Interplay between R-ratio and Lattice for $(g-2)_\mu$ HVP

Tom Blum (UCONN/RBRC), Taku Izubuchi (BNL/RBRC), Christoph Lehner (BNL)

for the RBC/UKQCD collaborations

RBC/UKQCD Collaboration

Thomas Blum(UConn/RBRC), Peter Boyle (Edinburgh),

Norman Christ (Columbia), Vera Guelpers (Southampton),

Masashi Hayakawa (Nagoya), James Harrison (Southampton),

Chulwoo Jung (BNL), Andreas Jüttner (Southampton),

Luchang Jin (Columbia), Christoph Lehner (BNL),

Antonin Portelli (Edinburgh), Matt Spraggs (Southampton)

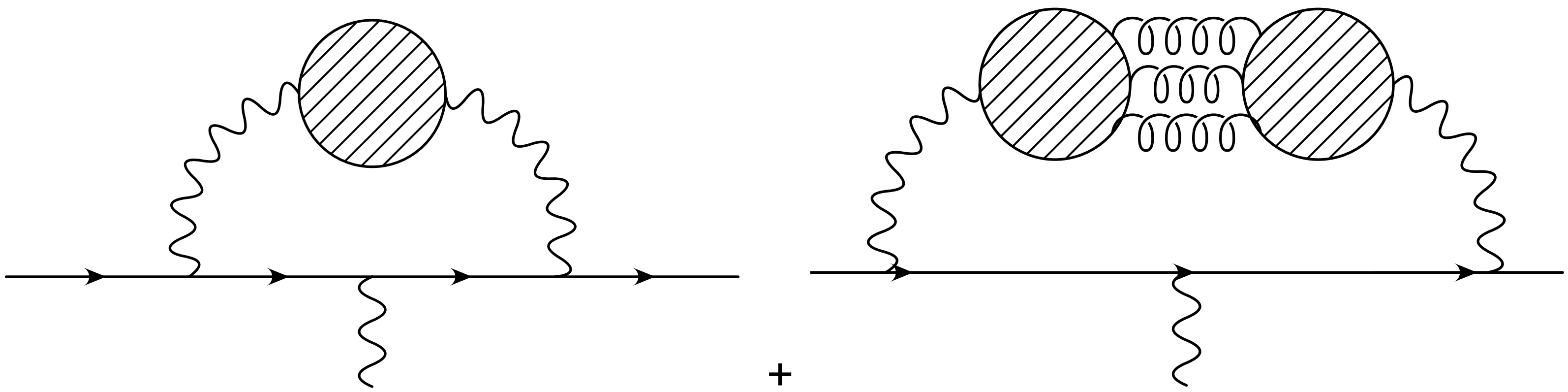
...

- Introduction
- Interplay between Lattice and R ratio for g-2 HVP
- (τ inclusive decay application)

Introduction

- How to compare and check a_μ^{HVP} from lattice QCD and R-ratio analysis ?
- How to *safely* maximize precision of a_μ^{HVP} from LQCD and R-Ratio (including pQCD, isospin corrected τ decay, etc) studies.
- Simplest way : compare two numbers, $a_\mu^{\text{HVP,LQCD}}$ and $a_\mu^{\text{HVP,R-ratio}}$, and average with the error weight.
- We could find more convinient rendez-vouz point ?

Lattice QCD method [Blum 2003, Lautrup et al. 1971]



Using lattice QCD and continuum, ∞-volume photon and lepton

$$a_\mu(HVP) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2\, f_{\mathsf{QED}}(q^2)\, \hat{\Pi}(q^2)$$

$f_{\mathsf{QED}}(q^2)$ is known, $\hat{\Pi}(q^2)$ is subtracted HVP, $\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$, computed non-perturbatively Euclidean space-time lattice

$$\begin{aligned} \Pi^{\mu\nu}(q) &= \int e^{iqx} \langle j^\mu(x) j^\nu(0) \rangle \qquad j^\mu(x) = \sum_i Q_i \bar{\psi}(x) \gamma^\mu \psi(x) \\ &= \Pi(q^2)(q^\mu q^\nu - q^2 \delta^{\mu\nu}) \end{aligned}$$

Euclidean Time Momentum Representation

[Bernecker Meyer 2011, Feng et al. 2013]

In Euclidean space-time, project verctor 2 pt to zero spacial momentum, $\vec{p} = 0$:

$$C(t) = \frac{1}{3} \sum_{x,i} \langle j_i(x) j_i(0) \rangle$$

g-2 HVP contribution is

$$a_\mu^{HVP} = \sum_t w(t) C(t)$$

$$w(t) = 2 \int_0^\infty \frac{d\omega}{\omega} f_{\mathrm{QED}}(\omega^2) \left[\frac{\cos \omega t - 1}{\omega^2} + \frac{t^2}{2} \right]$$

- Subtraction $\Pi(0)$ is performed.
  Noise/Signal $\sim e^{(E_{\pi\pi} - m_\pi)t}$, is improved [Lehner et al. 2015] .
- Corresponding $\hat{\Pi}(Q^2)$ has exponentially small volume error [Portelli et al. 2016] . $w(t)$ includes the continuum QED part of the diagram

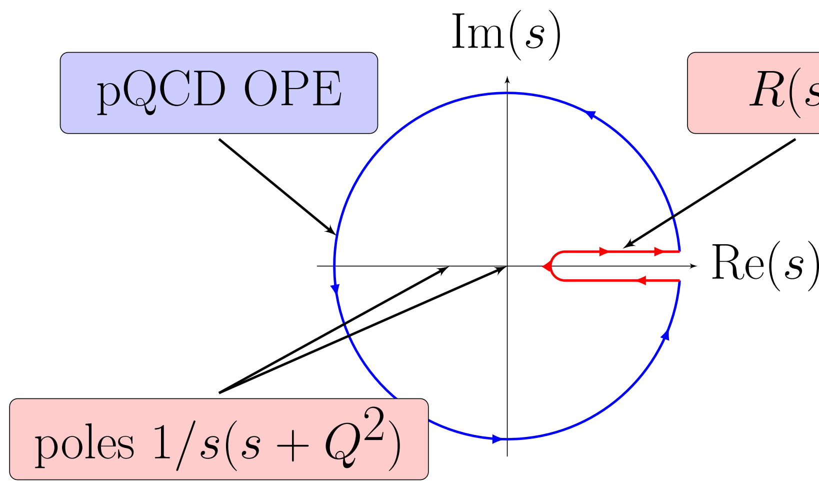
Euclidean time correlation from e^+e^- $R(s)$ data

From e^+e^- $R(s)$ ratio, using disparsive relation, zero-spacial momentum projected Euclidean correlation function $C(t)$ is obtained

$$\hat{\Pi}(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$$

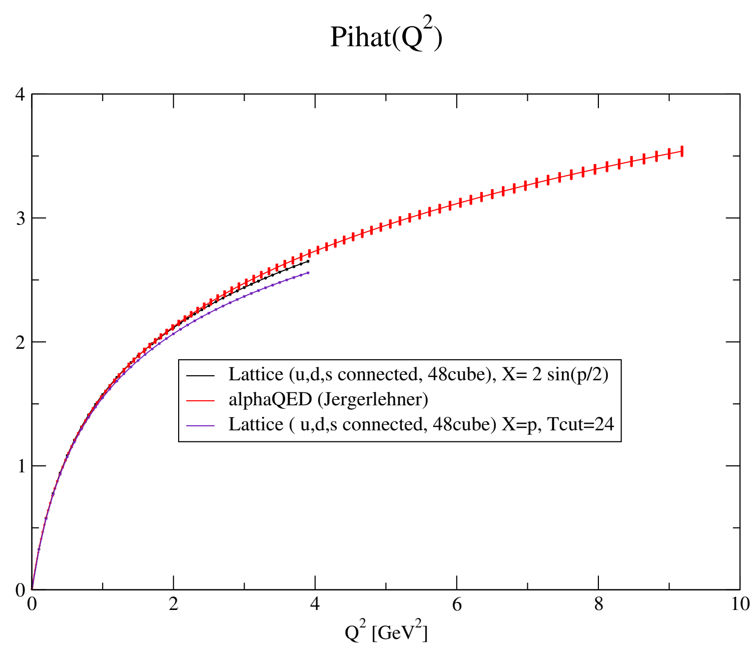
$$C^{\text{R-ratio}}(t) = \frac{1}{12\pi^2} \int_0^\infty \frac{d\omega}{2\pi} \hat{\Pi}(\omega^2) = \frac{1}{12\pi^2} \int_0^\infty ds\, \sqrt{s} R(s) e^{-\sqrt{s}t}$$

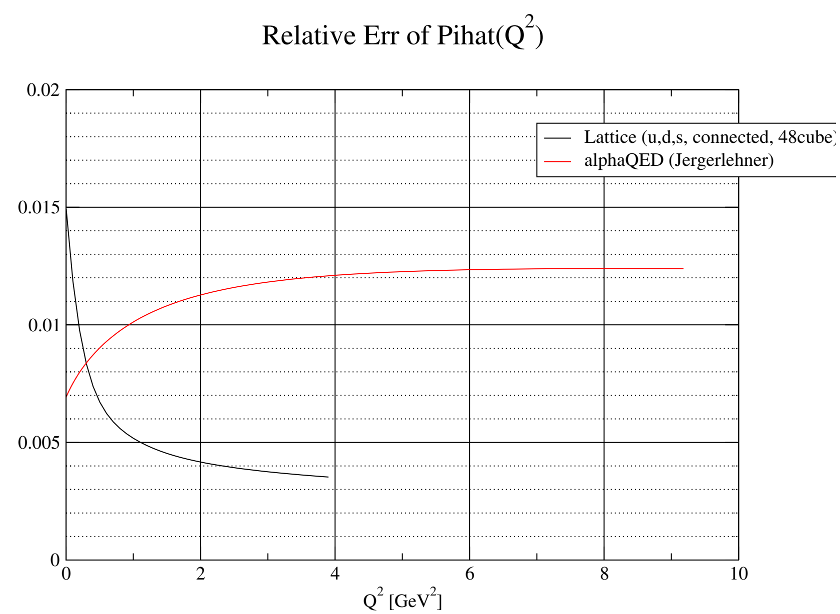
- $C(t)$ or $w(t)C(t)$ are directly comparable to Lattice results with the proper limits ($m_q \to m_q^{\text{phys}}, a \to 0, V \to \infty$, QED ...)
- Lattice: long distance has large statistical noise, (short distance: discretization error, removed by $a \to 0$ and/or pQCD)
- R-ratio : short distance has larger error



$\hat{\Pi}(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$

($1/a = 1.78$ **GeV,** **Relative statistical error)**





Simulation details [RBC/UKQCD 2015]

two gauge field ensembles generated by RBC/UKQCD collaborations

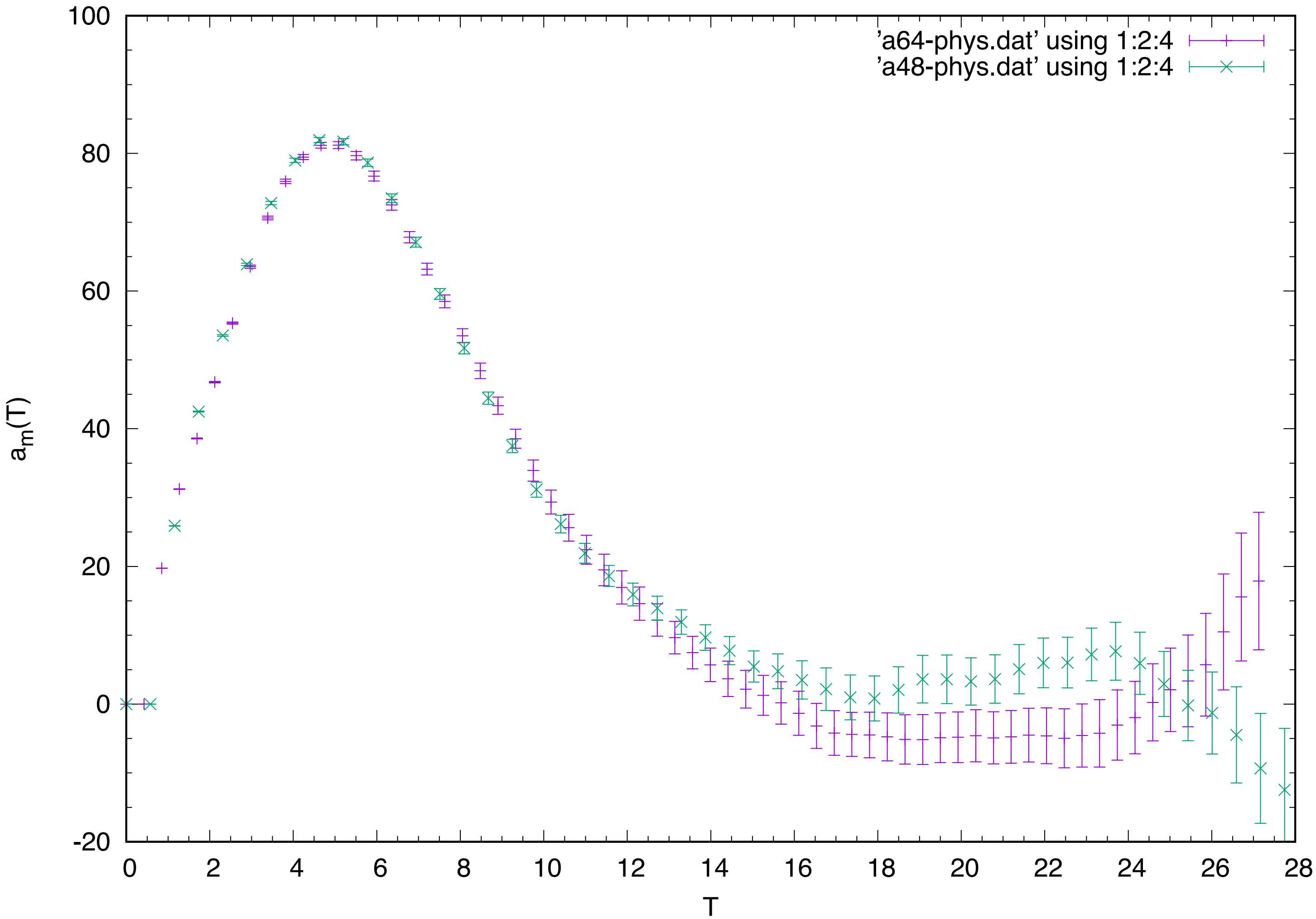
Domain wall fermions: chiral symmetry at finite a

Iwasaki Gauge action (gluons)

- pion mass $m_\pi = 139.2(2)$ and 139.3(3) MeV ($m_\pi L \lesssim 4$)
- lattice spacings $a = 0.114$ and 0.086 fm
- lattice scale $a^{-1} = 1.730$ and 2.359 GeV
- lattice size $L/a = 48$ and 64
- lattice volume $(5.476)^3$ and $(5.354)^3$ fm^3

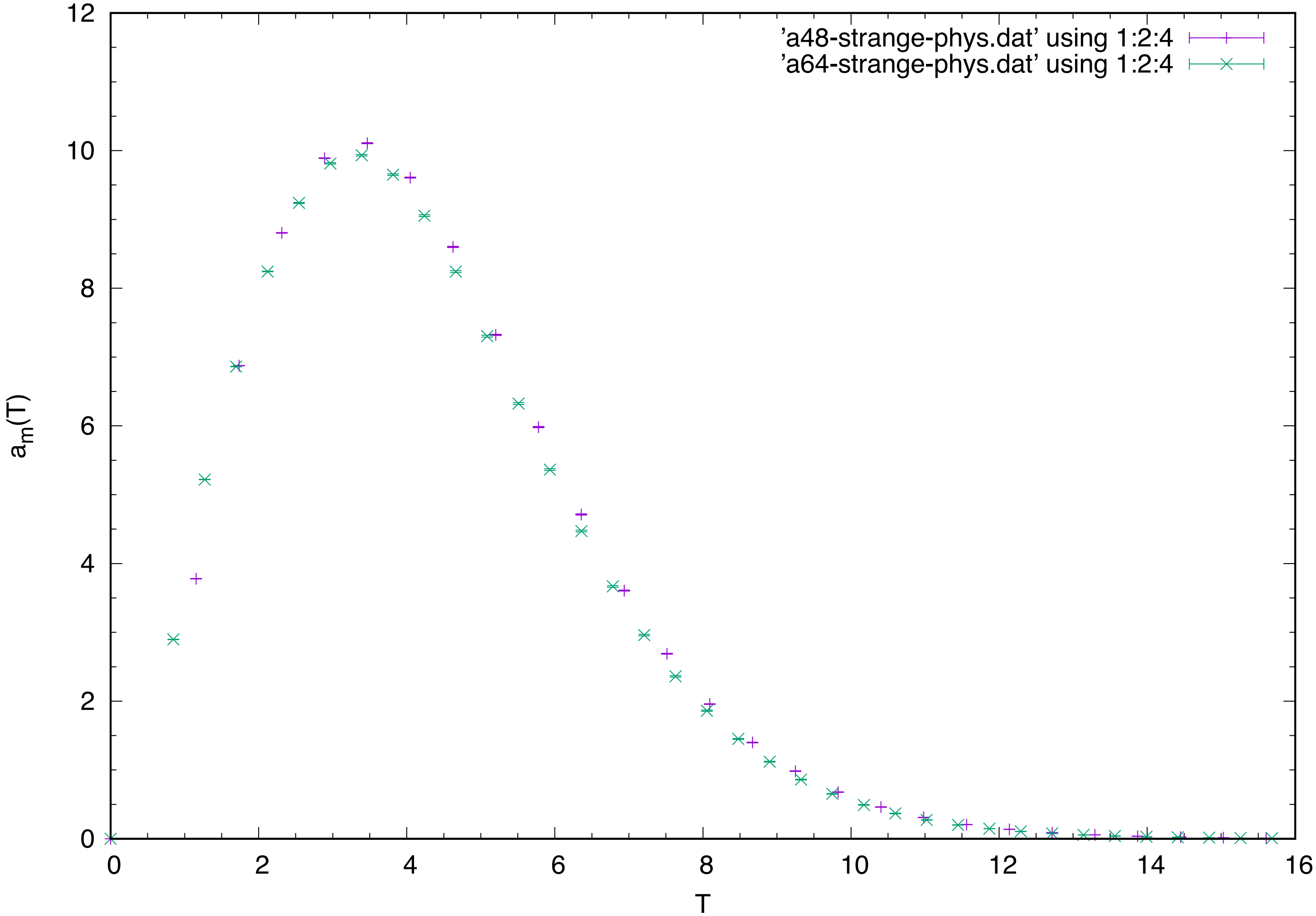
Use all-mode-average (AMA) [Blum et al 2012] and low-mode- averaging (LMA) [Giusti et al, 2004, Degrand et al 2005, Lehner 2016 for HVP] techniques for improved statistics by more than three orders of magnitudes compared to basic CG, and $\times 10$ smaller memory via multigrid-Lanczos [Lehner 2017] .

a_μ integrand $w(t)C(t)$



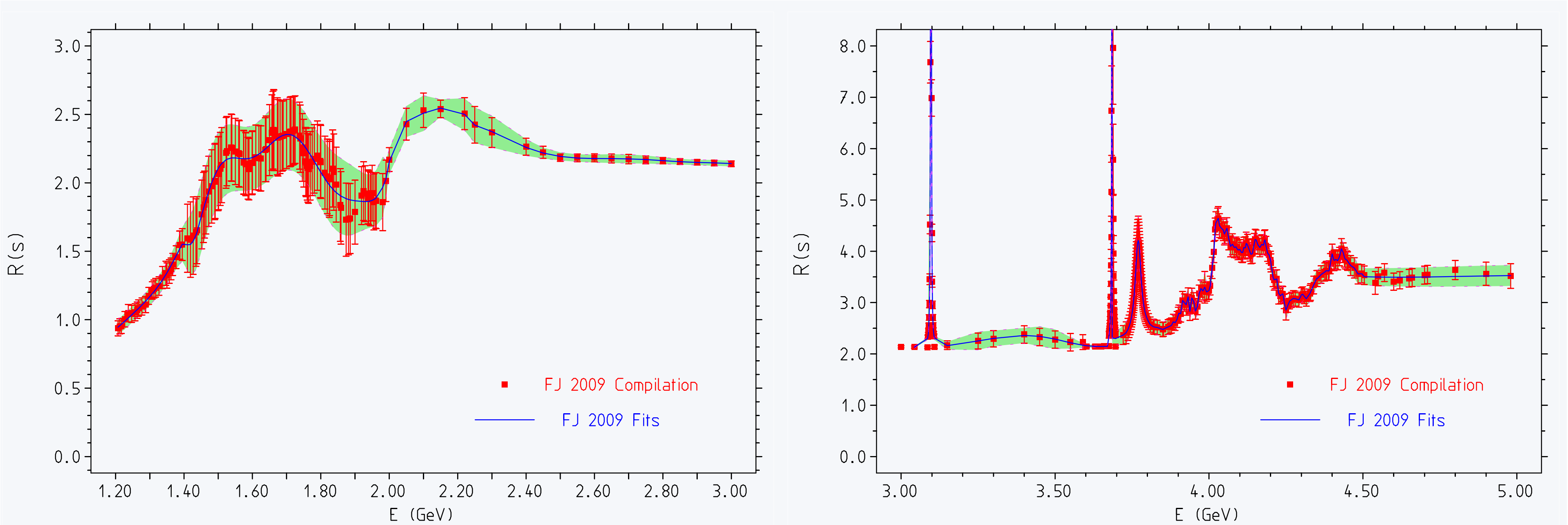
light quark (connected) contribution, RBC/UKQCD 48^3, 64^3 ensembles
Expotentially growing noise in long-distance region $\propto e^{(E_{\pi\pi}-m_\pi)t}$

strange quark contribution, [M. Spraggs et al 2016]



(RBC/UKQCD 48^3, 64^3 ensembles)

Comparison to R-ratio [Bernecker Meyer 2011]



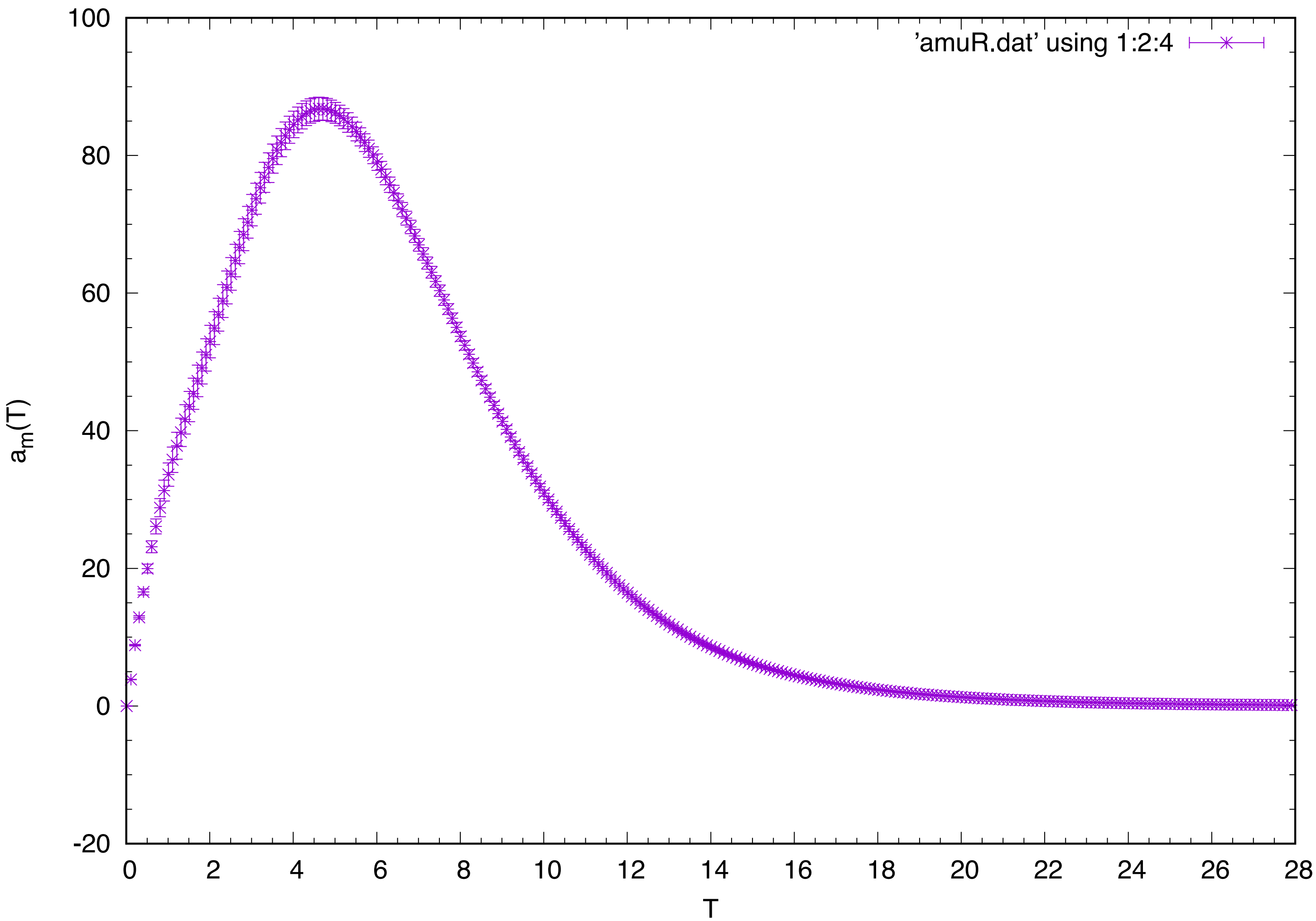
R(s) data and fit (piecewise chebysev polys) from F. Jegerlehner

We use a range of $\sqrt{s}$ of 2 m_π to 40 GeV

(χ PT used for low and pQCD for high $\sqrt{s}$ regions)

Don't have correlations between data points, covariant matrix from KNT17 will be very useful [A. Keshavarzi's talk]

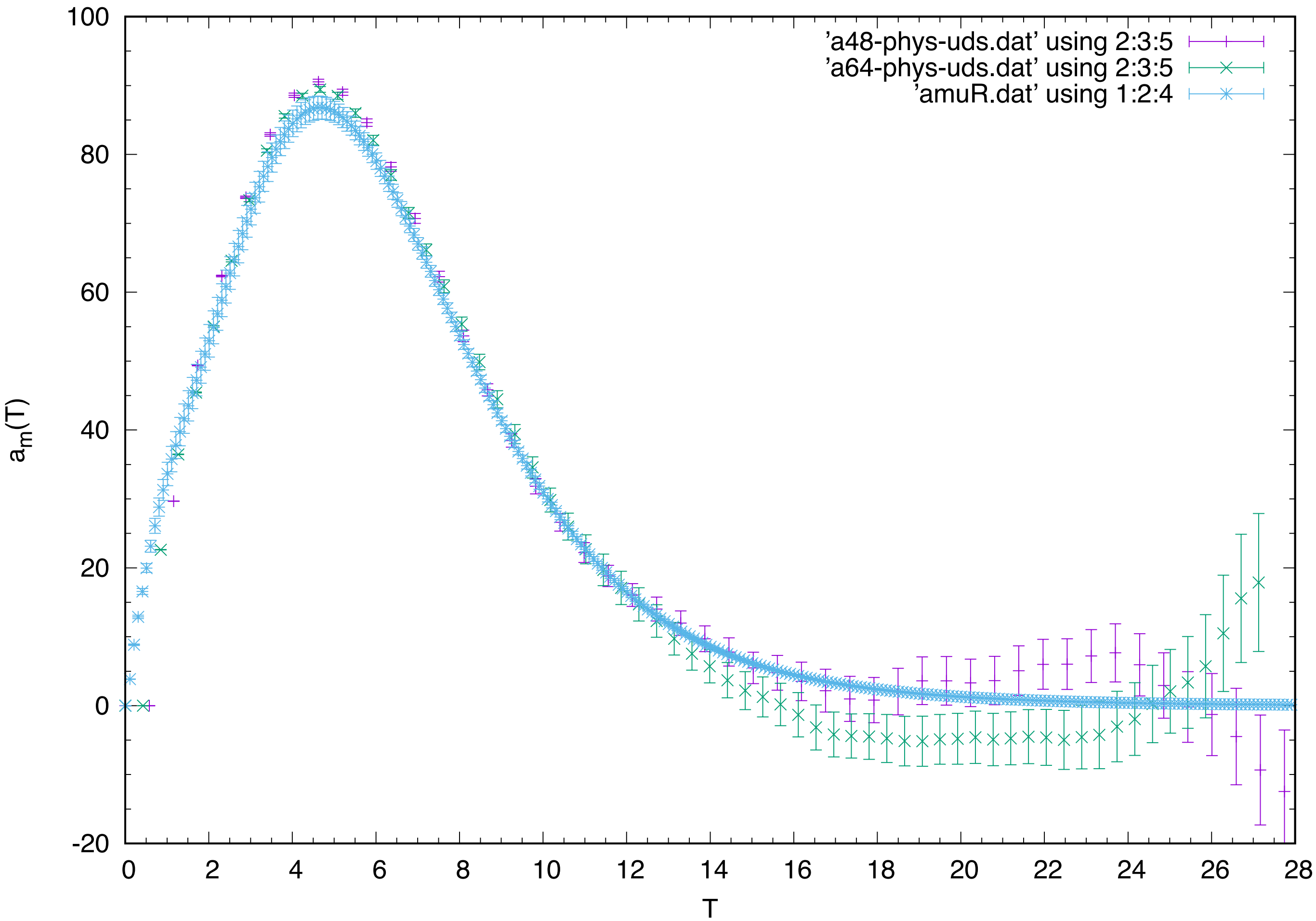
Comparison to R-ratio [?]



(Using R(s) from F. Jegerlehner's AphaQED 2012)

Comparison to R-ratio

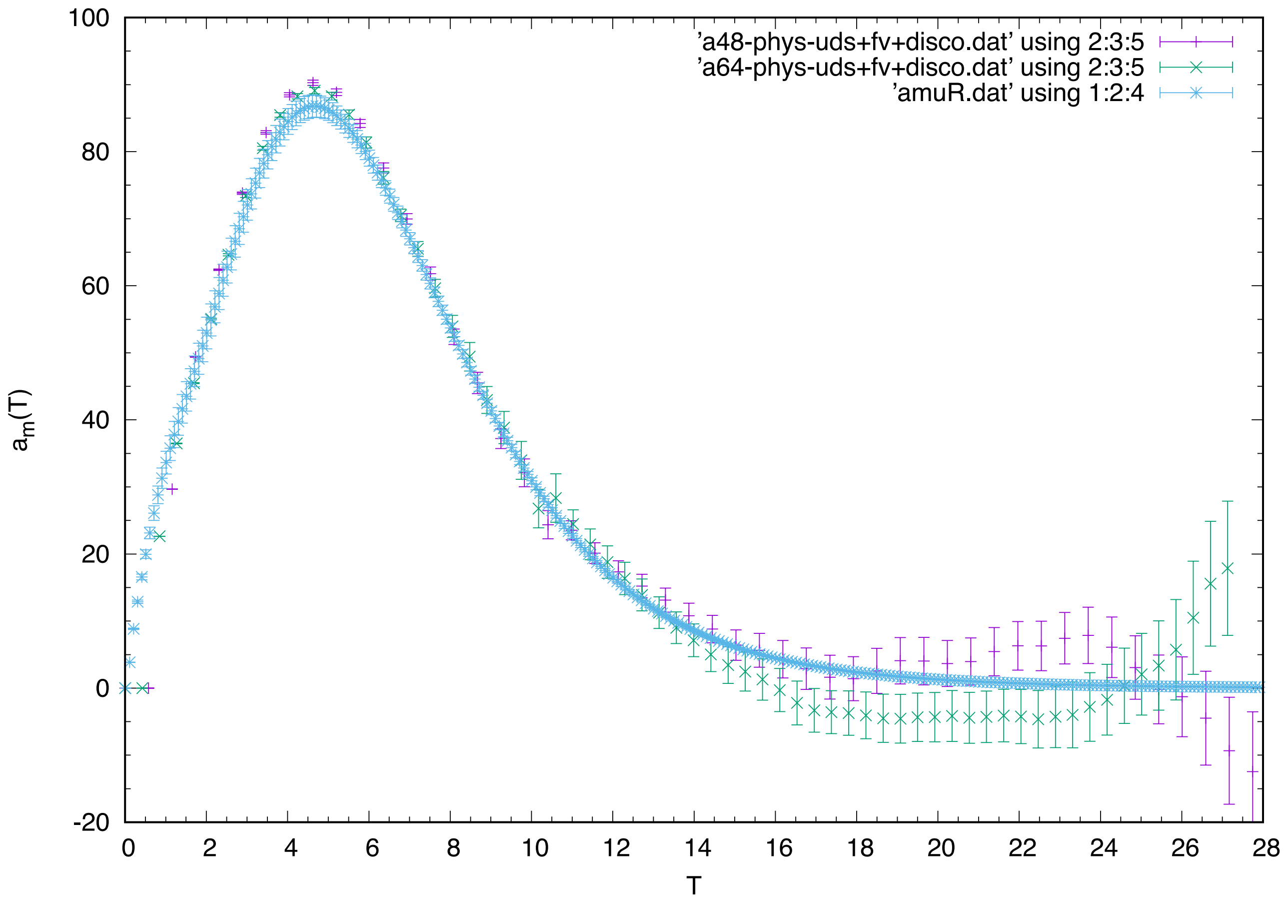
$u + d + s$



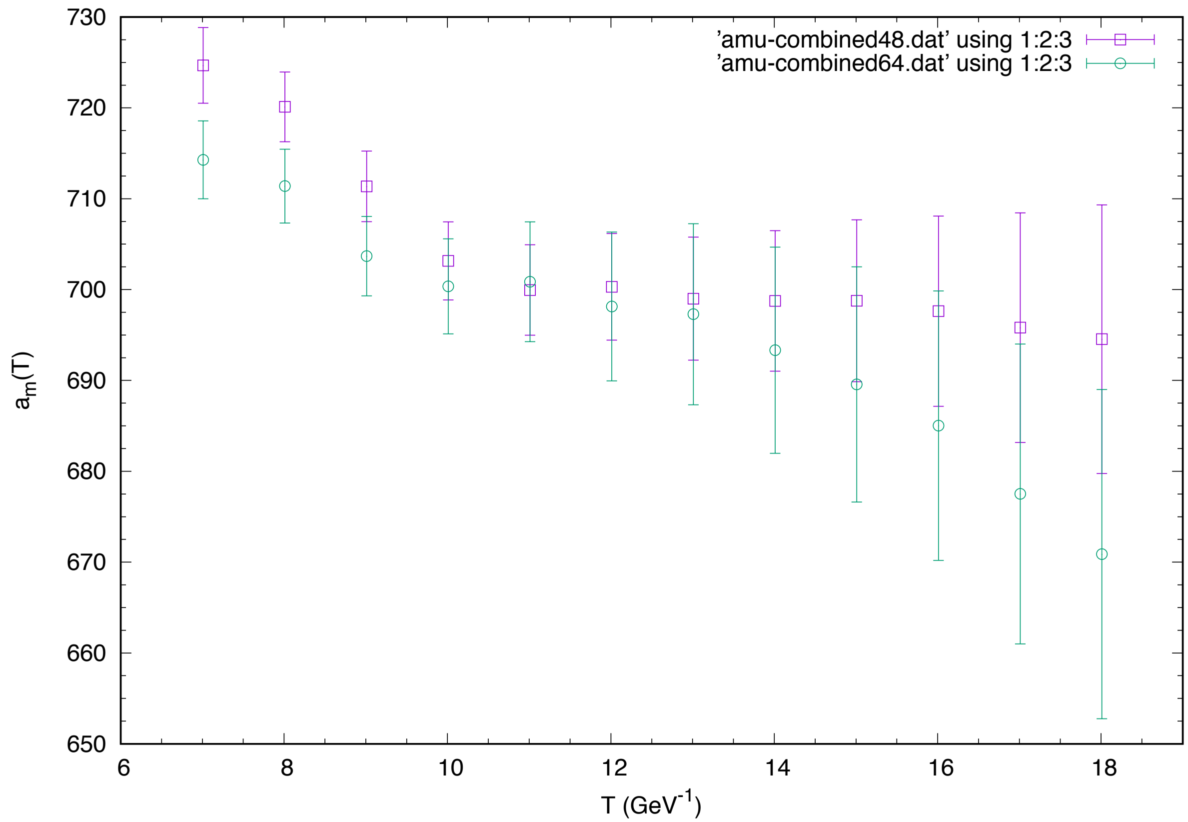
Lattice more precise at small T, R ratio at high

Comparison to R-ratio

$u+d+s$+ ChPT LO $\pi\pi$ FV correction + disconnected (48^3)(C. Lehner) [Blum et al 2015]



Combined lattice + R(s) result (no lattice charm)

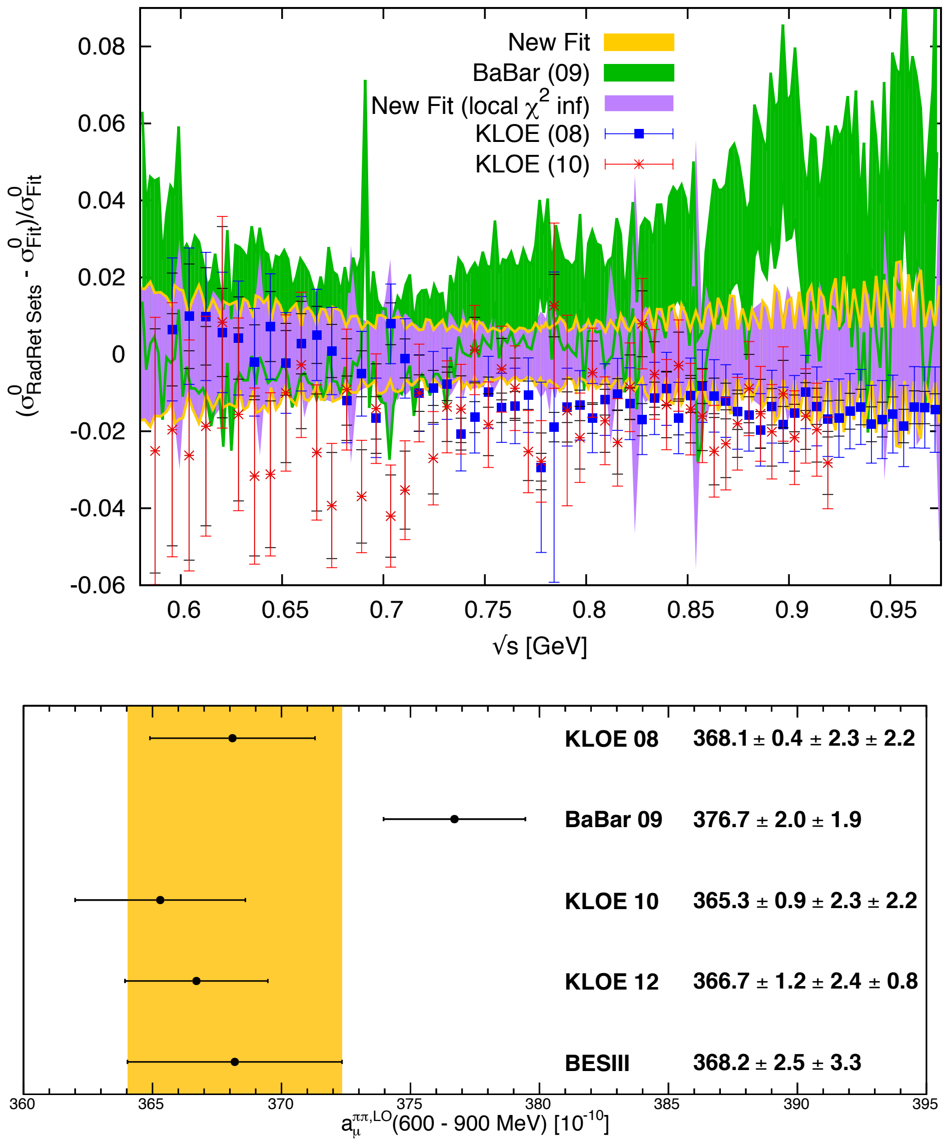


$$a_\mu^{HVP}(T) = \sum_{t=0}^{T} w(t) C^{\text{Lat}}(t) + \int_T^\infty dt w(t) C(t)$$

Errors range from $\sim$ 0.5 to 1.2 % for $T \lesssim$ 11 [GeV^{-1}] = 2.2 [fm]

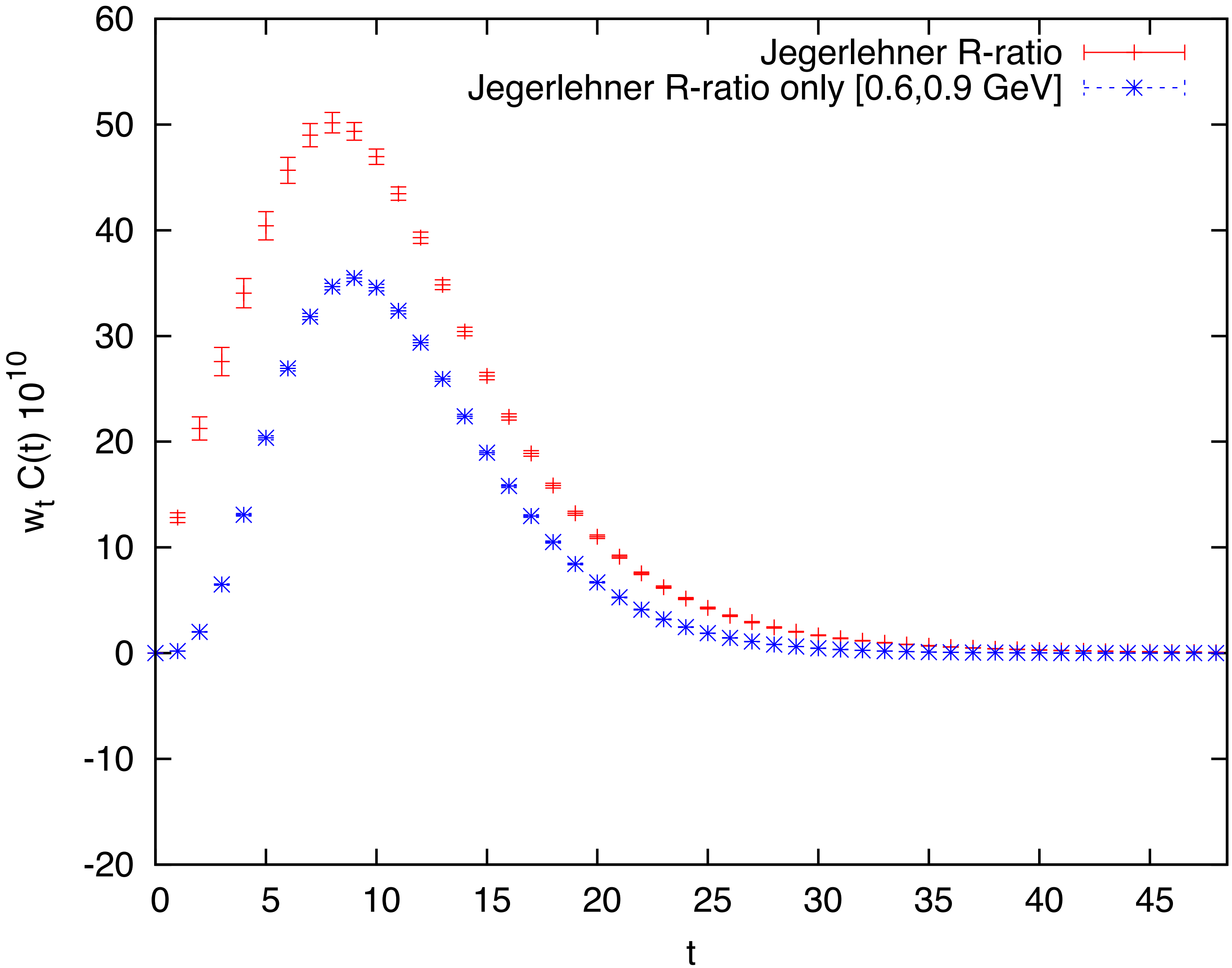
Comparison with R(s) of certain range

Near ρ peak, KLOE and Babar disagree



(Hagiwara, $et.al$) BESIII

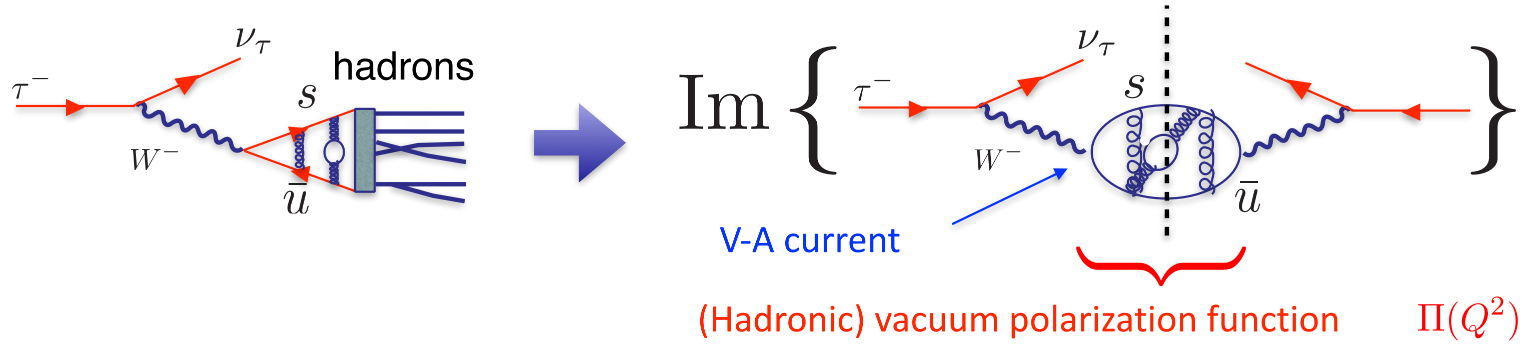
Careful comparison of R-ratio with lattice results may help



Summary and Discusssion for $g-2$ HVP

- a_μ^{HVP} purely from the first principle Lattice QCD(+QED) with only α_s, m_q as input number is certainly desireble.
- LQCD fighting with errors
  - Statistics : AMA, LMA, multigrid-Lanczos
  - Finite volume : Larger volume, or pion form factors(PACS (8.5 fm)^4 and (11 fm)^4)
  - Discretization error : a^2 scaling
  - Isospin corrections : Lattice QCD(+QED), $m_u \neq m_d$ corrections
- Alternatively, lattice and R-ratio could be combined in some forms to cross-check and may produce better results than either.
- Exploring one Rendez-Vouz point between Lattice and R-ratio : Euclidean time correlation function $C(t)$ or $w(t)C(t)$

Tau decay application [H. Ohki, K. Maltman, A. Portelli TI et al.]



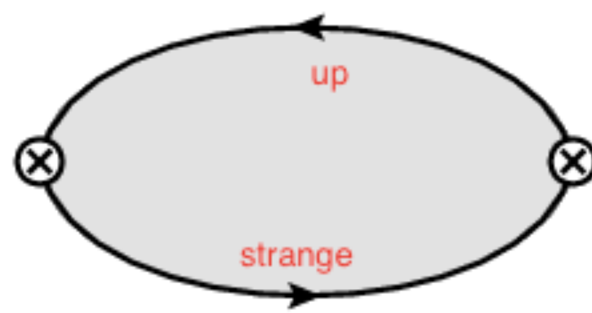
- $\tau \to \nu + had$ through V-A vertex. EW correction S_{EW} [Marciano, Sirlin]

$$R_{ij} = \frac{\Gamma(\tau^- \to \text{hadrons}_{ij}\ \nu_\tau)}{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}$$

$$= \frac{12\pi |V_{ij}|^2 S_{EW}}{m_\tau^2} \int_0^{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \underbrace{\left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im}\Pi^{(1)}(s) + \text{Im}\Pi^{(0)}(s)\right]}_{\equiv\ \text{Im}\,\Pi(s)}$$

- The Spin=0 and 1, vacuum polarization, Vector(V) or Axial (A) current-current two point

$$\Pi^{\mu\nu}_{ij;V/A}(q^2) = i \int d^4x e^{iqx} \left\langle 0 | T J^\mu_{ij;V/A}(x) J^{\dagger\mu}_{ij;V/A}(0) | 0 \right\rangle$$
$$= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}_{ij;V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij;V/A}$$



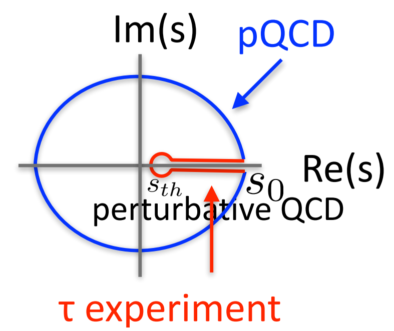
Finite Energy Sum Rule (FESR)

[Shifman, Vainstein, Zakharov 1979]

- Optical theorem relate S=-1 *diff.* crossection and HVP for given quantum number: flavor (us or ud), spin (0 or 1), parity (V or A)

- Do *finite* radius contour integral for arbitrary regular weight function $w(s)$

$$\int_{s_{th}}^{s_0} \mathrm{Im}\Pi(s)w(s) = -\frac{i}{2\pi}\oint_{|s|=s_0} ds\Pi(s)w(s)$$



- Real axis integral from experimental spectral function $\mathrm{Im}\Pi(s) = \rho(s)$ extracted in terms of dR_τ/ds $y_\tau = s/m_\tau^2$, $\omega_\tau(y) = (1-y)^2(1+2y)$, $\omega_L(y) = 2y(1-y)^2$. S_{EW} from EW correction [Marciano,Sirlin] .

$$\frac{dR_{ij;V/A}}{ds} = \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_\tau^2} \times \left[\omega_\tau(s)\rho^{(0+1)}_{ij;V/A}(s) - \omega_L(y_\tau)\rho^{(0)}_{ij;V/A}(s)\right], \quad (1)$$

- Use pQCD with OPE for the large circle integral (duality violation error)

V_{us} extraction from FESR

[E. Gamiz et al. 2005, 2002]

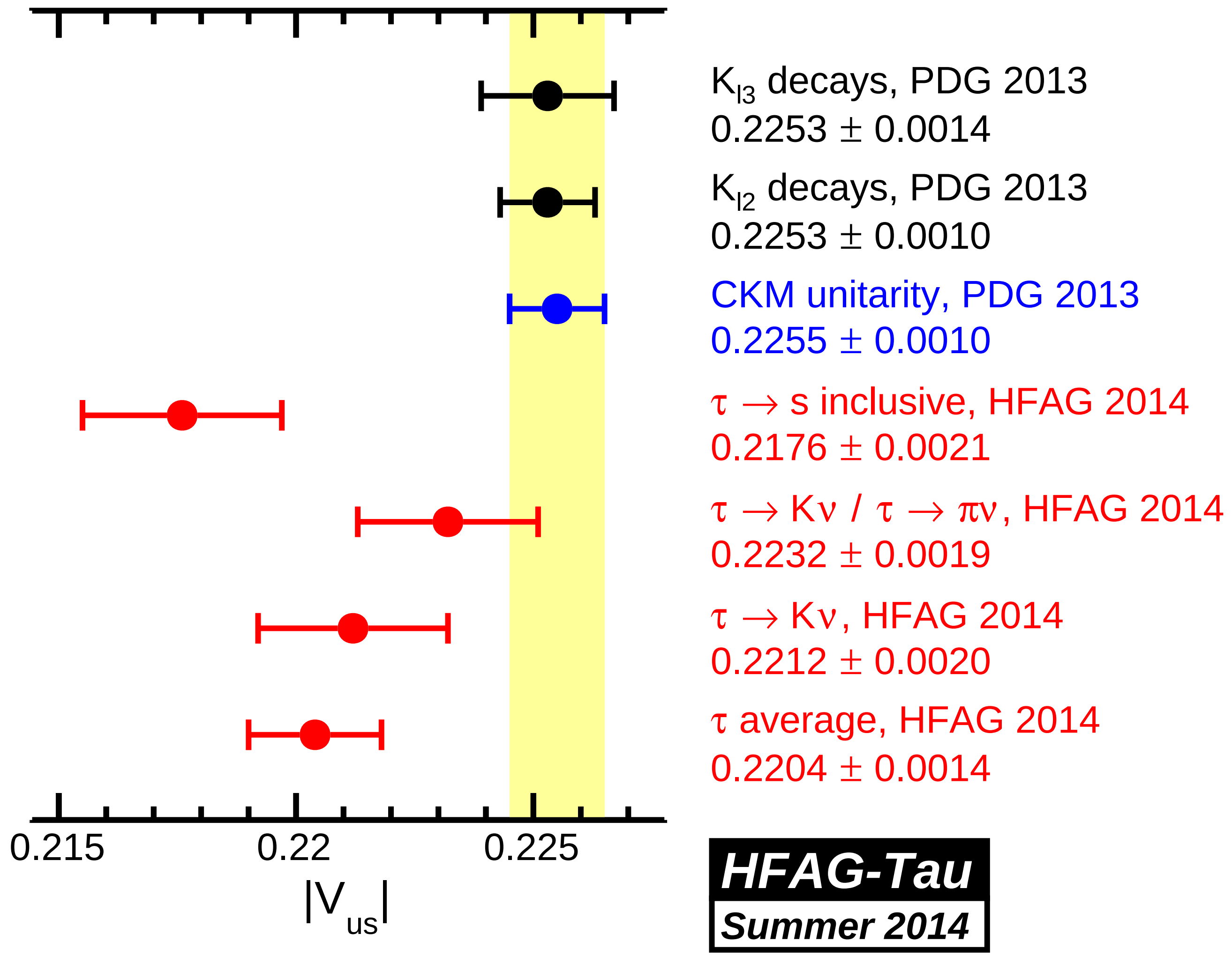
- extract V_{us} (and/or α_s, m_s) by FESR
- Use both, ud and us hadron final states and take their ratio
- Use OPE Vaccum Saturation assumption estimate for higher dimension
- 3+σ lower V_{us} from Kl3, Kl2 determinations

$$|V_{us}| = 0.2238(5)_{\text{exp}}(9)_{\text{lat}} \quad K_{l3} \text{ FNAL/MILC 2015}$$
$$|V_{us}| = 0.2233(5)_{\text{exp}}(9)_{\text{lat}} \quad K_{l3} \text{ RBC/UKQCD 2015}$$
$$|V_{us}| = 0.2250(4)_{\text{exp}}(9)_{\text{lat}} \quad K_{l2}/\pi_{l2} \text{ FLAG}$$

- Using τ-inclusive decay relation $[-3.4\sigma]$ from CKM Unitarity, or Kl3,Kl2

$$|V_{us}| = 0.2176(22) \quad \text{HFAG 2014}$$

- Trucation error of pQCD and Higher order contributions of OPE (Duality violation) ?



0.215
0.22
0.225
$|V_{us}|$
K_{l3} decays, PDG 2013
0.2253 ± 0.0014
K_{l2} decays, PDG 2013
0.2253 ± 0.0010
CKM unitarity, PDG 2013
0.2255 ± 0.0010
$\tau \to s$ inclusive, HFAG 2014
0.2176 ± 0.0021
$\tau \to K\nu$ / $\tau \to \pi\nu$, HFAG 2014
0.2232 ± 0.0019
$\tau \to K\nu$, HFAG 2014
0.2212 ± 0.0020
τ average, HFAG 2014
0.2204 ± 0.0014
HFAG-Tau
Summer 2014

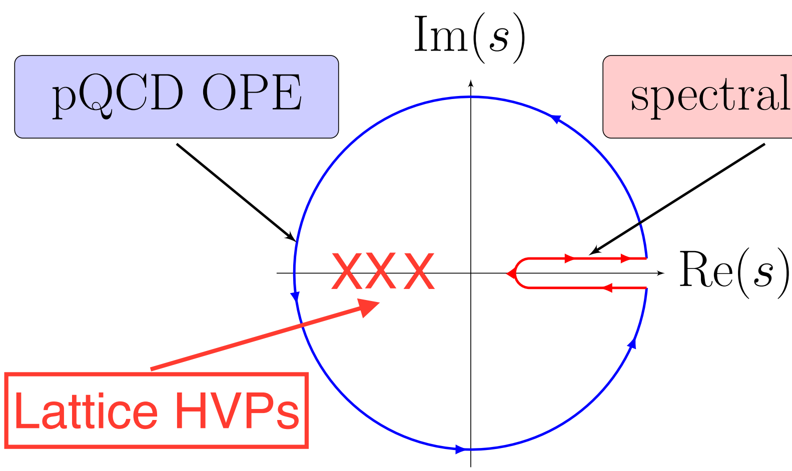
Our new method : Combining FESR and Lattice

- If we have a reliable estimate for $\Pi(s)$ in Euclidean (space-like) points, $s = -Q_k^2 < 0$, we could extend the FESR with weight function $w(s)$ to have poles there,

$$\int_{s_{th}}^{\infty} w(s)\mathrm{Im}\Pi(s) = \pi \sum_{k}^{N_p} \mathrm{Res}_k[w(s)\Pi(s)]_{s=-Q_k^2}$$

$$\Pi(s) = \left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s) \propto s \quad (|s| \to \infty)$$

- For $N_p \geq 3$, the $|s| \to \infty$ circle integral vanishes.



weight function $w(s)$

- Example of weight function

$$w(s) = \prod_k^{N_p} \frac{1}{(s+Q_k^2)} = \sum_k a_k \frac{1}{s+Q_k^2}, \quad a_k = \sum_{j\neq k} \frac{1}{Q_k^2 - Q_j^2}$$

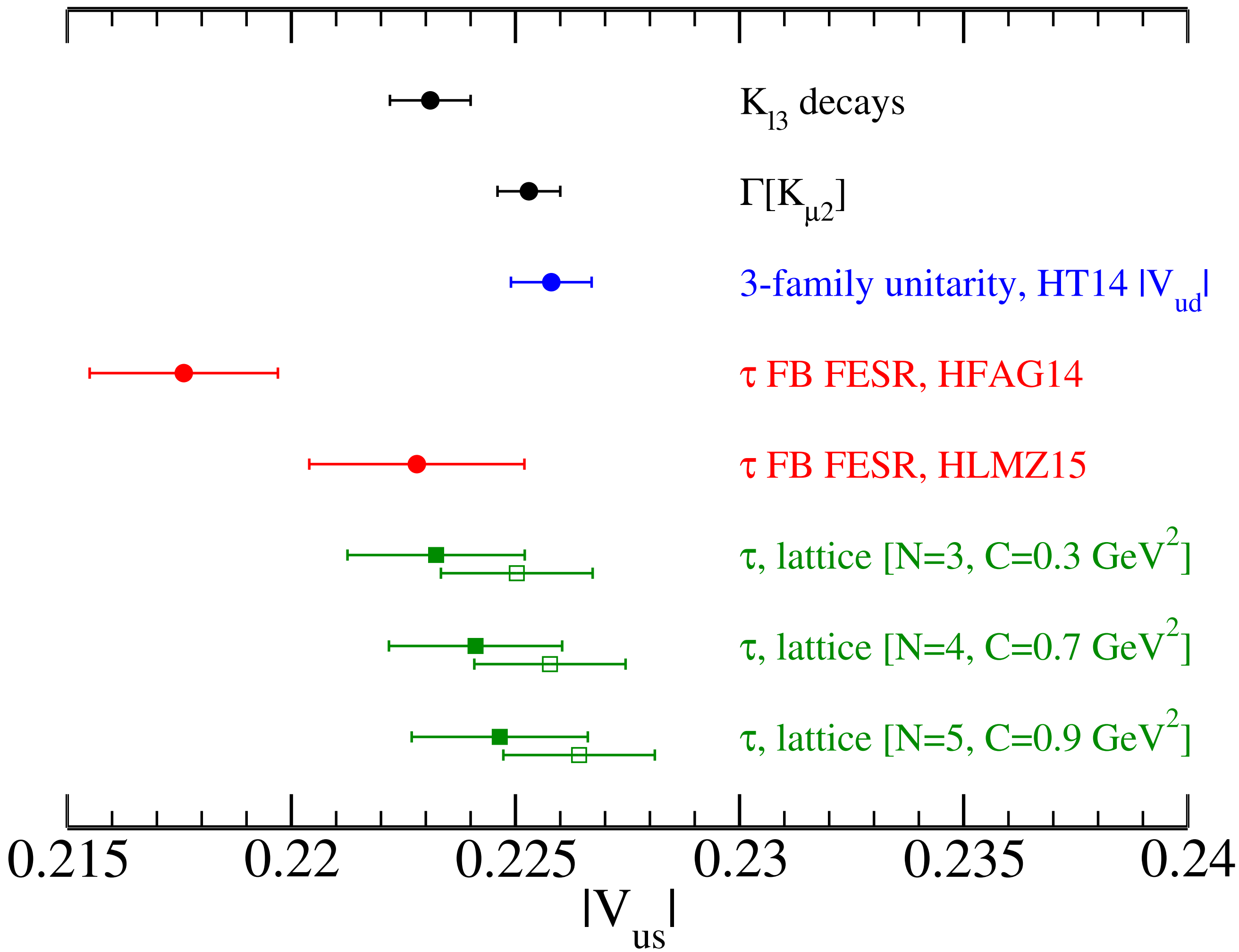
$$\Longrightarrow \sum_k (Q_k)^M a_k = 0 \quad (M = 0, 1, \cdots, N_p - 2)$$

- The residue constraints automatically subtracts $\Pi^{(0,1)}(0)$ and $s\Pi^{(1)}(0)$ terms.

- For experimental data, $w(s) \sim 1/s^n, n \geq 3$ suppresses
  - ▷ *larger error from higher multiplicity final states at larger* $s < m_\tau^2$
  - ▷ *uncertanties due to pQCD+OPE at* $m_\tau^2 < s$

- For lattice, Q_k^2 should be not too small to avoid large stat. error, $Q^2 \to 0$ extrapolation, Finite Volume error(?). Also not too larger than m_τ^2 to make the suppression in time-like, higher energy, higher multiplicity, region enhanced.

Our $|V_{us}|$ results Summary



Discussions and Conclusions for τ inclusive decay

- Our result from τ decay

$$|V_{us}| = \begin{cases} 0.2241(14)_{exp}(13)_{th}[0.85\%], & \text{full } \tau \text{ result} \\ 0.2258(10)_{exp}(13)_{th}[0.73\%], & \text{for using K pole from } K_{l2} \end{cases}$$

for $N = 4, C = 0.7$ [GeV^2], vs $V_{us} = 0.2253(7)[0.3\%]$ from K_{l2}/π_{l2}

- All thinkable systematics estimated.
- Lattice error is currently the largest (matter of time).
- Give a physics target for B factories.
- Systematically cross check by changing N, C (controlling ''inclusiveness'')
- Excluding K pole, the difference gets larger, $\sim$ 8% in integral, or $\sim$ 4% in V_{us}. Direction is smaller $|V_{us}|$ (closer to the current tau inclusive determinations).

Interplay Lattice and R ratio haver many interesting applications including g-2 HVP, and others

Oher applications

- lepton(s) initial states : τ or e^+e^-
    - *$\bar{u}d$ channel (FB analysis), and/or extract α_s*
    - *$\bar{q}q$ for $(g-2)_\mu$, combined analysis with Lattice and R-ratio to crosscheck and to extract better accuracy than either*

- $\rho \to \pi\pi + \ldots$ resonances analsys from HVP

- $K \to \pi\pi$

- $B \to had(charm)$ inclusive extraction for V_{cb}

- Use of excited state...

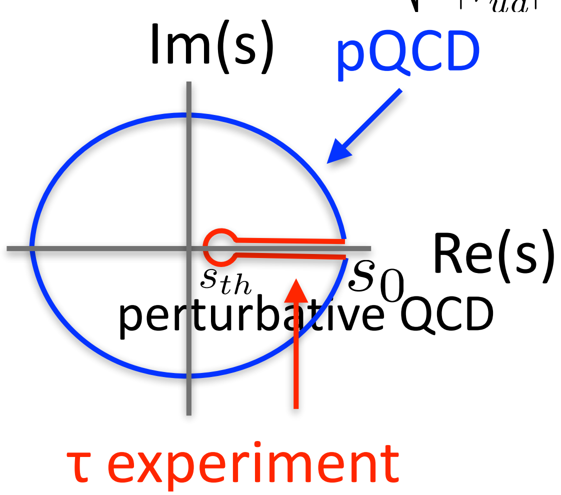
$|V_{us}|$ determination from FESR

- Inclusive differencial τ decay rate with wieght $w(s)$

$$R_{ij}^{\omega}(s_0) \sim \int_{s_{th}}^{s_0} ds \frac{dR_{ij}}{ds} \frac{\omega(s/s_0)}{\omega_\tau(s/m_\tau^2)}$$

- Take difference between up-down and up-strange channel (Flavor Breaking, FB difference) $\Delta R = R_{ud}/|V_{ud}|^2 - R_{us}/|V_{us}|^2$
- From $|V_{ud}|$ input,

$$|V_{us}| = \sqrt{\frac{R_{us}^{\omega}(s_0)}{\frac{R_{ud}^{\omega}(s_0)}{|V_{ud}|^2} - [\Delta R^{\omega}(s_0)]^{\text{pQCD}}}}$$



Experimental tau data

- Belle, BarBar, ALEPH, compilation by Kim Maltman, in the form of

$$\Pi(s) = \left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s)$$

- Unit normalized invariant hadron mass distribtuion $1/N \times dN(s)/ds$ and Branching Fraction

- higher multiplicity final states, at larger s, have larger errors

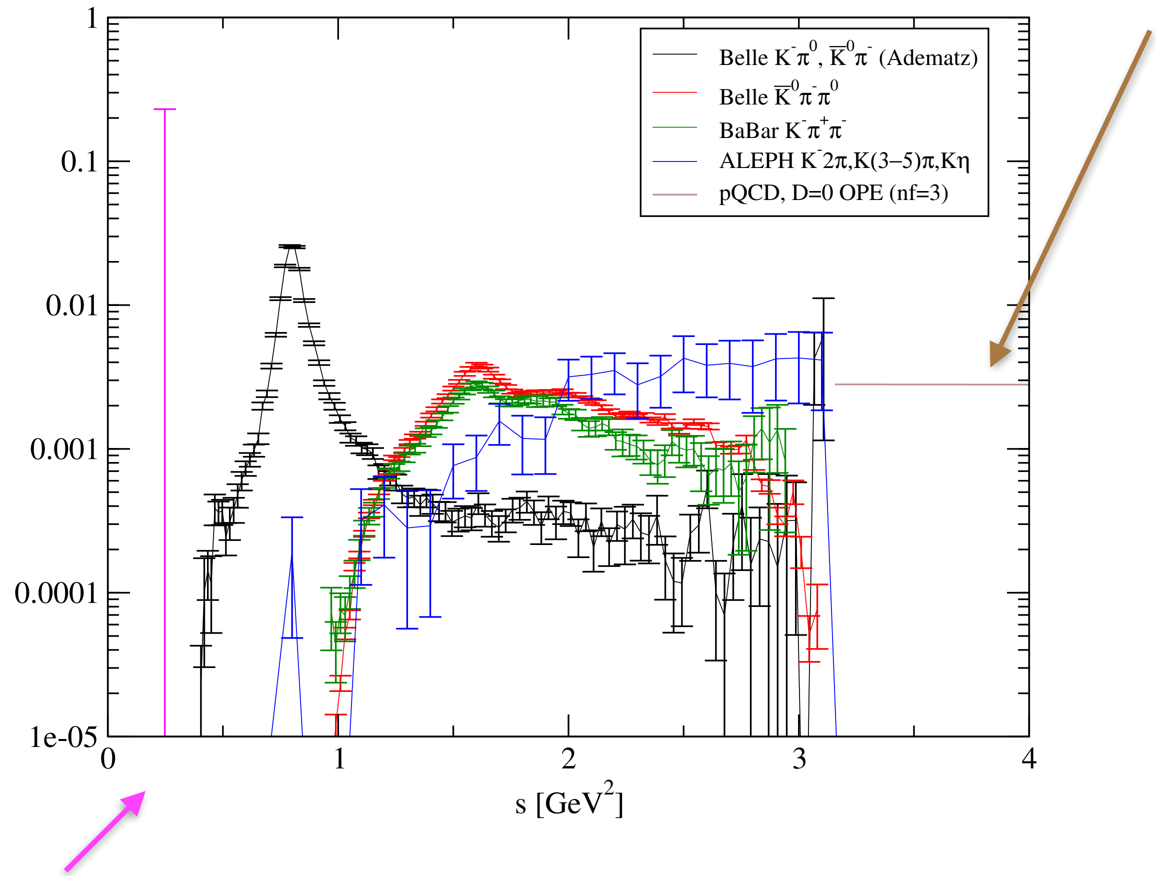
- In each channels, correlation matrix for unit normalized distribution are used for available channels ($K\pi$) and added by the Branching Fraction in quadrature. Error from all channel are added in quadrature.

τ inclusive decay experiment

$$\rho(s) \equiv |V_{us}|^2 \left[\left(1 + 2\frac{s}{m_\tau^2} \right) \mathrm{Im}\Pi^1(\mathrm{s}) + \mathrm{Im}\Pi^0(\mathrm{s}) \right]$$

To compare with experiments,

a conventional value of |Vus|=0.2253 is used



For K pole, we assume a delta function form, whose coefficient is obtained from the experimental value of K-> μ decay width

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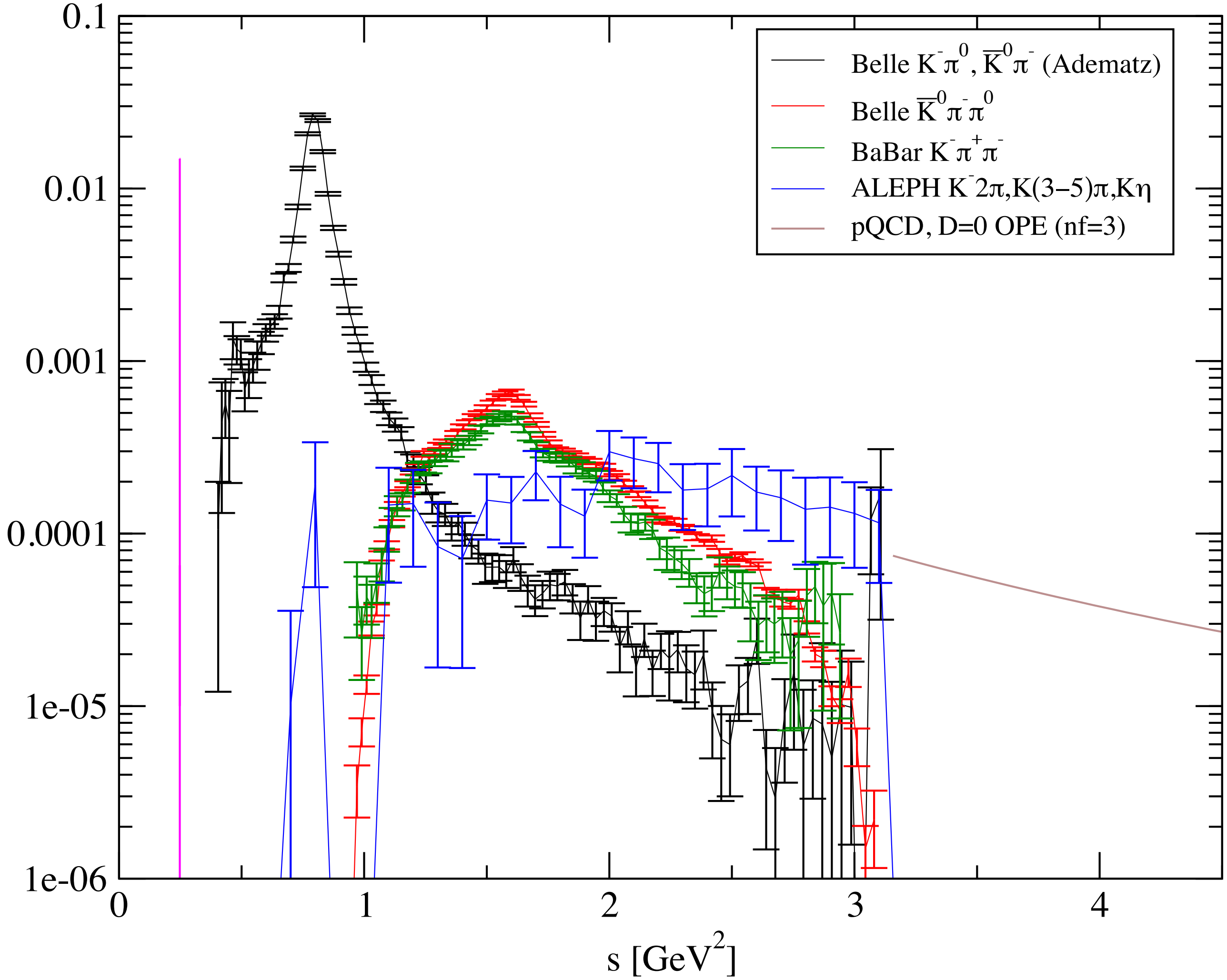
$$\delta(s - m_k^2)0.0012299(46) \ \sim 2 f_k^2 |V_{us}|^2$$

- example: N=3, $\{Q_1^2, Q_2^2, Q_3^2\} = \{0.1, 0.2, 0.3\}$ $[\mathrm{GeV}^2]$

$\omega(s)\rho(s)$



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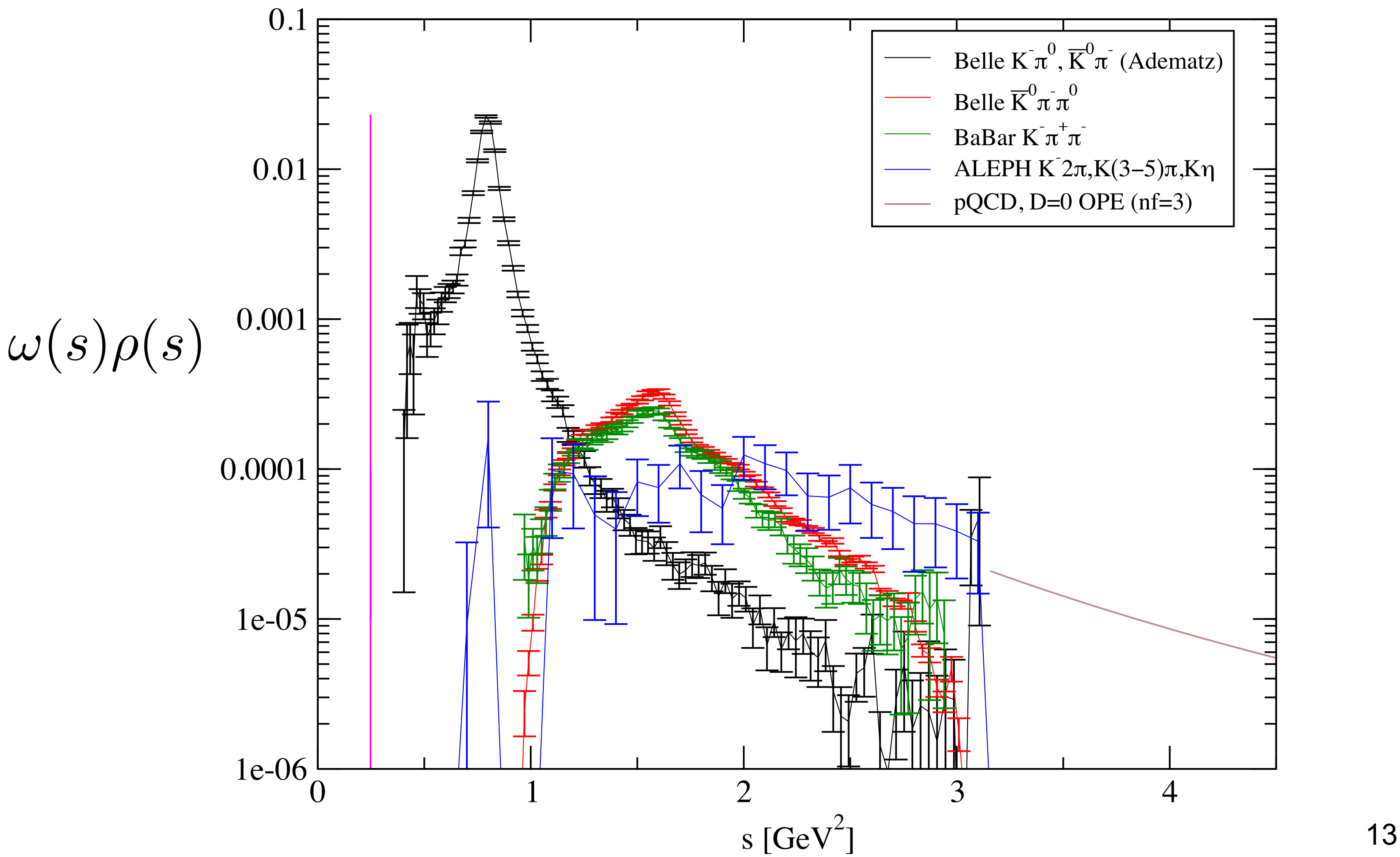
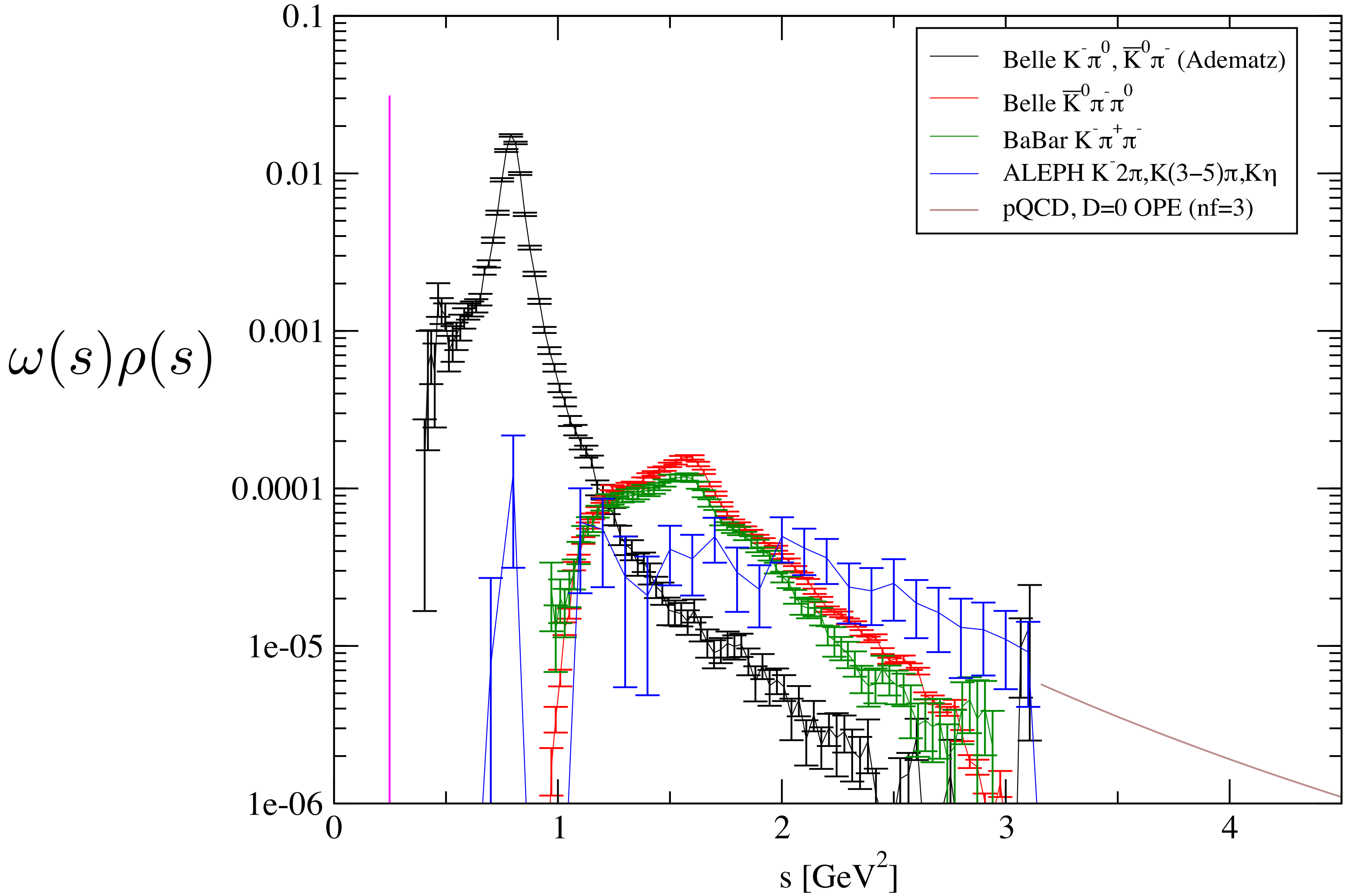
- example: N=4, $\{Q_1^2, Q_2^2, Q_3^2, Q_4^2\} = \{0.1, 0.2, 0.3, 0.4\}$ [GeV^2]



- example: N=5, $\{Q_1^2, Q_2^2, Q_3^2, Q_4^2, Q_5^2\} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$ $[\mathrm{GeV}^2]$



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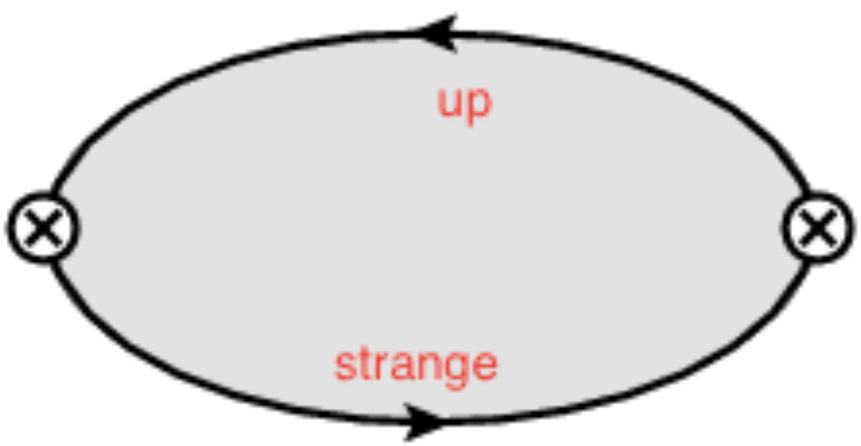
Lattice HVPs

- From Vector-Vector and Axial-Axial correlation function made of up and strange quarks $\Pi^{V/A}_{\nu\nu}(Q^2)$
- Local currents with appropriate renormalization factors, Z_V, Z_A computed non-perturbatively, or conserved currents.
- Tensor $\Pi_{\mu\nu}$ are decomposed into spin 1 and spin 0 components with tensor zero-mode subtraction
- Smaller Q^2 data has a larger relative error, while larger Q^2 suffers discretization errors.



QCD ensemble and statistics

- Main analysis is on two ensemble, at almost physical quark masses ($M_\pi \approx 140$ MeV, $M_K \approx 499$ MeV), V=(5 fm)^3.
- Correct the residual up and strange quark mass error by partially quenched calculation.
- Consistent with other heavier / smaller ensemble are used to estimate size and direction of discretization errors.

Vol	a^{-1} [GeV]	M_π [MeV]	M_K [MeV]	conf
$48^3 \times 96$	1.7295(38)	139	499	88
		135	496	5 (PQ-correction)
$64^3 \times 128$	2.359(7)	139	508	80
		135	496	5 (PQ-correction)

|Vus| from all channels

- 4 channels: Vector or Axial (V or A), spin 0 and 1
- A0 channel is dominated by K pole.

-> For the kaon decay contribution

we use $f_K^{phys} = 0.15551(83)$ [GeV] [RBC/UKQCD, 2014] in place of A0.

- Other channels :

Lattice HVPs for A1, V1, V0 (& residual A0) < - > multi hadron states & pQCD

- We take the continuum limit using the data L=48 and 64

$$V_1 + V_0 + A_1 + A_0 \;\; : |V_{us}^{V_1+V_0+A_1+A_0}| = \sqrt{\frac{\rho_{exp}^{\text{K-pole}} + \rho_{exp}^{others}}{(f_K^{phys})^2 \omega(m_K^2) + F_{lat}(\Pi_{\text{others}}) - \rho_{\text{pQCD}}}}$$

$$\rho_{exp}^{others} = |V_{us}|^2 \int_{s_{th}}^{m_\tau^2} ds\omega(s)\mathrm{Im}\Pi(\mathrm{s})$$

$$F_{lat} = \sum_{k=1}^{N} \mathrm{Res}(\omega(-\mathrm{Q_k^2}))\Pi_{\mathrm{lat}}(-\mathrm{Q_k^2}) \qquad \rho_{pQCD} = \int_{m_\tau^2}^{\infty} ds\omega(s)\Pi_{OPE}(s)$$

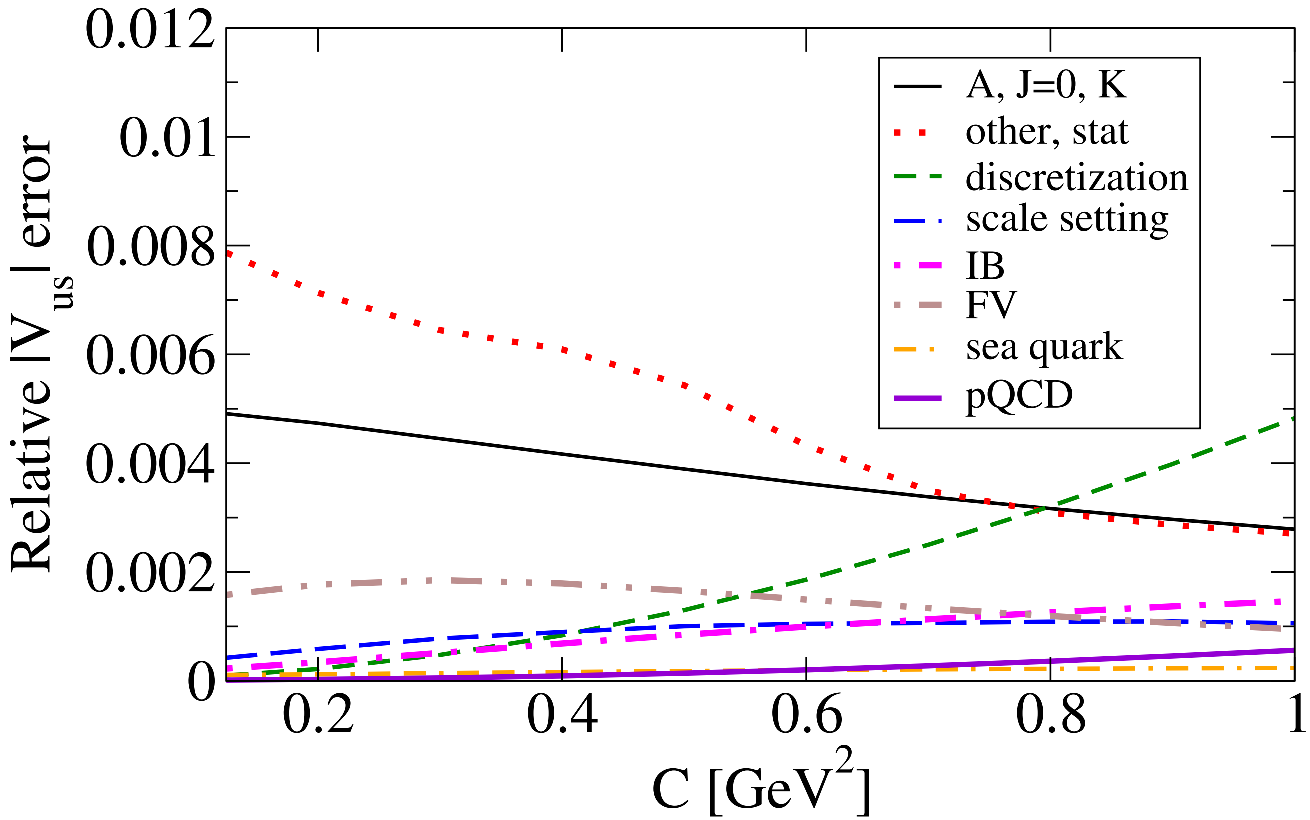
Error budget

- Lattice stat errors are still significant (could be improved)
- Discretization error from $O(a\Lambda_{QCD})^4$
- Finite Volume error, estimated from ChPT in FV for $K\pi$ channel
- Isospin breaking effects, estimated from IB in K_{l2} and $K \to \pi$ analysis [Antonelli, Cirigliano, Lusiani, Passemar, JHEP10(2013)070]
- pQCD uncertanty 2% of pQCD contribution
- Experimental specral funcation error is currently comparable or smaller than theory error.

Error budget

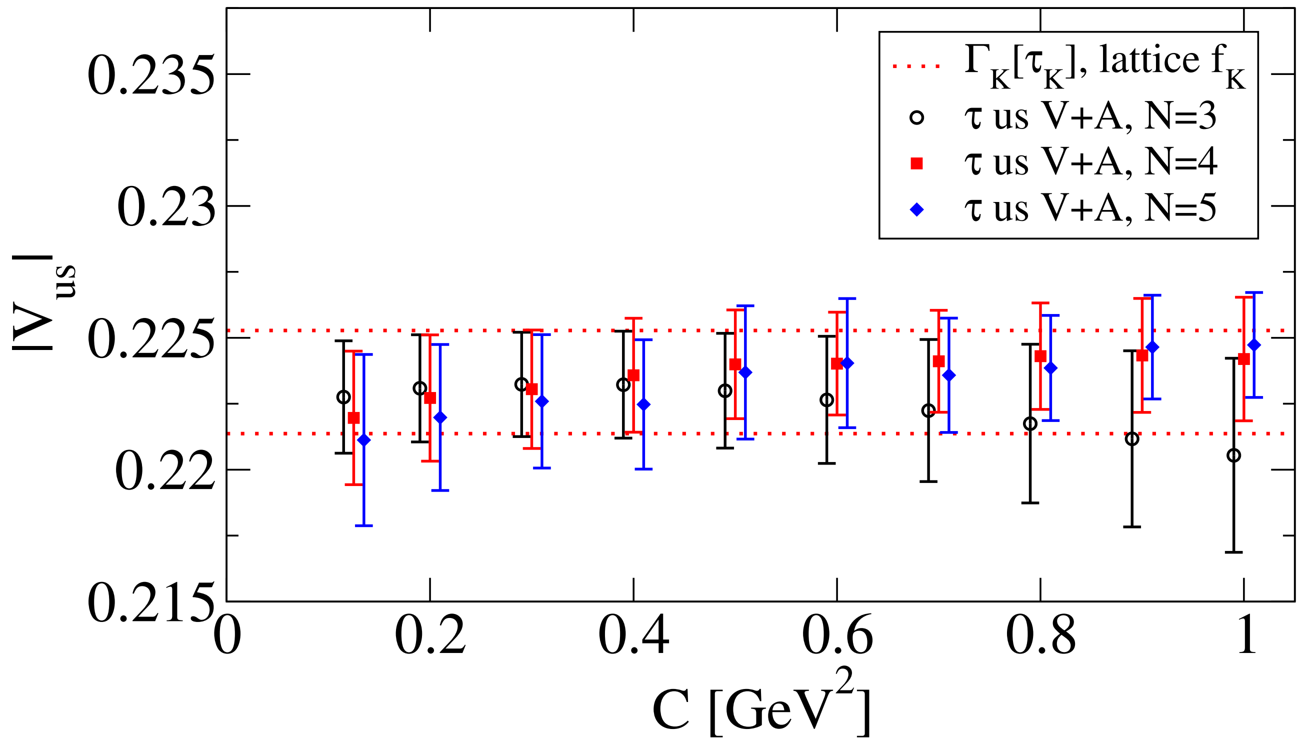
contribution		error ($\times 10^4$)			
	$[N, C]$	$[3, 0.3]$	$[3, 1]$	$[4, 0.7]$	**[5, 0.9]**
theory	f_K	8.2	4.4	7.6	8.2
	others, stat.	9.7	4.1	7.8	9.4
	discretization	2.2	17.6	5.6	6.1
	scale setting	2.1	1.9	2.4	2.6
	IB	2.2	4.6	2.5	2.3
	FV	2.3	0.9	3.0	4.1
	sea quark	0.7	0.9	0.5	0.3
	pQCD	1.2	5.8	0.6	0.2
experiment	K	10.5	5.8	9.7	10.4
	$K\pi$	3.9	6.4	4.7	4.4
	$K^-\pi^+\pi^-$	1.0	2.7	1.1	0.8
	$\bar{K}^0\pi^-\pi^0$	0.7	2.1	0.8	0.6
	residual	9.0	29.3	9.2	6.2

Errors Breakup $|V_{us}|$



$N = 4$ case

$|V_{us}|$ Results



- $N = 3, 4, 5$. Full error. Holizontal dots is exclusive $\tau \to K$ detemination using f_K.
- All estimated systematic erros included