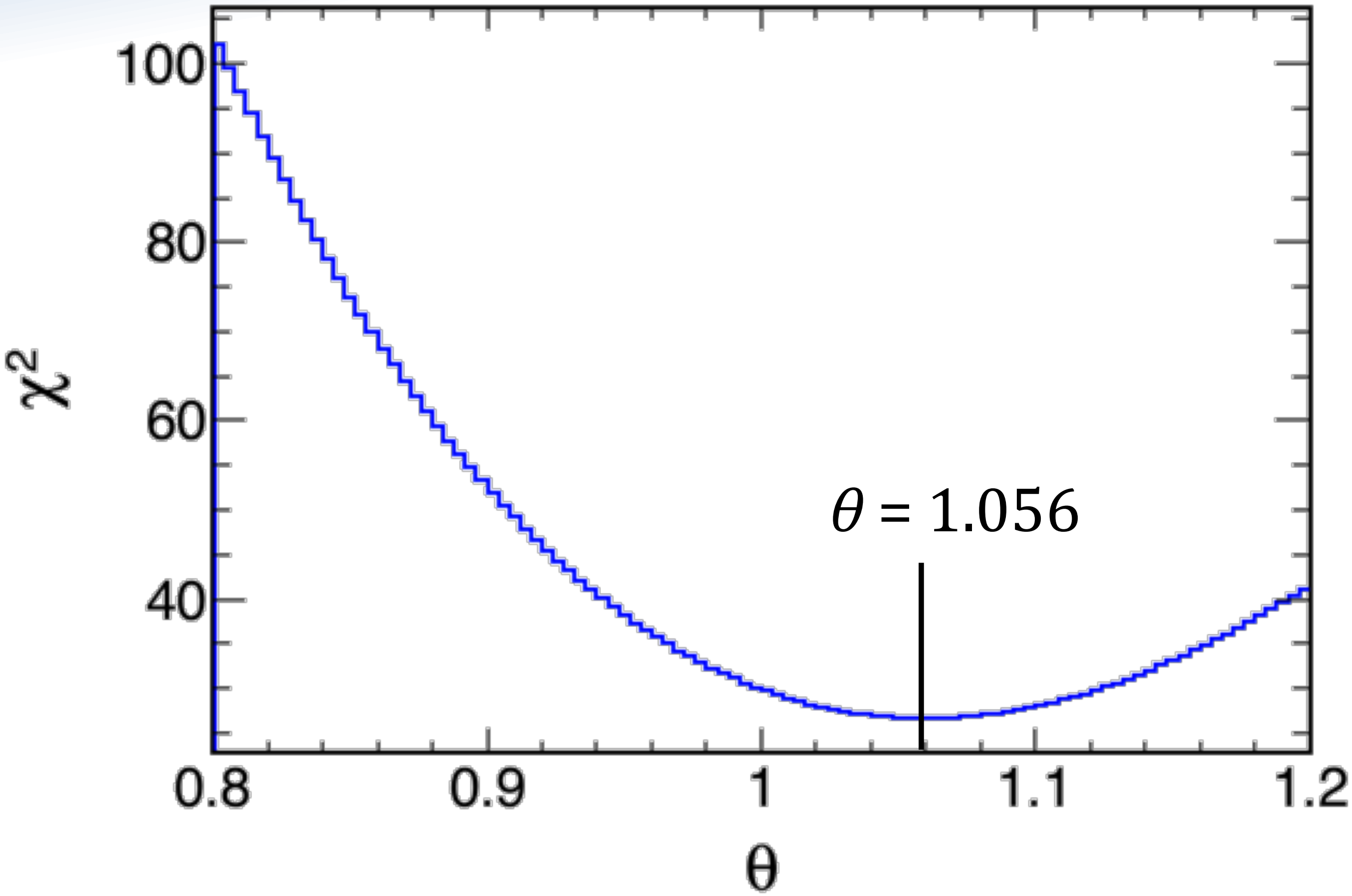
- Here we'll discuss "frequentist" confidence intervals, because that's what you will most often see.

Definition of an Confidence Interval at level α:

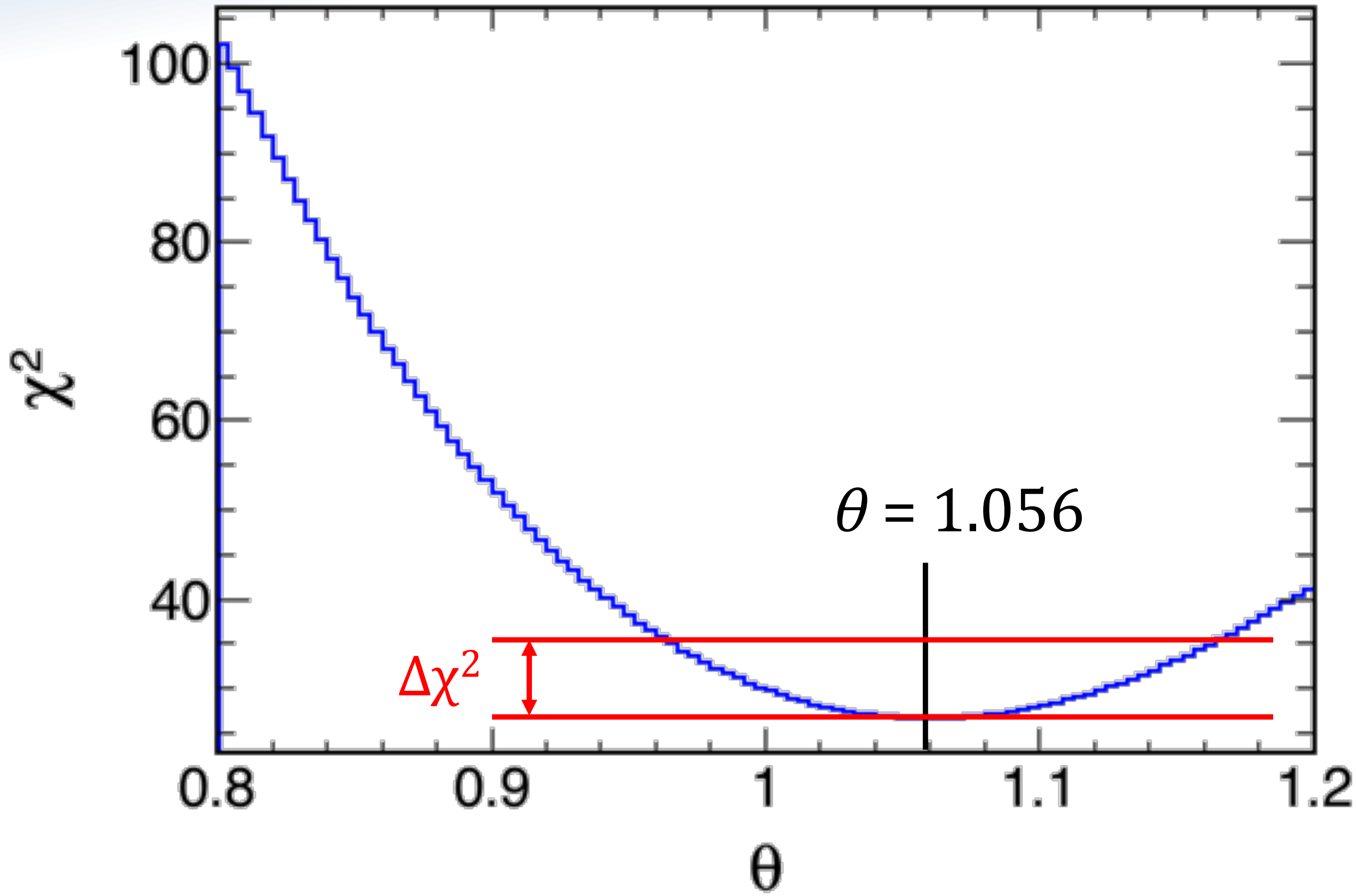
If we repeat the experiment numerous times, α of the intervals we draw will cover the true value.

- This isn't really what you wanted to know, but it has been rigorously defined.
- There are many ways to construct CI's depending on the circumstance.

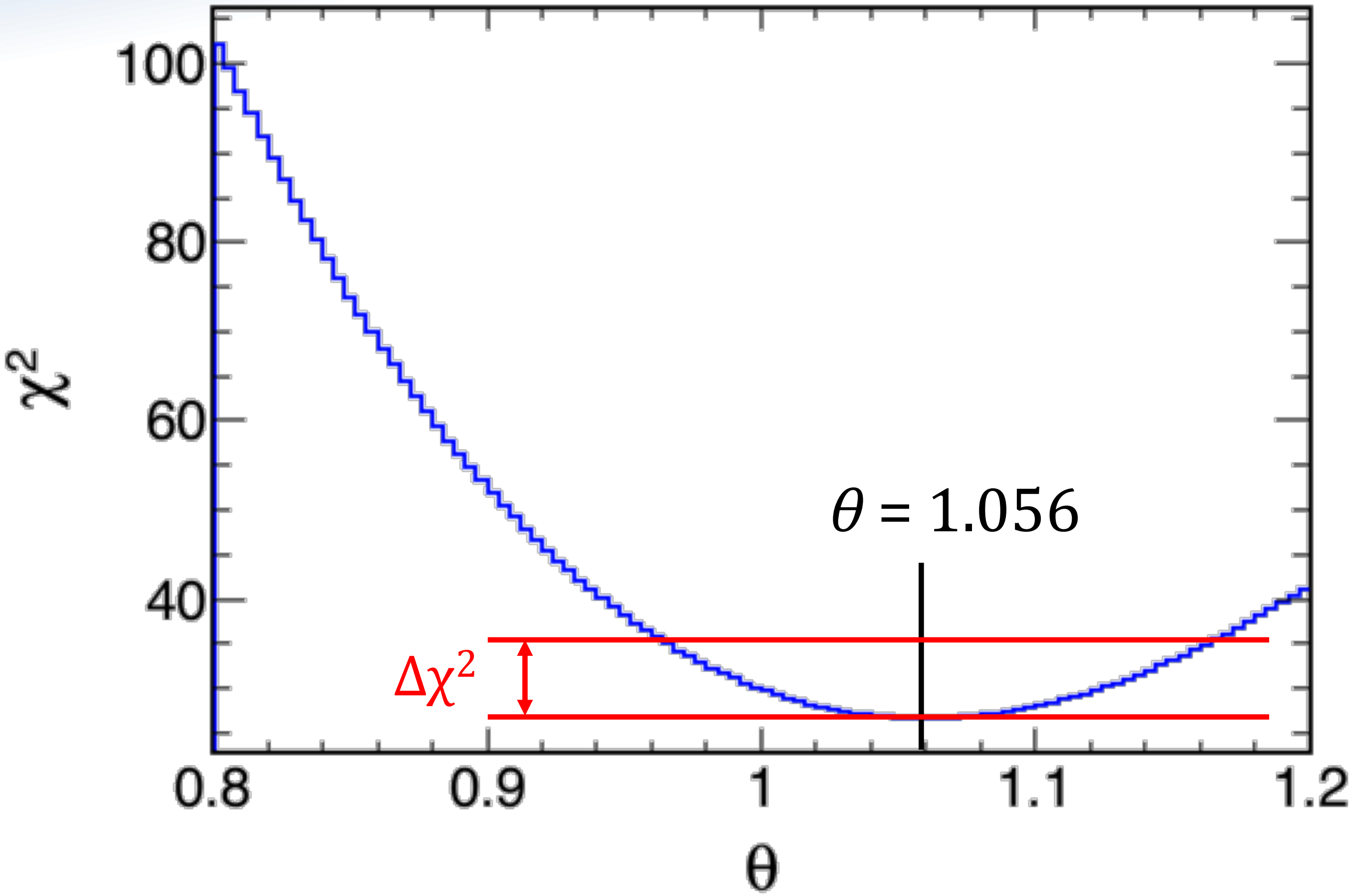




- If you problem has all Gaussian errors, then the distribution of the estimator of the parameter is *also* Gaussian.
  - Presented without proof, since that's what the PDG does, too.
  - This is the case for our example, too.



- We will use the likelihood distribution to draw the CI.
- We allow inside our CI any values of θ with small values $\Delta\chi^2$ relative to the best fit, and we exclude values of θ with larger values of $\Delta\chi^2$.



- The question you should be asking:
- How do I know what "up value" to choose to know which θ's are in and which are out?

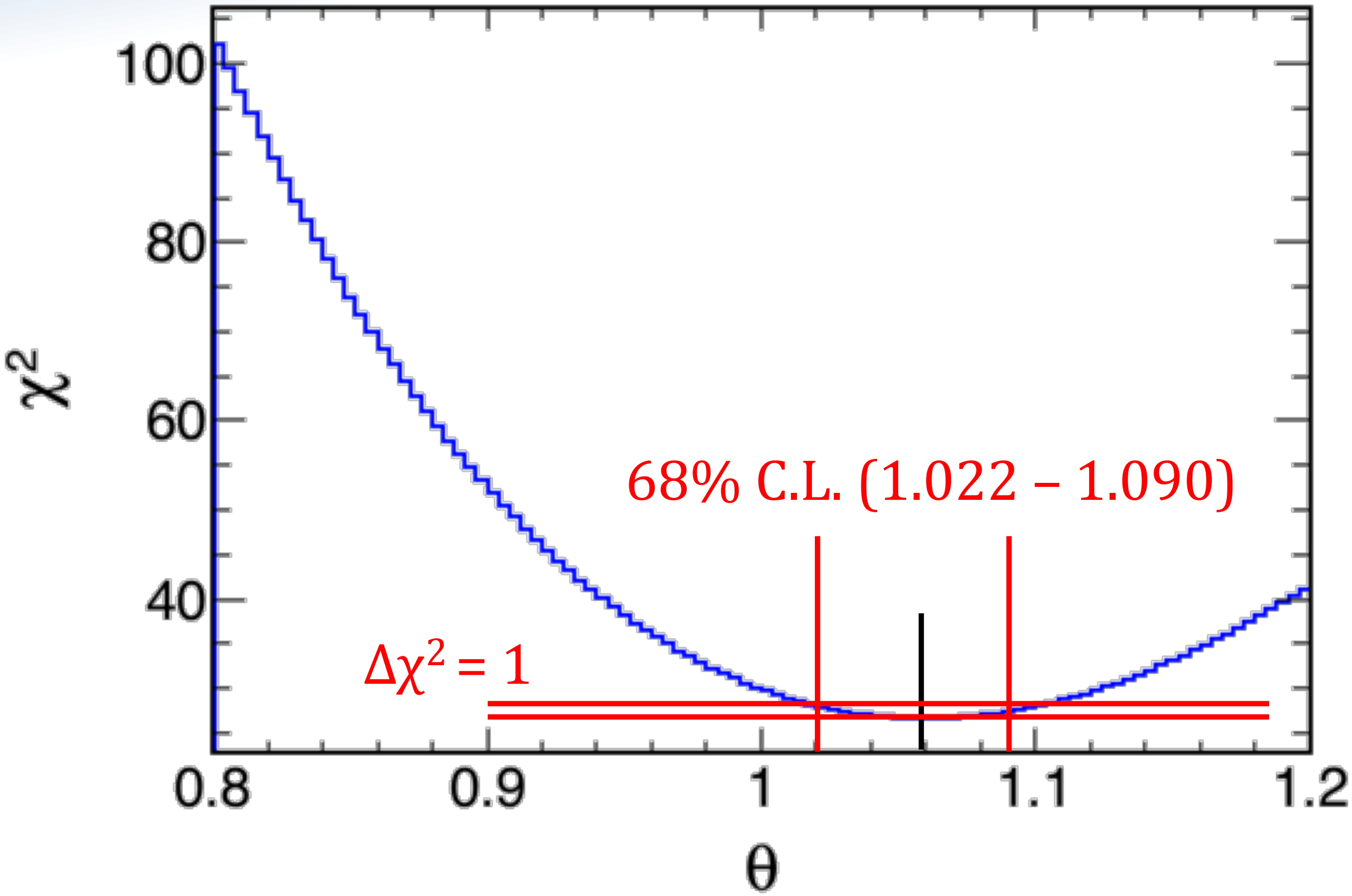
- Here is where we take advantage of everything being Gaussian.
- As with the hypothesis tests, we know what distribution $\Delta\chi^2$ should have, so we can look it up.
- This table comes from the PDG:

Table 37.2: Values of $\Delta\chi^2$ or $2\Delta \ln L$ corresponding to a coverage probability $1-\alpha$ in the large data sample limit, for joint estimation of m parameters.

$(1-\alpha)$ (%)	$m=1$	$m=2$	$m=3$
68.27	$\boxed{1.00}$	2.30	3.53
90.	2.71	4.61	6.25
95.	3.84	5.99	7.82
95.45	4.00	6.18	8.03
99.	6.63	9.21	11.34
99.73	9.00	11.83	14.16

The level of the CI you want to draw.

The number of dimensions.



- This 68% (e.g. 1σ) C.L. is what we generally report as an error band.
- So, in Stats-ese: ML Estimate 1.056 with 68% CL 1.022-1.090
- In Physic-ese: 1.056 ± 0.034

A little more realism

- Choice of likelihood function
  - It's rare in neutrino physics that we have so much data that χ^2 is valid.
  - Instead, we use an L which is based on bins with Poisson errors.

$$-2\ln\mathcal{L}(\theta) = \sum^{N} \frac{\left(O_i - E_i(\theta)\right)^2}{\sigma^2} = \chi^2$$

$$-2\ln\lambda(\boldsymbol{\theta}) = 2\sum_{i=1}^{N}\left[\mu_i(\boldsymbol{\theta}) - n_i + n_i \ln\frac{n_i}{\mu_i(\boldsymbol{\theta})}\right] ,$$

If you have bins with < 30 entries, you probably need this. Just look it up in the PDG.

- More variables?
  - If you have 2 variables, and you want to show 2 variables, then it's straightforward.
  - Just pick the right up value, and points below it are in.

$(1-\alpha)$ (%)	$m=1$	$m=2$	$m=3$
68.27	1.00	2.30	3.53
90.	2.71	4.61	6.25
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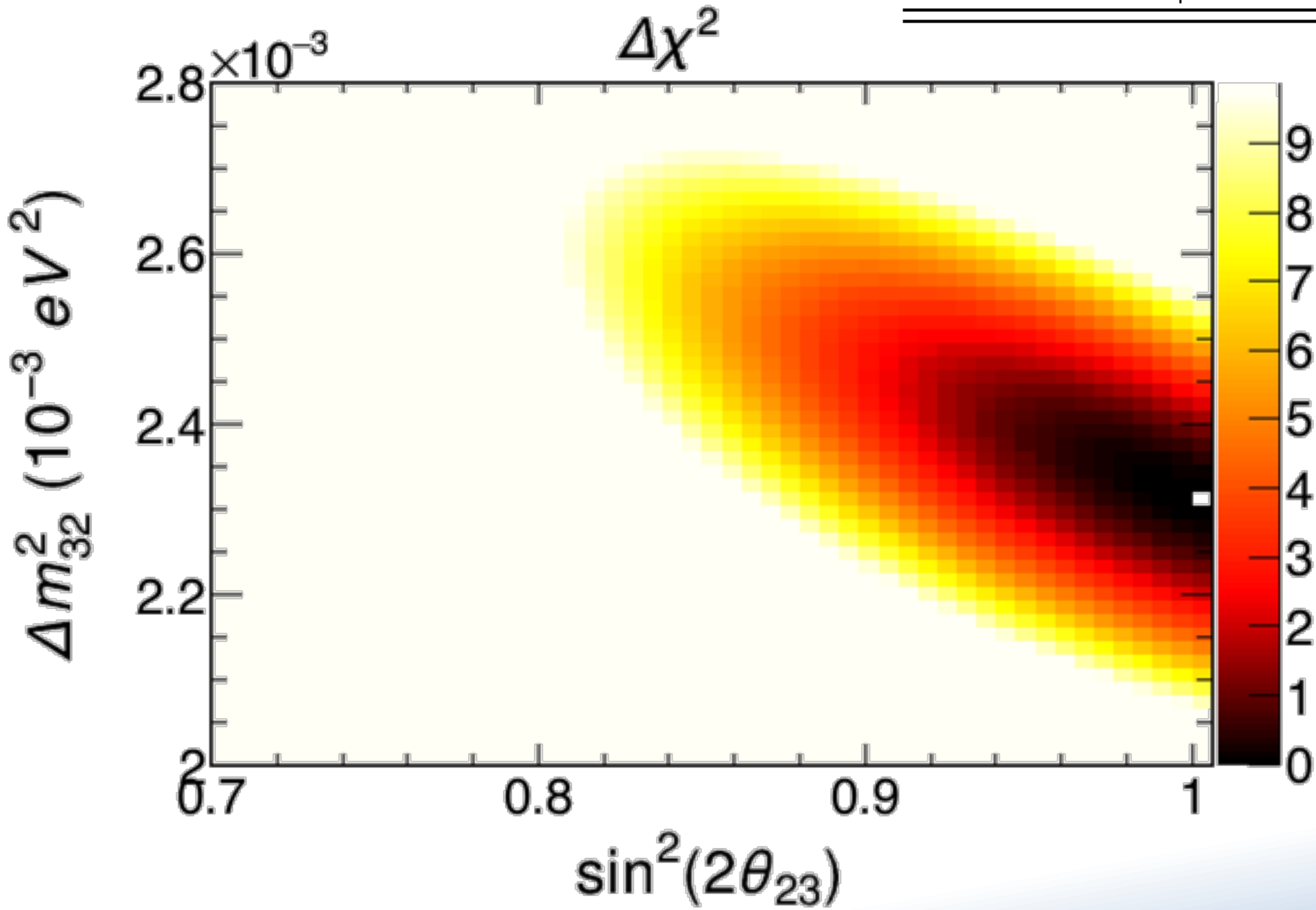

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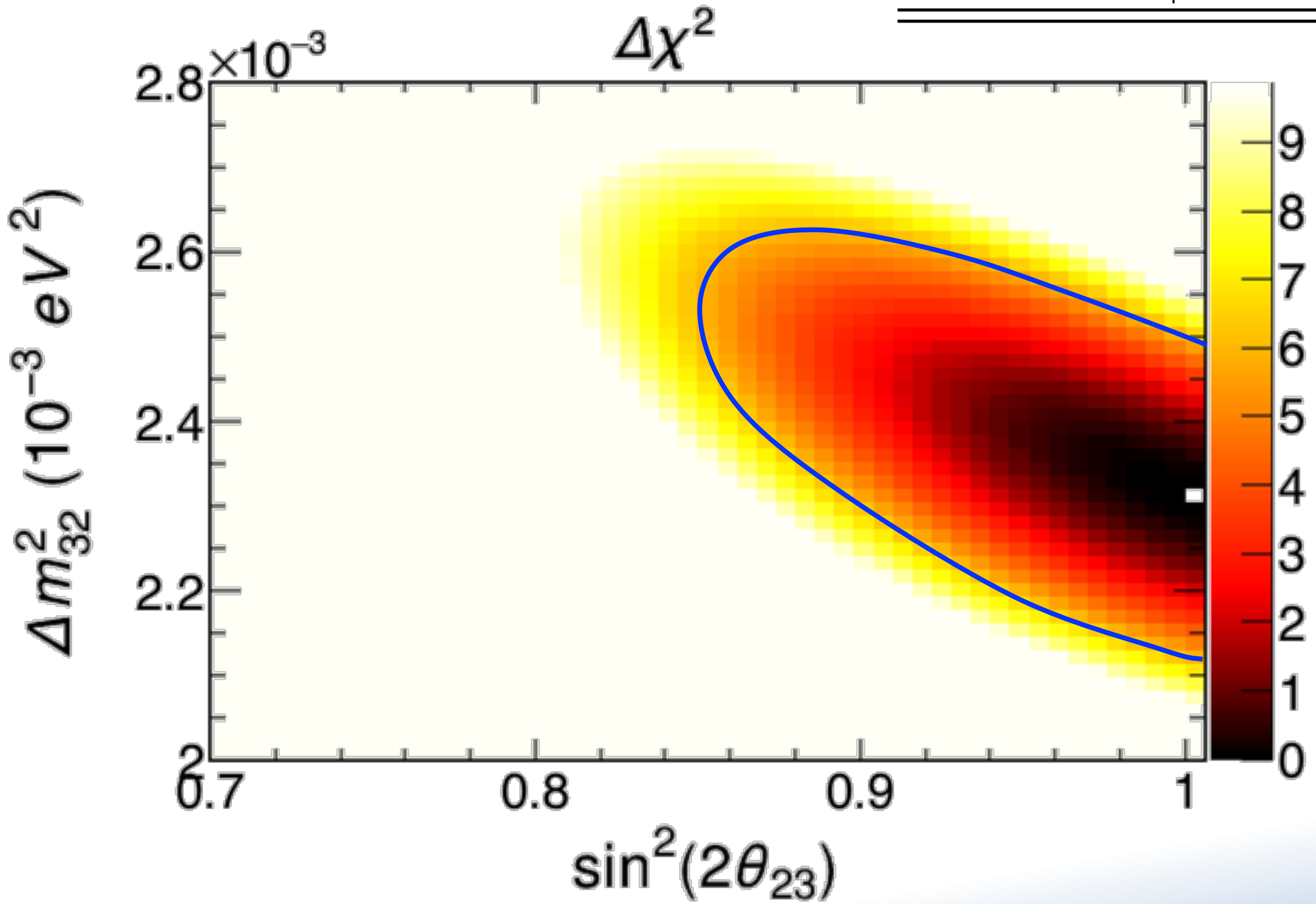

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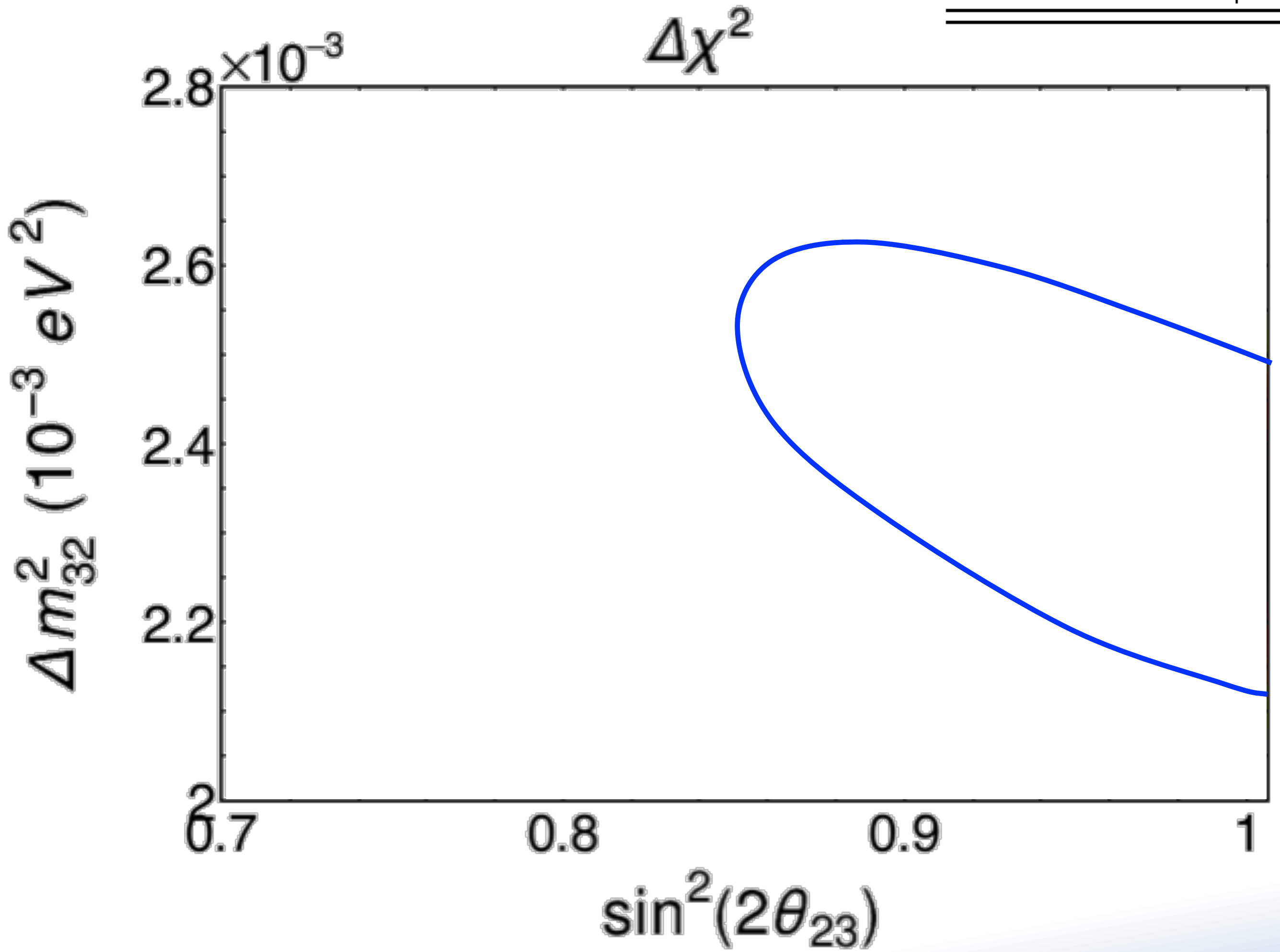
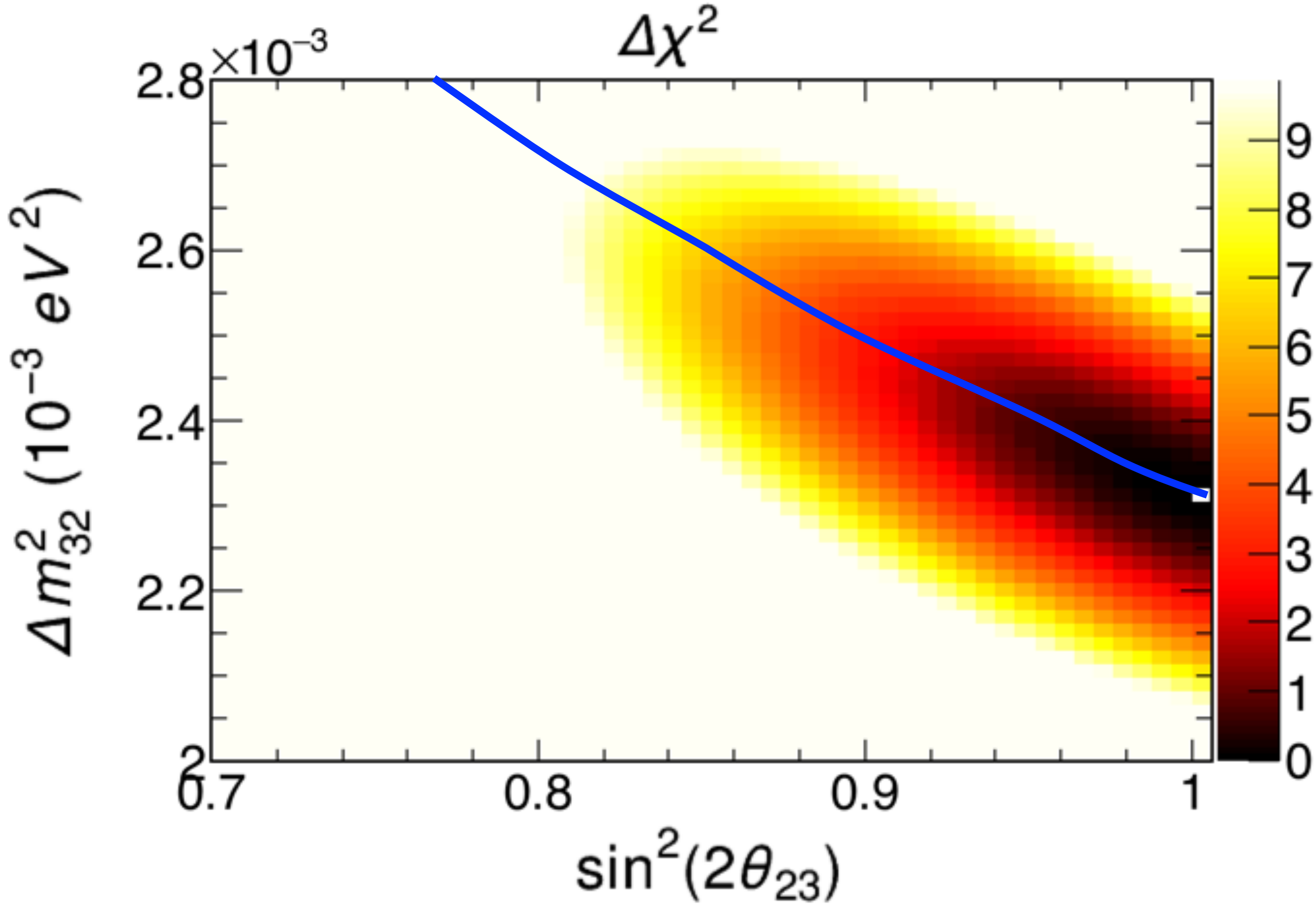


- Nuisance parameters
  - Often your likelihood depends on more parameters than you want to present.
  - Extra parameters can be physics or systematic uncertainties.

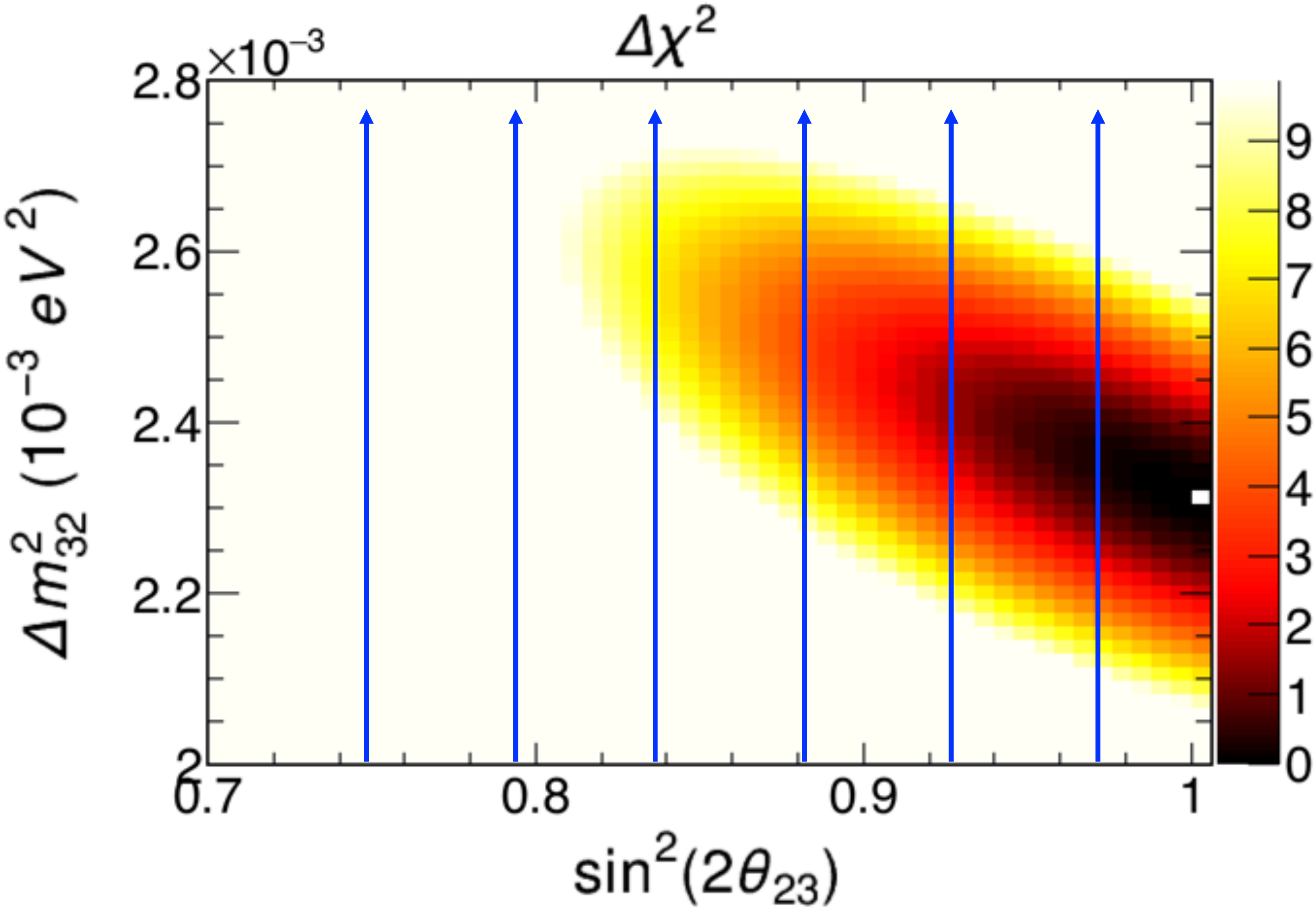
- For example, in the NOvA joint fit we do:
  $(\Delta m^2, \theta_{23}, \theta_{13}, \delta,$ systematic errors$) \rightarrow (\theta_{23}, \delta)$

- Two different approaches:
  - Profiling
  - Marginalizing

- Profiling
  - Take the best fit in all parameters you are not showing *at each point* you do show.
  - More common, works under certain assumptions.



- Marginalizing
  - Integrate up all the values you are not showing.
  - Shows up more in Bayesian analyses.



$(1-\alpha)$ (%)	$m=1$	$m=2$	$m=3$
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99.	6.63	9.21	11.34
99.73	9.00	11.83	14.16

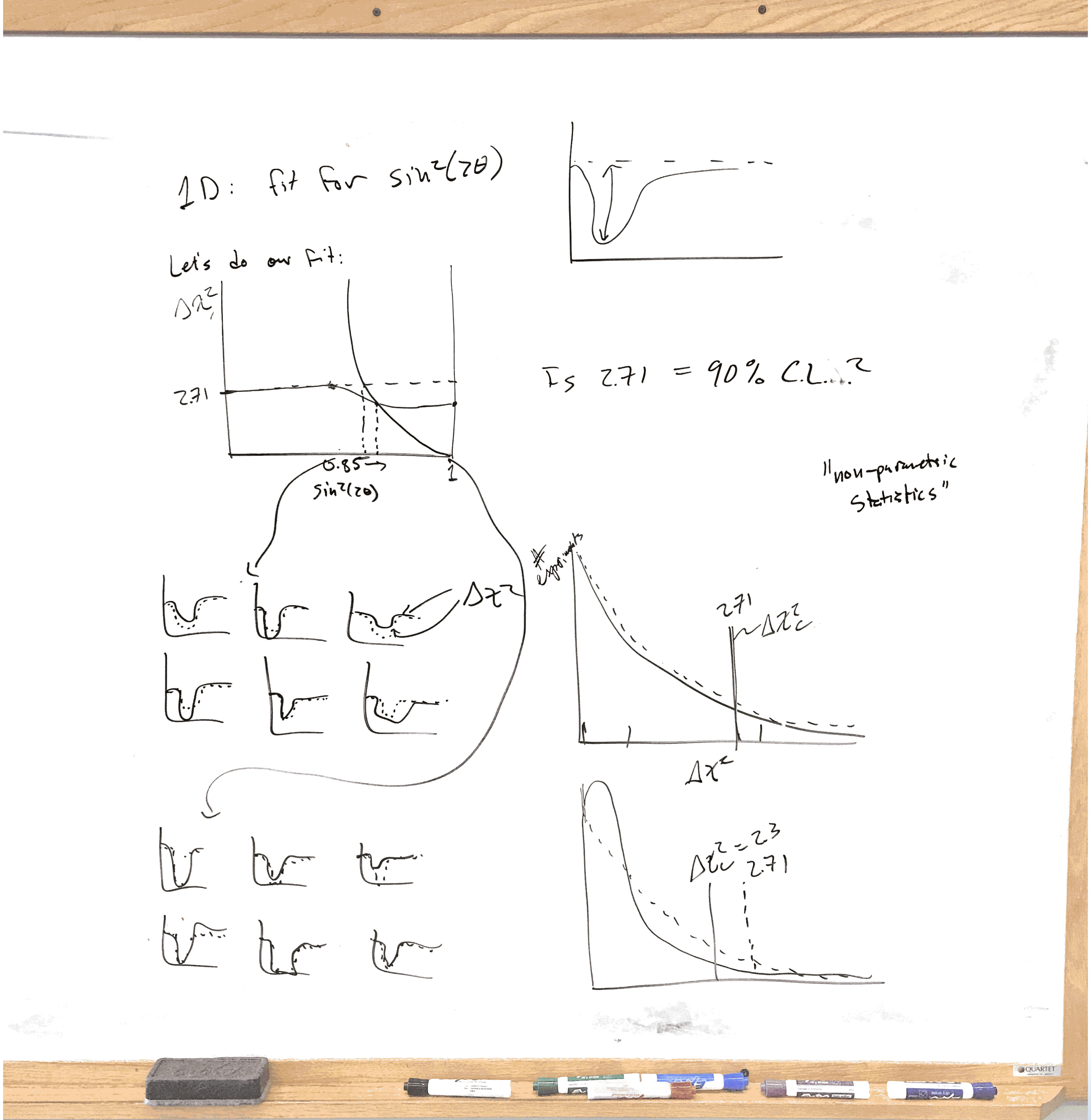
- What if you can't trust the values from the PDG?
  - They don't have the right coverage:
    a 90% C.L. is actually an 85% C.L.
- Commonly happens when statistics are low and the problem has a physical boundary.
  - Happens a lot in neutrino physics since $\mathbf{0 < \sin^2 2\theta < 1}$

Feldman-Cousins

- The solution is a technique called **Feldman-Cousins**

- From a paper called "A Unified Approach to the Classical Statistical Analysis of Small Signals"
  - by Gary Feldman and Bob Cousins
  - Phys. Rev. **D**57 (1998) 3873

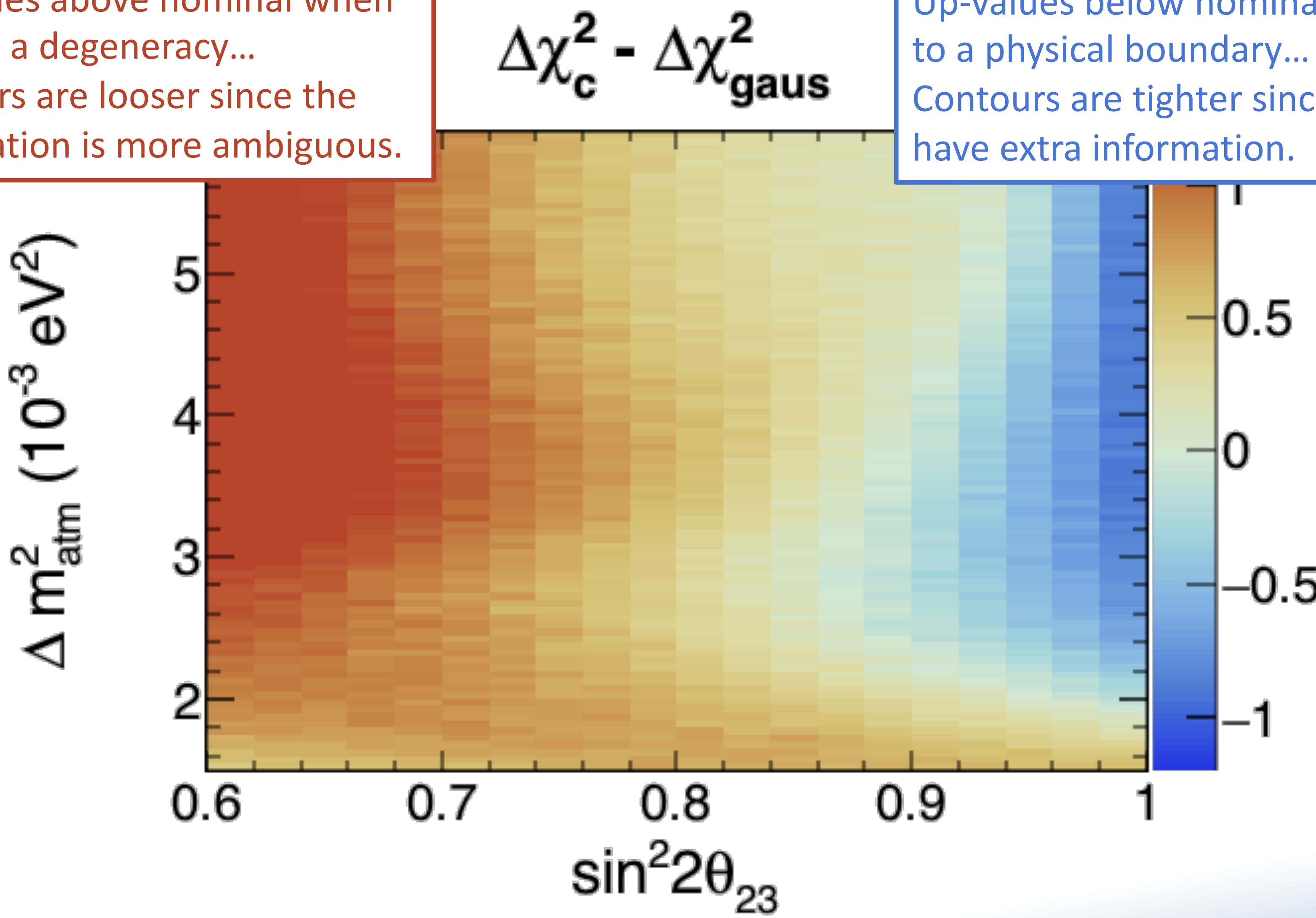
- Let's walk through an example.


1D: fit for sin²(2θ)
Let's do our fit:
Δχ²
2.71
0.85
sin²(2θ)
1
Is 2.71 = 90% C.L...?
"non-parametric statistics"
experiments
Δχ²
2.71
Δχ²c
Δχ²
Δχ²c = 2.3
2.71

- A real-life example from the MINOS anti-ν_μ disappearance analysis circa 2010.

Up-values above nominal when there is a degeneracy… Contours are looser since the information is more ambiguous.

Up-values below nominal close to a physical boundary… Contours are tighter since we have extra information.



Frequentist

- The true value of a measurement is **an unknown constant**.
- Report the **probability of experimental outcomes**, given a value of that constant.
- Use that to construct a confidence interval which will contain the true value in α fraction of experiments.

Bayesian

- The true value of a measurement is a **random variable**.
- **Before** the measurement, have a "prior" PDF of that variable.
- After the measurement, **update to a "posterior" PDF** using the data collected.

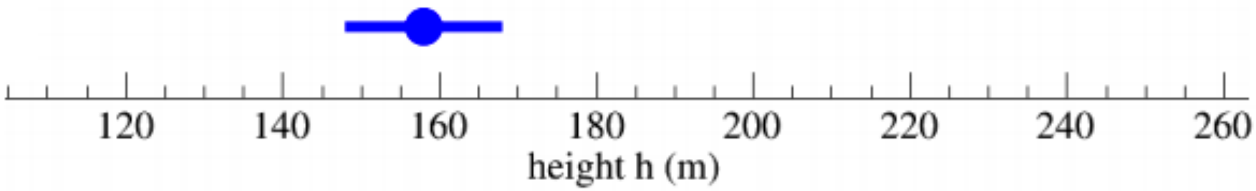
Frequentist

- Apply solid mathematical rigor to answer a question that nobody cares about.

Bayesian

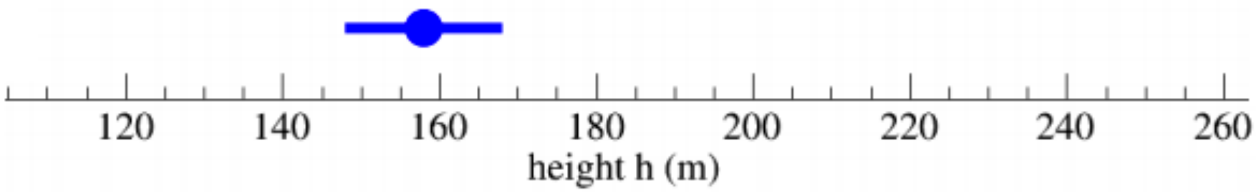
- Answers the question everyone is really interested in using assumptions no one believes.

Frequentist



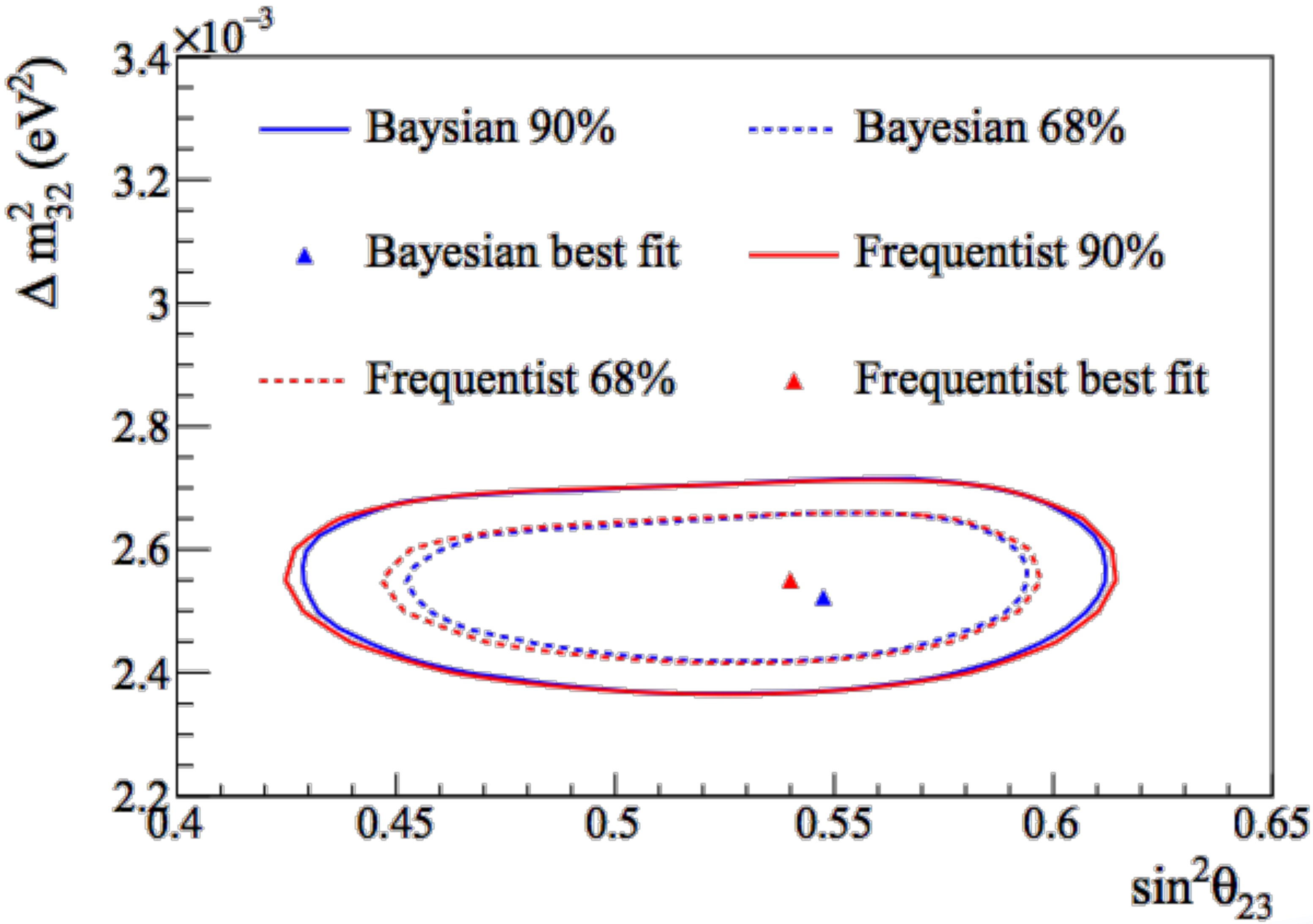
$$h = (158 \pm 20)\ \mathrm{m}$$

Bayesian



$$h = (158 \pm 20)\ \mathrm{m}$$

- In the real world:
  - This is from the latest T2K PRD, arXiv:1707.01048



Conclusion

- I’ve tried to show the statistical underpinnings of some of the most common statistical techniques we use.
  - But there are many, many more possible techniques.
  - There are numerous alternative ways to do everything I have presented here.
- Some general advice: **use the simplest method that is correct, but no simpler**.
  - If you use a technique that requires assumptions that you cannot meet, your results will be questioned.
  - But, if you use a more complicated technique, be prepared to explain how it works and why you chose it.
- I highly recommend the PDG statistics section as a place to find statistical techniques which are “commonly accepted” in physics.

Backups

$$\mu_1 = 2.5 \pm 0.1 \quad \mu_2 = 3.1 \pm 0.3$$

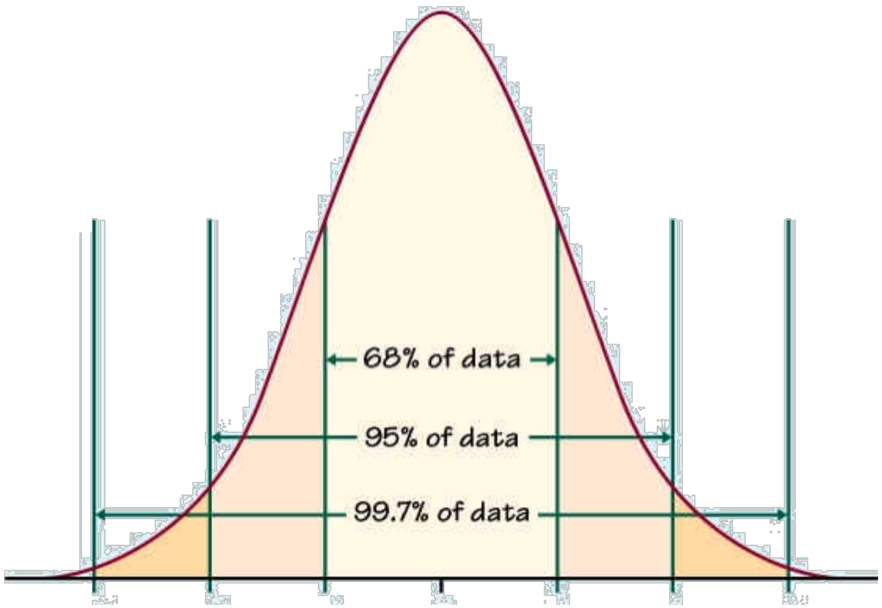
$$X = \mu_2 - \mu_1 = 0.6$$

- We know, from the central limit theorem, that means are normally distributed.
  - The difference between means is, too.
  - The standard deviation of that difference is:

$$\sigma_{\mu_2 - \mu_1} = \sqrt{\sigma^2_{\mu_2} + \sigma^2_{\mu_1}} = 0.32$$

- Now the question is: Is 0.6 significantly different from 0 if it comes from a normal distribution with $\sigma = 0.32$?

- What we are asking is: **how likely** is it that we would get our result, or something more extreme **assuming the null hypothesis is true**?
  - This is the definition of p-value, which we compare to our α.
- "different from" means we are making a "two-sided" test:
  - If we set an α = 0.05, we want to know if our value falls into the central 1-α or 95% of the distribution.



- To start, we calculate a “Z-score,” which effectively converts from our specific normal distribution to the canonical $\mathcal{N}(0,1)$:

$$Z = \frac{X - \mu_0}{\sigma} = \frac{0.6 - 0}{0.32} = 1.88$$

This is what we mean when we say “1.88 σ”

- But, what does that Z-score mean?
  - In other words, what is it's p-value we can compare to α?
  - What fraction of values in the distribution are **more extreme** than ours?

$$p = \int_{-\infty}^{-Z} \mathcal{N}(x, 1) dx + \int_{Z}^{\infty} \mathcal{N}(x, 1) dx$$

- But infinity is hard, so we can take advantage of the fact that probabilities all add up to 1 to do the inverse:

$$p = 1 - \int_{-Z}^{Z} \mathcal{N}(x, 1) dx$$

$$p = 1 - \int_{-Z}^{Z} \mathcal{N}(x, 1) dx$$

- This integral doesn't have an analytical solution, but we need it all the time, so it's results are readily available as the "error function"

```
// Z-score (sigmas) -> p-value
root [4] 1 - TMath::Erf(1.88 / TMath::Sqrt(2))
(Double_t) 0.0601081
```

$$\mu_1 = 2.5 \pm 0.1 \quad \mu_2 = 3.1 \pm 0.3$$

- With $\boldsymbol{p}$ **= 0.06**, we have failed to reject the null hypothesis at $\boldsymbol{\alpha}$ **= 0.05**.
- "These two means are consistent at the 95% level."
- Or, we might say:
  - "These means differ by 1.88σ" or
  - "They are consistent at the 94% level"

- Now, we need to choose a test statistic.
  - There are several choices for this problem.
  - Which one is the right one depends on the circumstance.

- A good first guess: try a chisquare (χ^2) test.
  - This is what the test statistic looks like:

$$T = \sum^{N} \frac{(O_i - E_i)^2}{\sigma^2}$$

  - Squared difference between the histograms, normalized by the expected uncertainty.

- Why this test statistic?
  - Let's see how it behaves assuming H_0.
  - The data is drawn randomly from the model, so each bin, O_i, should be drawn randomly from the model:

$$O_i \sim \mathcal{N}(E_i, \sigma)$$

- Given that, the argument in the sum of the chisquare should look familiar – it is a Z-score, squared.

$$T = \sum^{N} \frac{(O_i - E_i)^2}{\sigma^2} \qquad Z = \frac{X - \mu_0}{\sigma}$$