Dispersion relation for hadronic light-by-light scattering

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in collaboration with G. Colangelo, M. Hoferichter, and M. Procura

JHEP **04** (2017) 161, [arXiv:1702.07347 [hep-ph]]
arXiv:1701.06554 [hep-ph] (to appear in PRL)
JHEP **09** (2015) 074, [arXiv:1506.01386 [hep-ph]]
JHEP **09** (2014) 091, [arXiv:1402.7081 [hep-ph]]

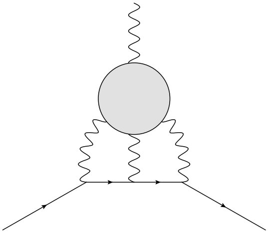
and with G. Colangelo, M. Hoferichter, B. Kubis, and M. Procura

Phys. Lett. **B738** (2014) 6, [arXiv:1408.2517 [hep-ph]]

June 5, 2017

First Workshop of the Muon $g-2$ Theory Initiative, St. Charles

Hadronic light-by-light (HLbL) scattering



- up to now only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- with recent progress on vacuum polarisation, HLbL starts to dominate theory error

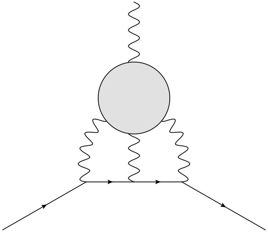
Model calculations of HLbL

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3
total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

→ Jegerlehner, Nyffeler (2009)

- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties

How to improve HLbL calculation?



- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities

The HLbL tensor: definitions

- hadronic four-point function:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$$
$$= -i \int dxdydz e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0 | T j^\mu_{\text{em}}(x) j^\nu_{\text{em}}(y) j^\lambda_{\text{em}}(z) j^\sigma_{\text{em}}(0) | 0 \rangle$$

- EM current:

$$j^\mu_{\text{em}} = \sum_{i=u,d,s} Q_i \bar{q}_i \gamma^\mu q_i$$

The HLbL tensor: definitions

- helicity amplitudes for the process

  $\gamma^*(q_1, \lambda_1)\gamma^*(q_2, \lambda_2) \to \gamma^*(-q_3, \lambda_3)\gamma(q_4, \lambda_4)$:

$$H_{\lambda_1\lambda_2\lambda_3\lambda_4} = \epsilon_\mu^{\lambda_1}\epsilon_\nu^{\lambda_2}\epsilon_\lambda^{\lambda_3 *}\epsilon_\sigma^{\lambda_4 *}\Pi^{\mu\nu\lambda\sigma}$$

- Mandelstam variables:

  $s = (q_1 + q_2)^2$, $t = (q_1 + q_3)^2$, $u = (q_2 + q_3)^2$

- for $(g-2)_\mu$, the external photon is on shell:

  $q_4^2 = 0$, where $q_4 = q_1 + q_2 + q_3$

The HLbL tensor

- a priori 138 ‘naive’ Lorentz structures:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,k,l,m} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \sum_{i,j} g^{\lambda\sigma} q_i^\mu q_j^\nu \Pi^5_{ij} + \dots$$

- in 4 space-time dimensions: 2 linear relations among the 138 Lorentz structures $\to$ Eichmann et al. (2014)
- six dynamical variables, e.g. two Mandelstam variables s, t and the photon virtualities q_1^2, q_2^2, q_3^2, q_4^2

HLbL tensor: gauge invariance

- Ward identities

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\}\Pi_{\mu\nu\lambda\sigma} = 0$$

  imply 95 linear relations between scalar functions Π_i

- off-shell basis: 138 – 95 – 2 = 41 structures
- corresponding to 41 helicity amplitudes
- relations between Π_i imply kinematic zeros

HLbL tensor: Lorentz decomposition

Solution for the Lorentz decomposition, following a recipe by Bardeen, Tung (1968) and Tarrach (1975):

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- crossing symmetry manifest: only 7 distinct structures, 47 follow from crossing
- scalar functions Π_i free of kinematic singularities $\Rightarrow$ ideal quantities for a dispersive treatment

Master formula: contribution to $(g-2)_\mu$

- from gauge invariance:

$$\Pi_{\mu\nu\lambda\rho} = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}$$

- for $(g-2)_\mu$: afterwards take $q_4 \to 0$
- no kinematic singularities in scalar functions: perform these steps with the derived Lorentz decomposition
- only 12 linear combinations of the scalar functions Π_i contribute to $(g-2)_\mu$

Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\text{HLbL}} = e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1,q_2;p)\hat{\Pi}_i(q_1,q_2,-q_1-q_2)}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known integration kernel functions
- five loop integrals can be performed with Gegenbauer polynomial techniques

  → Knecht, Nyffeler (2002); Jegerlehner, Nyffeler (2009), Bijnens, Zahiri-Abyaneh (2012); Bijnens, Relefors (2016)
- Wick rotation possible even in the presence of anomalous thresholds

Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

- T_i: known integration kernels
- $\bar{\Pi}_i$: linear combinations of the scalar functions Π_i
- Euclidean momenta: $Q_i^2 = -q_i^2$
- $Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$

Analytic properties of scalar functions

- right- and left-hand cuts in each Mandelstam variable
- double-spectral regions (box topologies)
- anomalous thresholds for large photon virtualities

Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \dots$$

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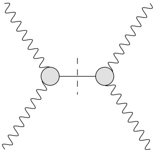
one-pion intermediate state:



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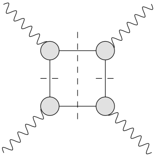
two-pion intermediate state in both channels:



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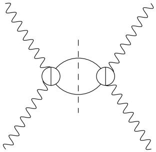
two-pion intermediate state in first channel:



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future work: higher intermediate states

Mandelstam representation

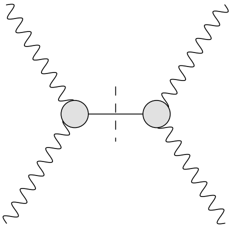
- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \dots$$

- the limit $q_4 \to 0$ for $(g-2)_\mu$ is taken in the end

Pion pole

→ talk by M. Hoferichter



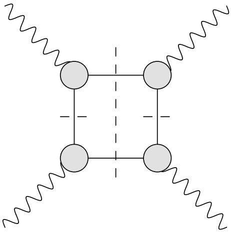
$$\Pi_1^{\pi^0\text{-pole}} = \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)}{s - M_\pi^2}$$

$\Pi_{2,3}^{\pi^0\text{-pole}}$ via crossing symmetry

- input: doubly-virtual and singly-virtual pion transition form factors $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- pion is on shell
- dispersive analysis of transition form factor:

  → Hoferichter et al., EPJC **74** (2014) 3180

Box contributions



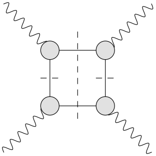
- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed

$$\Pi_i^{\pi\text{-box}} = \frac{1}{\pi^2}\int ds' dt' \frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

- q^2-dependence: pion vector form factors $F_\pi^V(q_i^2)$ for each off-shell photon factor out

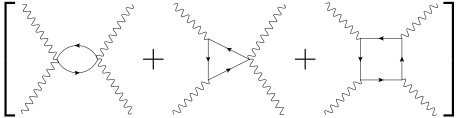
Box contributions

- sQED loop projected on BTT basis fulfils the same Mandelstam representation
- only difference are factors of F_π^V
- $\Rightarrow$ box topologies are identical to FsQED:



$$\equiv F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)$$

$\times$



- model-independent definition of pion loop

Box contributions

Very simple expressions for box contributions in terms of Feynman parameter integrals

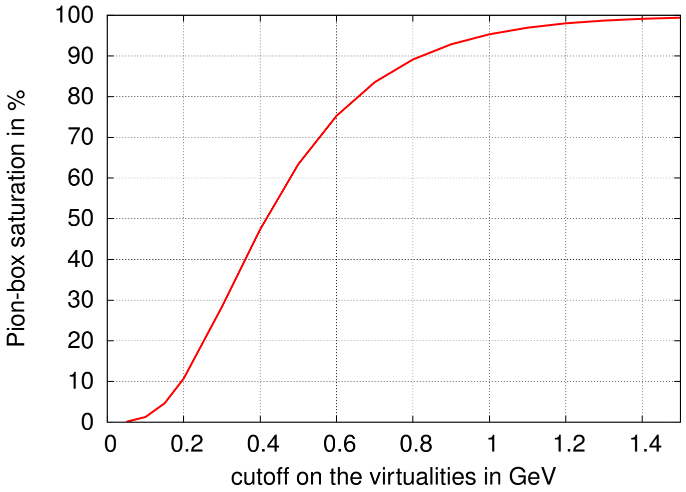
$$\Pi_i^{\pi\text{-box}}(q_1^2, q_2^2, q_3^2) = F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \times \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \, I_i(x,y),$$

with e.g.

$$I_7(x,y) = -\frac{4}{3} \frac{(1-2x)^2(1-2y)^2 y(1-y)}{\Delta_{123}^3},$$

$$\Delta_{ijk} = M_\pi^2 - xyq_i^2 - x(1-x-y)q_j^2 - y(1-x-y)q_k^2.$$

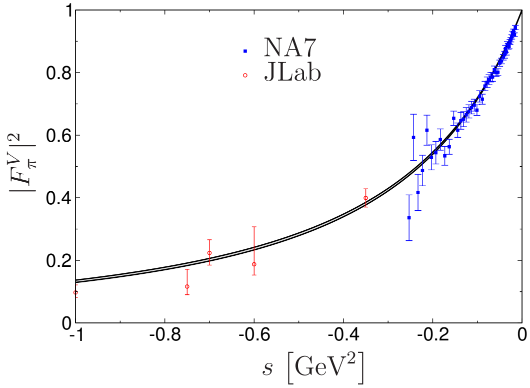
Pion-box saturation with photon virtualities



Box contributions

F_π^V: fit of dispersive representation to time- and space-like data

Result: $a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$



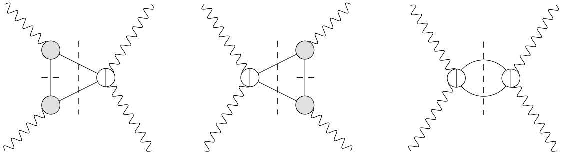
Helicity formalism and sum rules

- construction of singly-on-shell basis: unphysical helicity amplitudes drop out, 27 elements remain
- uniform asymptotic behaviour of the full tensor together with BTT tensor decomposition leads to 15 HLbL sum rules
- sum rules derived for general $(g-2)_\mu$ outer kinematics (not forward scattering $\rightarrow$ talk by I. Danilkin):

$$0 = \int ds' \mathrm{Im}\check{\Pi}_i(s') \Big|_{t=q_2^2, q_4^2=0}$$

- can be expressed in terms of helicity amplitudes

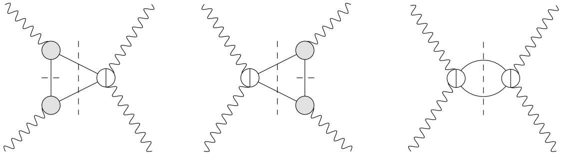
Rescattering contribution



- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel:

$$\Pi_i^{\pi\pi} = \frac{1}{2}\Bigg(\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{t'-t} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{u'-u}$$
$$+ \text{fixed-}t$$
$$+ \text{fixed-}u\Bigg)$$

Rescattering contribution



- unitarity gives imaginary parts in terms of helicity amplitudes for $\gamma^*\gamma^{(*)} \to \pi\pi$
- basis change to helicity amplitudes calculated
- expansion into partial waves
- framework valid for arbitrary partial waves
- resummation of PW expansion reproduces full result: checked for pion box

Convergence of partial-wave expansion

Relative deviation from full result: $1 - \frac{a_{\mu,J_\text{max}}^{\pi\text{-box, PW}}}{a_\mu^{\pi\text{-box}}}$

J_max	fixed-s	fixed-t	fixed-u	average
0	100.0%	-6.2%	-6.2%	29.2%
2	26.1%	-2.3%	7.3%	10.4%
4	10.8%	-1.5%	3.6%	4.3%
6	5.7%	-0.7%	2.1%	2.4%
8	3.5%	-0.4%	1.3%	1.5%
10	2.3%	-0.2%	0.9%	1.0%
12	1.7%	-0.1%	0.7%	0.7%
14	1.3%	-0.1%	0.5%	0.6%
16	1.0%	-0.0%	0.4%	0.4%

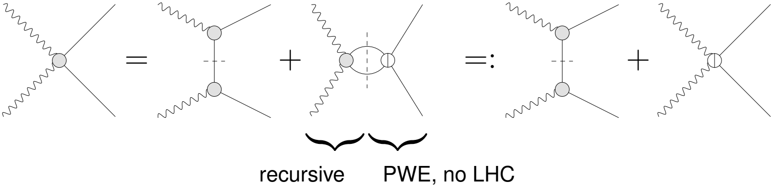
The subprocess

Helicity amplitudes for $\gamma^*\gamma^* \to \pi\pi$: dispersive solution of the S-wave unitarity relation with Omnès methods

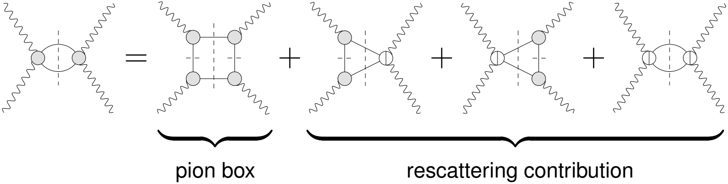
- pion-pole approximation to left-hand cut
  $\Rightarrow q^2$-dependence again given by F_π^V
- phase shifts based on modified inverse-amplitude method
- low-energy properties accurately reproduced, including $f_0(500)$ parameters
- fully consistent with $\pi^\pm$ polarisabilities
- result for S-waves: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

Topologies in the rescattering contribution

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



Two-pion contributions to HLbL:



Results for two-pion contributions

Pion-box contribution:

$$a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$$

S-wave rescattering contribution:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

Summary

- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states: π^0-pole and $\pi\pi$-cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- precise numerical evaluation of two-pion contributions
- a step towards a model-independent calculation of a_μ

Outlook

- higher pseudoscalar poles can be directly included

  $\rightarrow$ talk by B. Kubis
- two-particle intermediate states:
  - include kaons in a coupled-channel system
  - numerics for D-waves
  - generalisation to heavier left-hand cuts
- higher intermediate states in direct channel
  - framework needs to be extended
  - e.g. $3\pi \Rightarrow$ axials
- match the total to OPE/pQCD constraints

Backup

HLbL tensor: BTT Lorentz decomposition

Problem: find a decomposition

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

with the following properties:

- Lorentz structures $T_i^{\mu\nu\lambda\sigma}$ manifestly gauge invariant:

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} T^i_{\mu\nu\lambda\sigma} = 0$$

- scalar functions Π_i free of kinematic singularities and zeros

HLbL tensor: BTT Lorentz decomposition

Recipe by Bardeen, Tung (1968) and Tarrach (1975):

- construct gauge projectors:

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

- gauge invariant themselves, e.g.

$$q_1^\mu I_{\mu\nu}^{12} = 0$$

- leave HLbL tensor invariant, e.g.

$$I_{12}^{\mu\mu'} \Pi_{\mu'\nu\lambda\sigma} = \Pi^\mu{}_{\nu\lambda\sigma}$$

HLbL tensor: BTT Lorentz decomposition

Following Bardeen, Tung (1968):

- apply gauge projectors to the 138 initial structures: 95 immediately project to 0
- remove $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles by taking appropriate linear combinations
- BT basis: degenerate in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$

HLbL tensor: BTT Lorentz decomposition

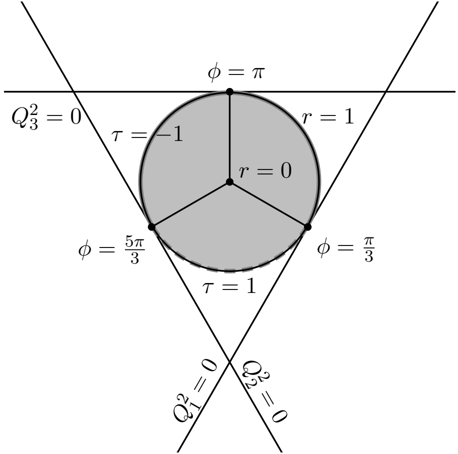
According to Tarrach (1975):

- degeneracies in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$:

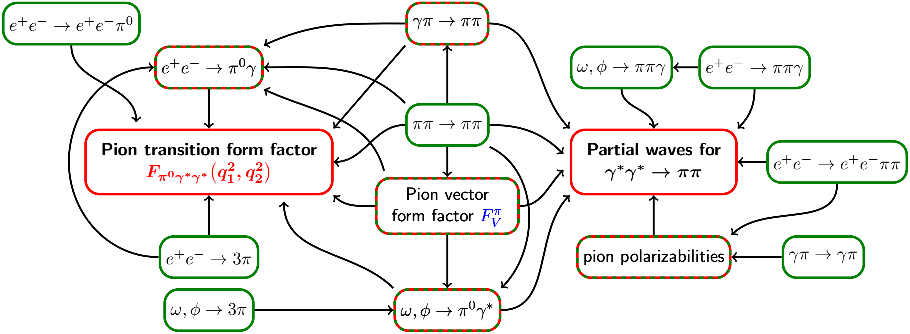
$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

- extend basis by additional structures $X_i^{\mu\nu\lambda\sigma}$, $Y_i^{\mu\nu\lambda\sigma}$ taking care of remaining kinematic singularities
- equivalent: implementing crossing symmetry

$(g-2)_\mu$ integration region in polar coordinates



A roadmap for HLbL



→ flowchart by M. Hoferichter