

Update on muon g-2 HVP from the $N_f=2+1+1$ MILC HISQ ensembles

Ruth Van de Water
(for the Fermilab Lattice, HPQCD,
& MILC Collaborations)

First workshop of Muon g-2 Theory Initiative
June 4, 2017

Motivation

- ♦ Muon anomalous magnetic moment ($g-2$) provides sensitive probe of physics beyond the Standard Model:
 - ❖ Mediated by quantum-mechanical loops
 - ❖ Known to very high precision of 0.54ppm
- ♦ Measurement from BNL E821 disagrees with Standard-Model theory expectations by more than 3σ
- ♦ Muon $g-2$ Experiment being mounted at Fermilab to reduce the experimental error by a factor of four
 - ❖ Will begin running this year, and expect first results in Spring 2018!
- ❖ **Theory error must be reduced to a commensurate level to identify definitively whether any deviation observed between theory and experiment is due to new particles or forces**



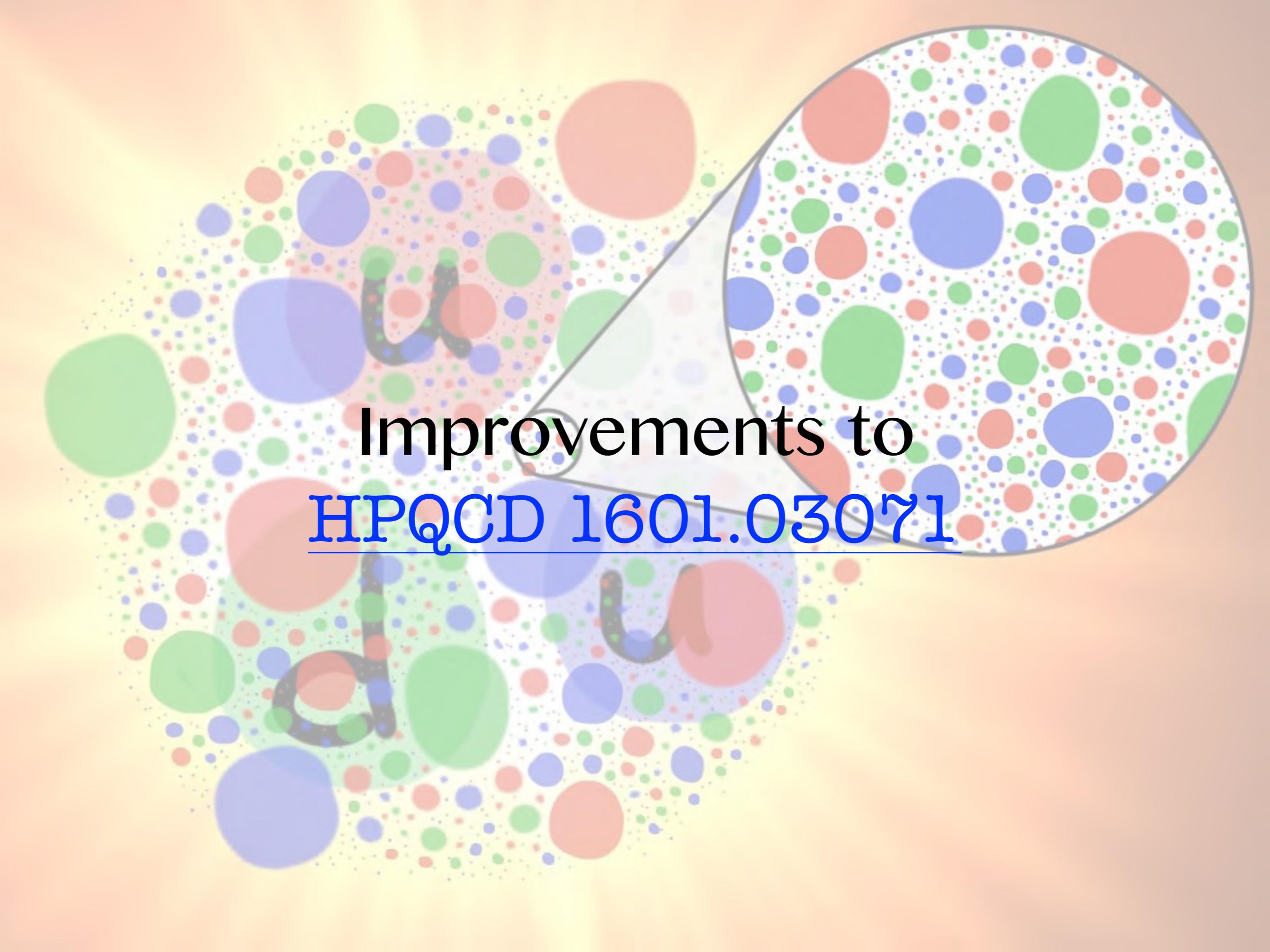
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Improvements to HPQCD 1601.03071

Ensembles & parameters

- ◆ Employ large set of **MILC ensembles** with four flavors of dynamical HISQ sea quarks with:
 - ❖ **Three lattice spacings, multiple spatial volumes, physical light-quark masses**
- ◆ Vector-current correlator data from **HPQCD+RV [1601.03071]** plus:
 - ❖ $a \sim 0.15 \text{ fm}$ physical-mass data with better-tuned quark masses & higher statistics

$\approx a$ (fm)	$am_l^{\text{sea}}/am_s^{\text{sea}}/am_c^{\text{sea}}$	w_0/a	$Z_{V,\bar{s}s}$	M_{π_5} (GeV)	$(\frac{L}{a})^3 \times (\frac{T}{a})$	$N_{\text{conf.}}$
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“Time-momentum representation”

- ◆ Calculate a_μ from weighted integral of Euclidean electromagnetic-current correlator
- ◆ Checks moments + Padé approach & eliminates systematic due to truncating at [n,n] Padé approximant
- ◆ (Moments expression for a_μ^{HVP} can be obtained by expanding $\tilde{R}(t)$ about $\omega=0$)
- ◆ Kernel $\tilde{K}(t)$ proportional to t at small t and to $1/t$ at large t , suppressing contributions from large times

$$a_\mu^{\text{HLO}} = 4\alpha^2 m_\mu \int_0^\infty dt t^3 G(t) \tilde{K}(t),$$
$$G(t) \equiv \int d\mathbf{x} \langle j_z^{\text{em}}(t, \mathbf{x}) j_z^{em\dagger}(0) \rangle,$$
$$\tilde{K}(t) \equiv \frac{2}{m_\mu t^3} \int_0^\infty \frac{d\omega}{\omega} K_E(\omega^2) \left[\omega^2 t^2 - 4 \sin^2 \left(\frac{\omega t}{2} \right) \right],$$
$$K_E(s) = \frac{1}{m_\mu^2} \cdot \hat{s} \cdot Z(\hat{s})^3 \cdot \frac{1 - \hat{s}Z(\hat{s})}{1 + \hat{s}Z(\hat{s})^2},$$
$$Z(\hat{s}) = -\frac{\hat{s} - \sqrt{\hat{s}^2 + 4\hat{s}}}{2\hat{s}}, \quad \hat{s} = \frac{s}{m_\mu^2}$$

[Bernecker & Meyer,
EPJA47 (2011) 148, [arXiv:1107.4388](https://arxiv.org/abs/1107.4388)]

a_μ^{HVP} integrand from $e^+e^- \rightarrow \text{hadrons}$

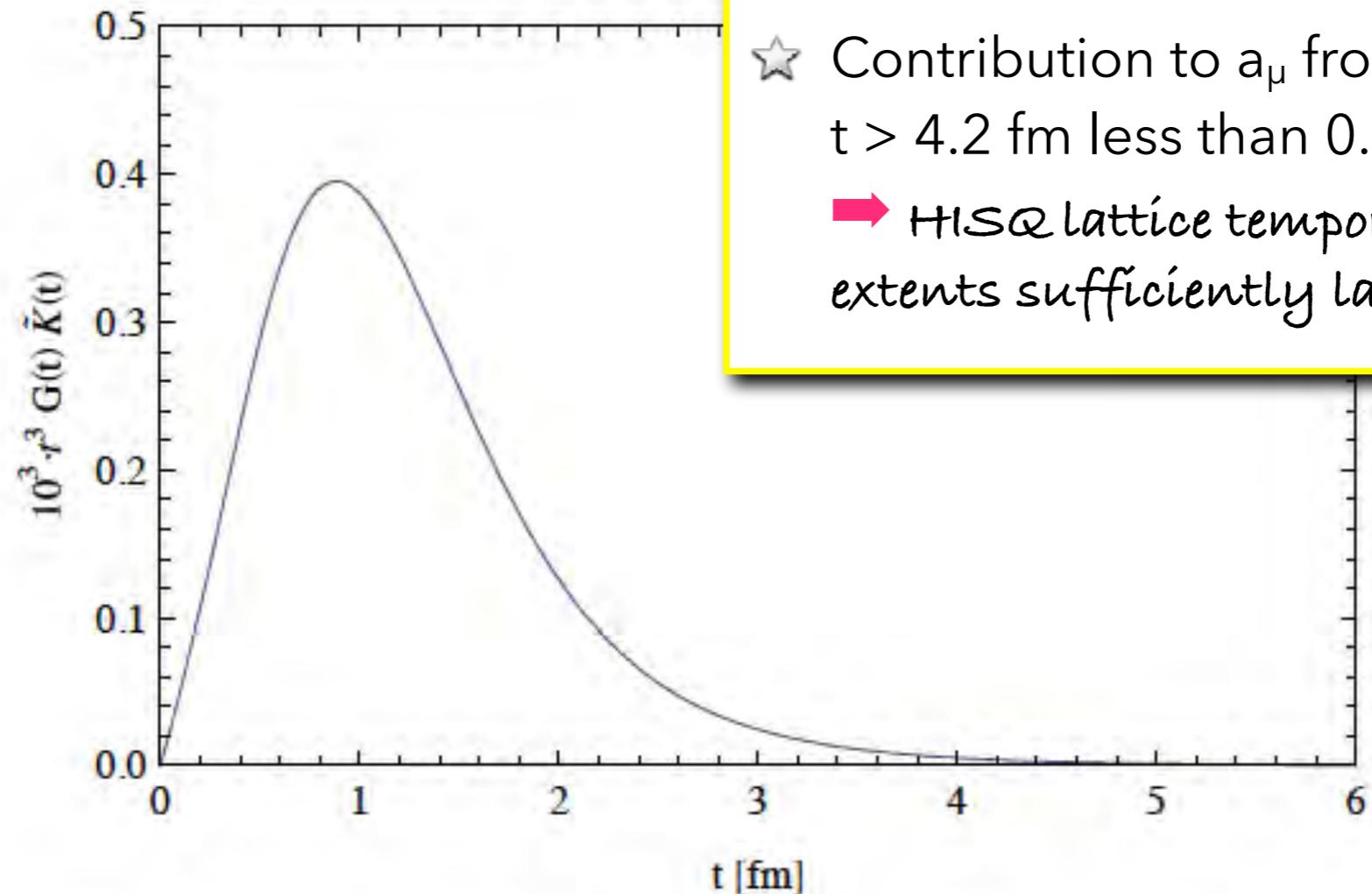


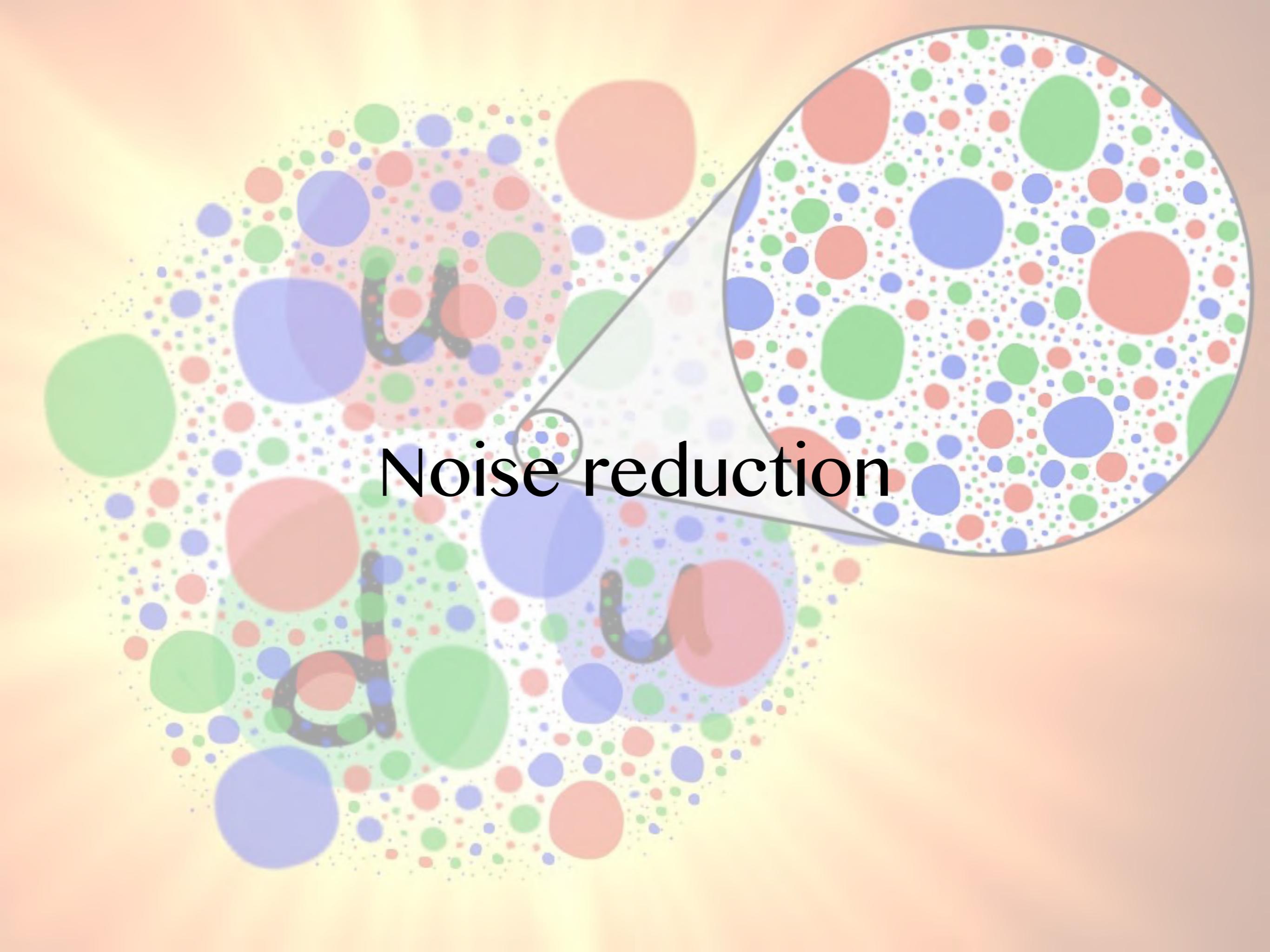
Fig. 4: The integrand of Eq. (84) in infinite volume, obtained from the phenomenological parametrization (18) of the $R(s)$ ratio.

Check: TMR vs. moments + Padés

- ♦ a_μ^{HVP} computed via time-momentum-representation integral and via time-moments + [3,3] Padé approximants agree to better than 0.1×10^{-10}

→ Confirms reliability of moments + Padé approach

Ensemble	$10^{10} a_\mu$	TMR integral	$\Pi_s + \text{Padé}$	Difference
l1648f211b580m01300m0650m838	444.5(4.6)	444.5(4.6)	[3,3]	-0.0027(35)
l2448f211b580m0064m0640m828	510.2(6.3)	510.2(6.3)	[3,3]	-0.0277(90)
l3248f211b580m00235m0647m831	561.3(8.3)	561.3(8.3)	[3,3]	-0.021(18)
l3248f211b580m002426m06730m8447	554.3(7.1)	554.3(7.1)	[3,3]	-0.018(13)
l2464f211b600m01020m0509m635	451.1(5.0)	451.1(5.0)	[3,3]	-0.0312(94)
l2464f211b600m00507m0507m628	500.9(7.0)	500.9(7.0)	[3,3]	-0.041(18)
l3264f211b600m00507m0507m628	513.9(6.1)	513.9(6.1)	[3,3]	-0.023(14)
l4064f211b600m00507m0507m628	520.5(6.5)	520.5(6.5)	[3,3]	-0.020(21)
l4864f211b600m00184m0507m628	569.1(7.5)	569.1(7.5)	[3,3]	-0.034(21)
l3296f211b630m00740m0370m440	441.9(5.6)	441.8(5.6)	[3,3]	0.014(29)
l4896f211b630m00363m0363m430	520.0(6.8)	520.1(6.8)	[3,3]	-0.066(20)



Noise reduction

Strategy

- ◆ Calculate time-momentum-representation weighted sum $a_\mu = \sum w(t)G(t)$
- ◆ Reduce statistical errors in a_μ by:

(1) Simultaneous fit of 4 combinations of (local, smeared) correlators G_{ij}

(2) Replacing $G_{\text{data}}(t)$ with $G_{\text{fit}}(t)$ for $t > t^*$

- ◆ Fit lattice correlators to cosh form that accounts for periodic temporal boundary conditions
- ◆ Correct for finite temporal extent by calculating $G_{\text{fit}}(t)$ using 2-point fit parameters in exponential parameterization and extending times in $G_{\text{fit}}(t)$ to $2 \times T$

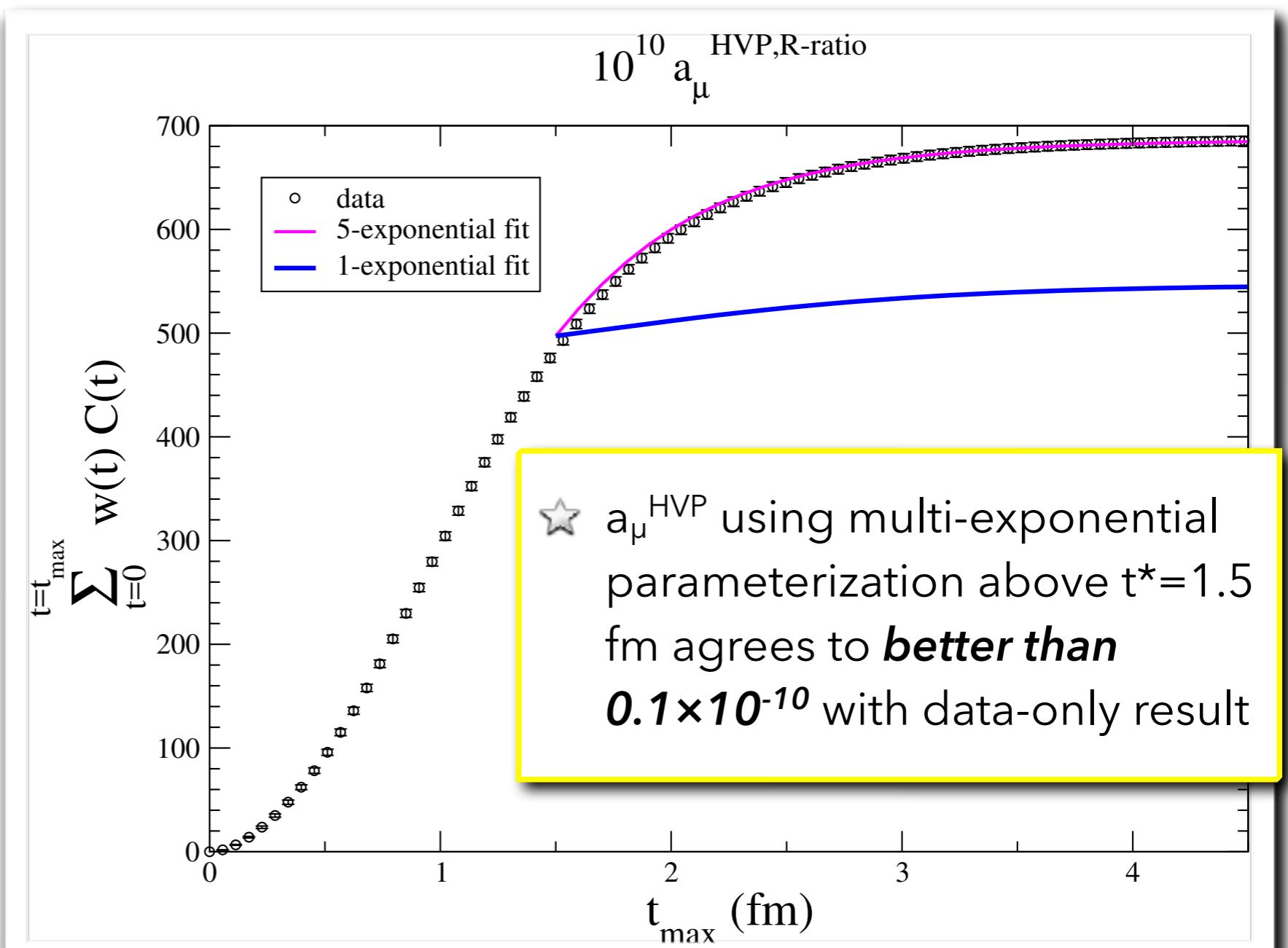
$$a_\mu = 2Z_V^2 \left(\sum_{t=0}^{t^*} w(t)G_{\text{data}}(t) + \sum_{t=t^*+1}^{2T} w(t)G_{\text{fit}}(t) \right)$$

Test with e^+e^- ("R-ratio") data

- ♦ Fit e^+e^- data to multi-exponential parameterization and compute a_μ using data below 1.5 fm and fit above 1.5 fm

→ Method yields correct result provided use of multi-exponential parameterization that accurately describes data above transition time

★ Thanks to F. Jegerlehner for compiling e^+e^- data in public [alphaQED fortran package!](#)



Correlator fits



Fit refinements yield results consistent with 1601.03071

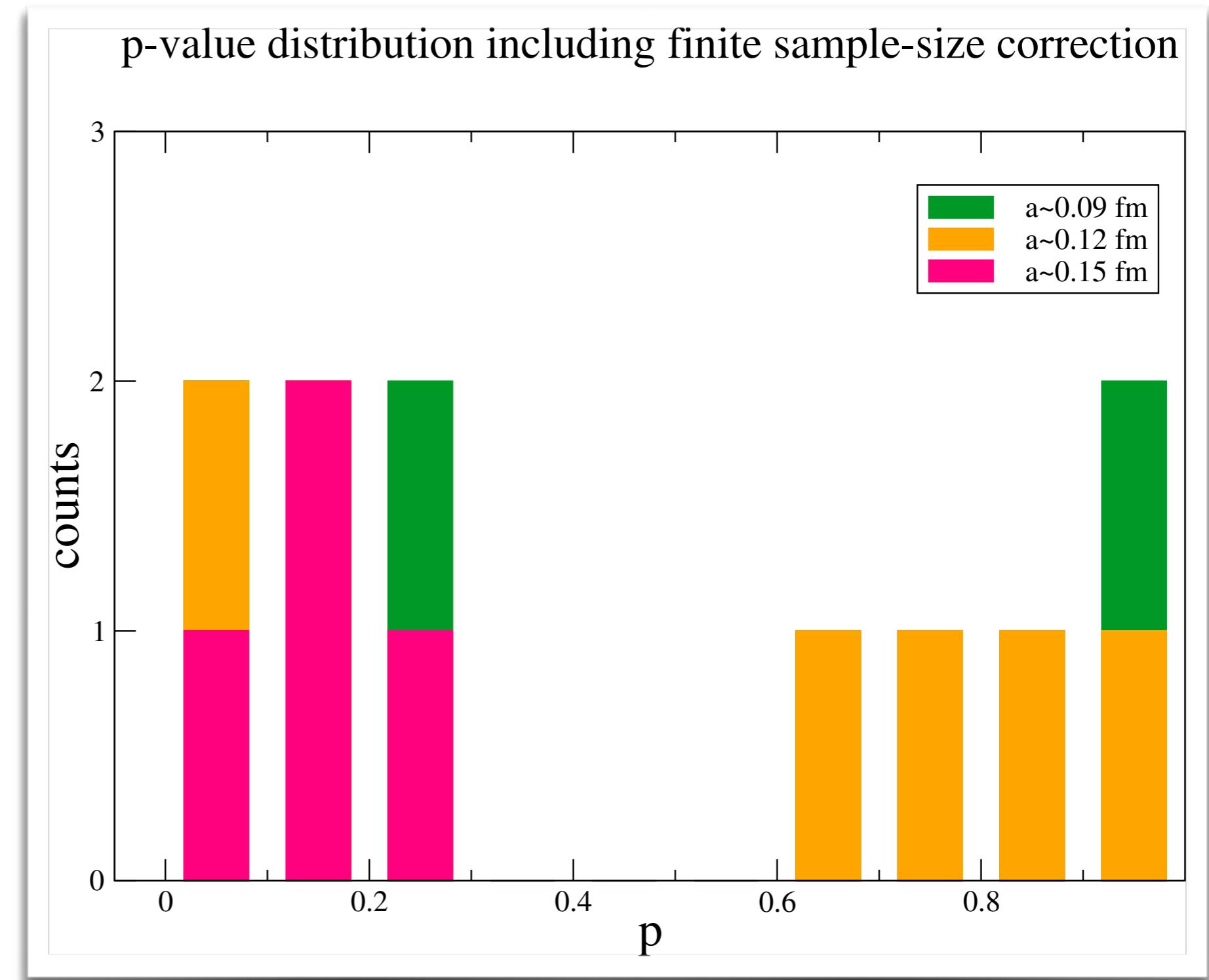
Simultaneous fit of four combinations of (local, smeared) correlators

$$G_{ij}(t) = a^3 \sum_{k=0}^{N-1} b_i^{(k)} b_j^{(k)} \left(e^{-E^{(k)} t} + e^{-E^{(k)} (T-t)} \right) - (-1)^t a^3 \sum_{k=0}^{N-1} d_i^{(k)} d_j^{(k)} \left(e^{-\tilde{E}^{(k)} t} + e^{-\tilde{E}^{(k)} (T-t)} \right)$$

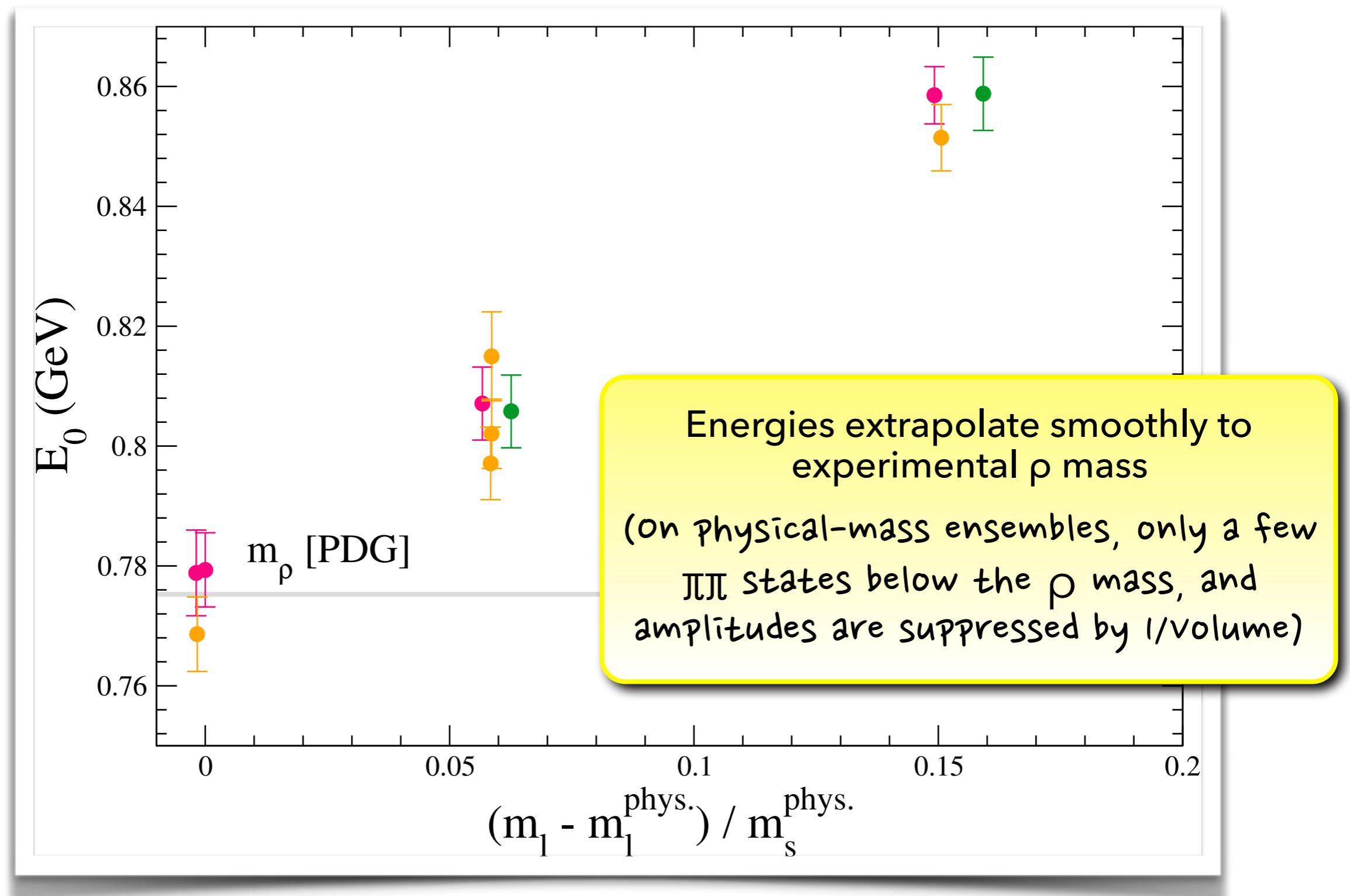
- ♦ Constrain energies & amplitudes with Gaussian priors
- ♦ Employ SVD cuts to reduce d.o.f. and improve reliability of correlation matrix
 - ❖ Conservative approach replaces eigenvalues of correlation matrix below SVD cut by the SVD cut times the maximum eigenvalue, thereby increasing fit error
- ♦ Choose number of states & fit range based on stability of E_0 , A_0 , E_1 , & goodness-of-fit
 - ❖ Fitted ground-state energies & errors (mostly) insensitive to t_{\max} → **fit between $(t_{\min}, T-t_{\min})$** to ensure that fit describes correlator over entire lattice time extent
 - ❖ $t_{\min}/a = [3,4,5]$ for $a \sim [0.15, 0.12, 0.09]$ fm → **$t_{\min} \sim 0.45$ fm for all lattice spacings**
 - ❖ **For all ensembles, obtain good χ^2/dof and stable fit results with $N_{\text{states}} \geq 3$**

Goodness-of-fit

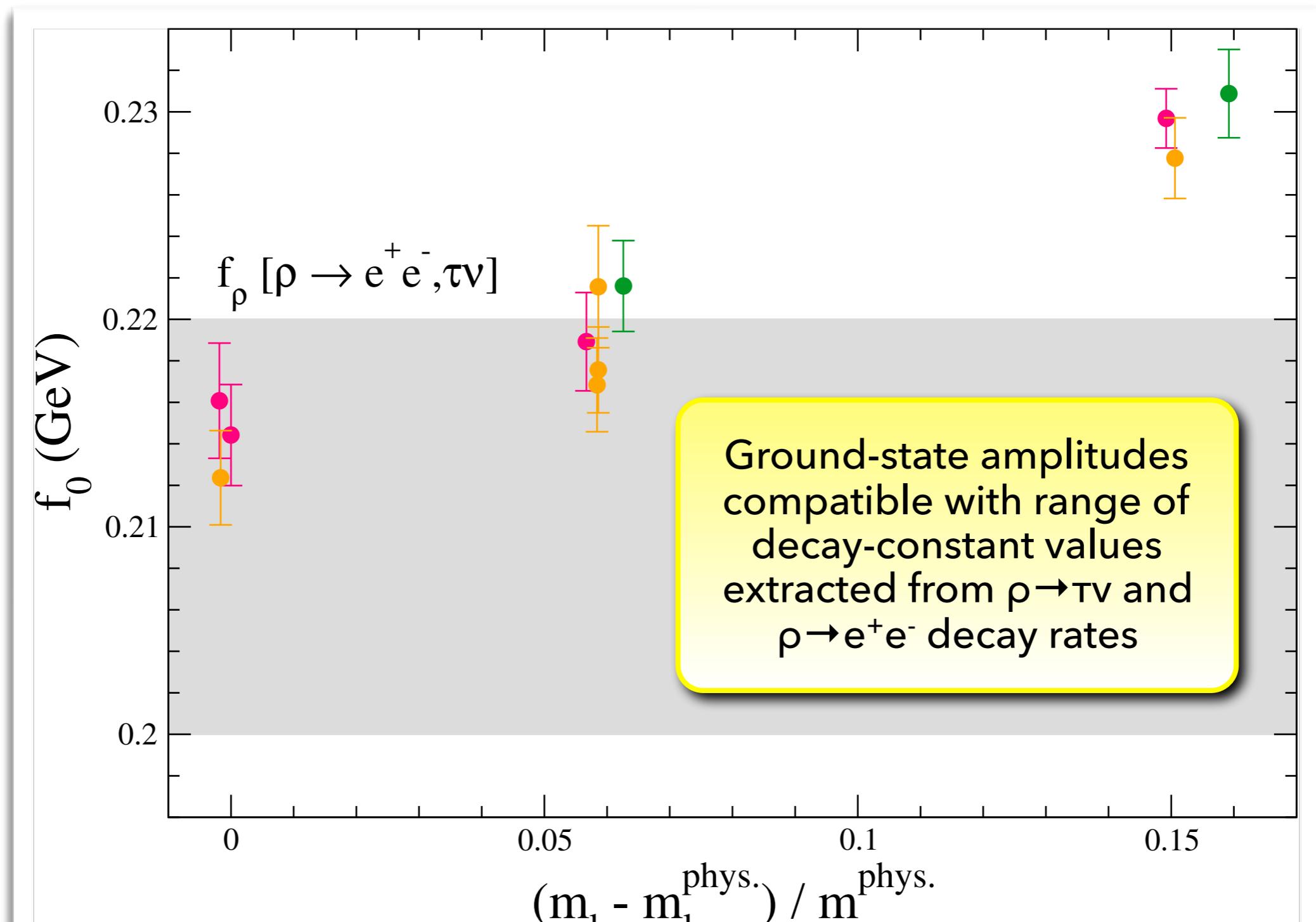
- ◆ Two-point fits provide good description of data as measured by $\chi^2_{\text{data}}/(N_{\text{data}} - N_{\text{params}})$ and confidence level
- ◆ p-value distribution reasonably uniform given small number of fits



2-pt. fit check: ground-state energies

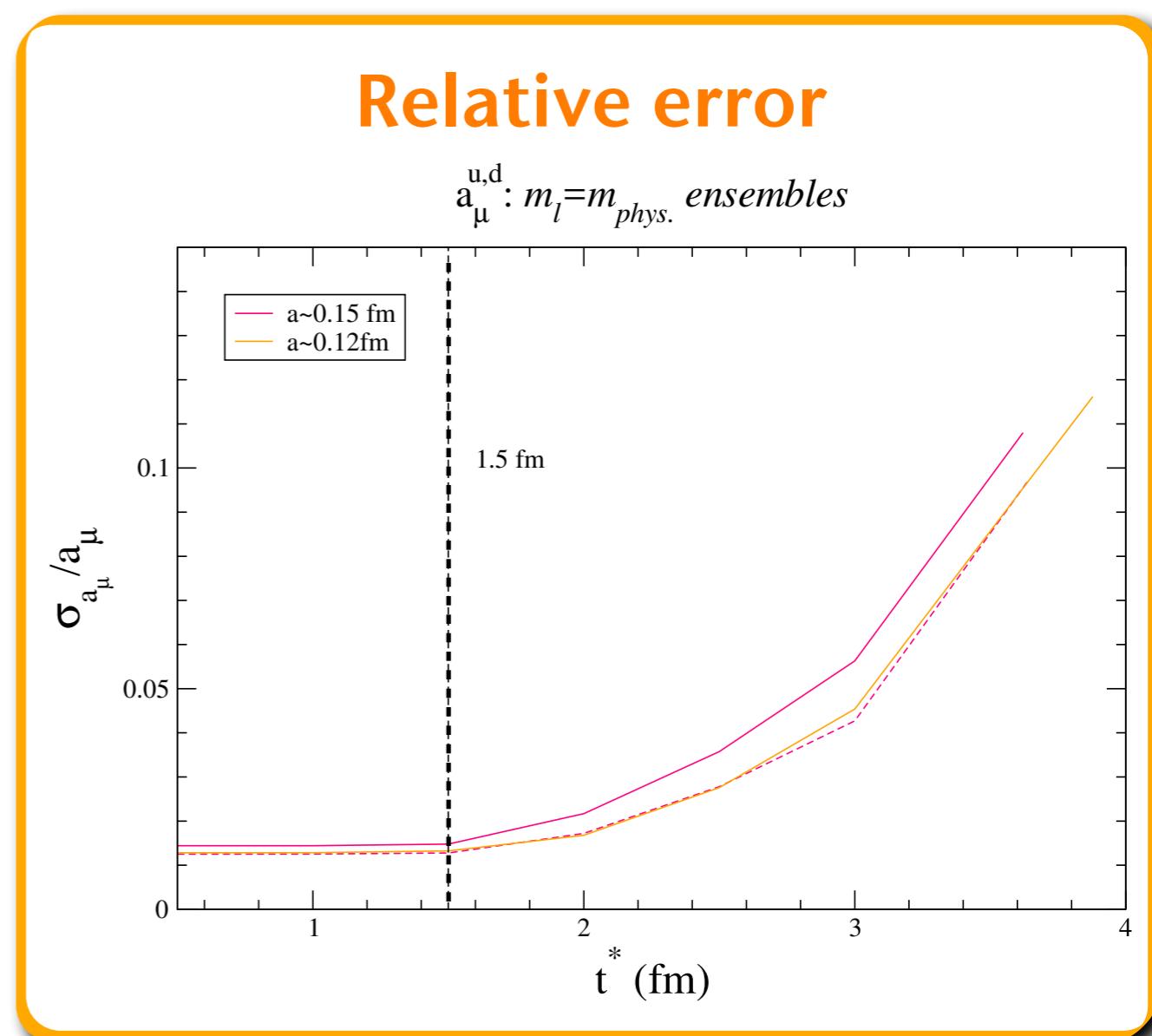
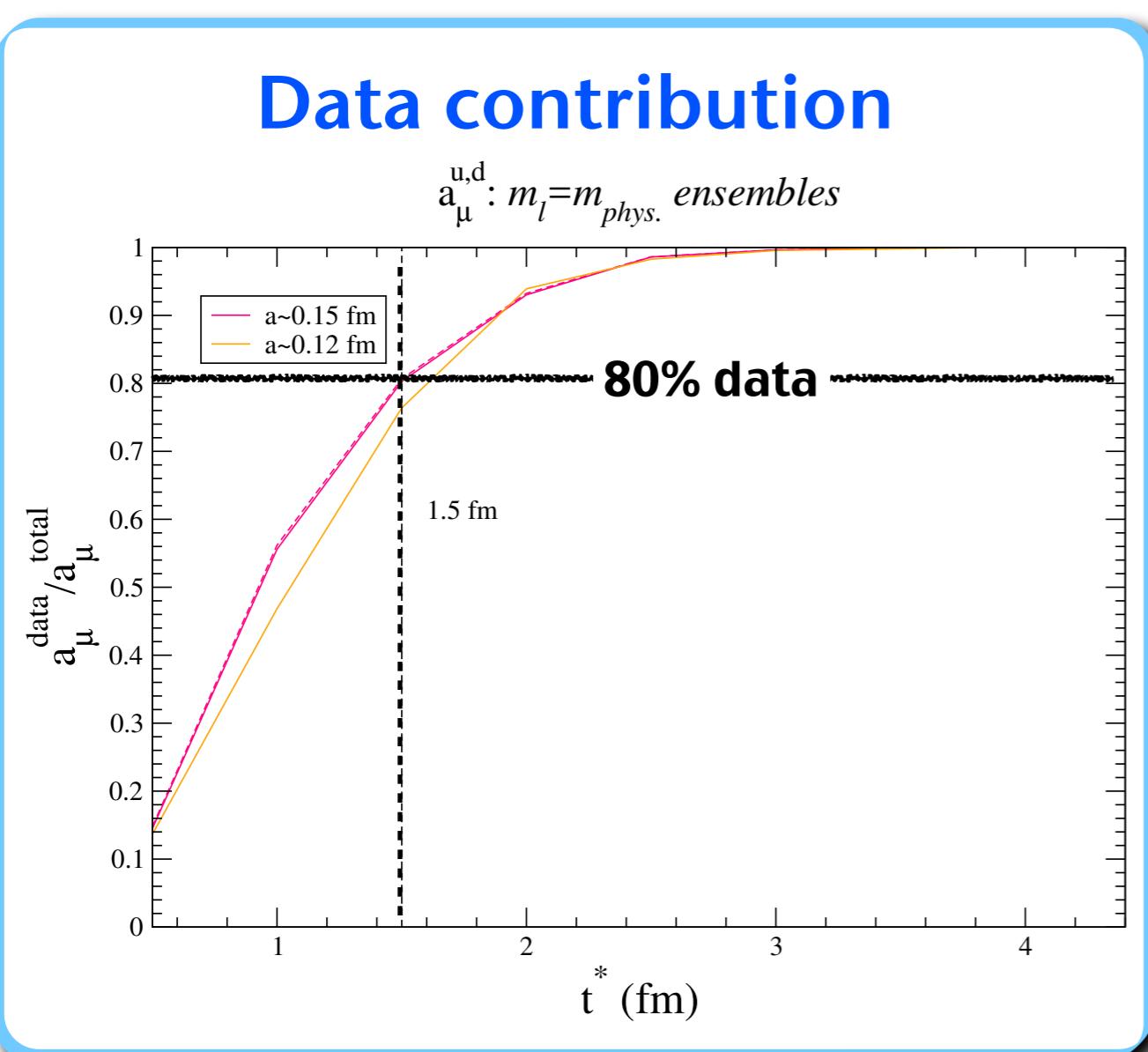


check 2: ground-state amplitudes



Selection of t^*

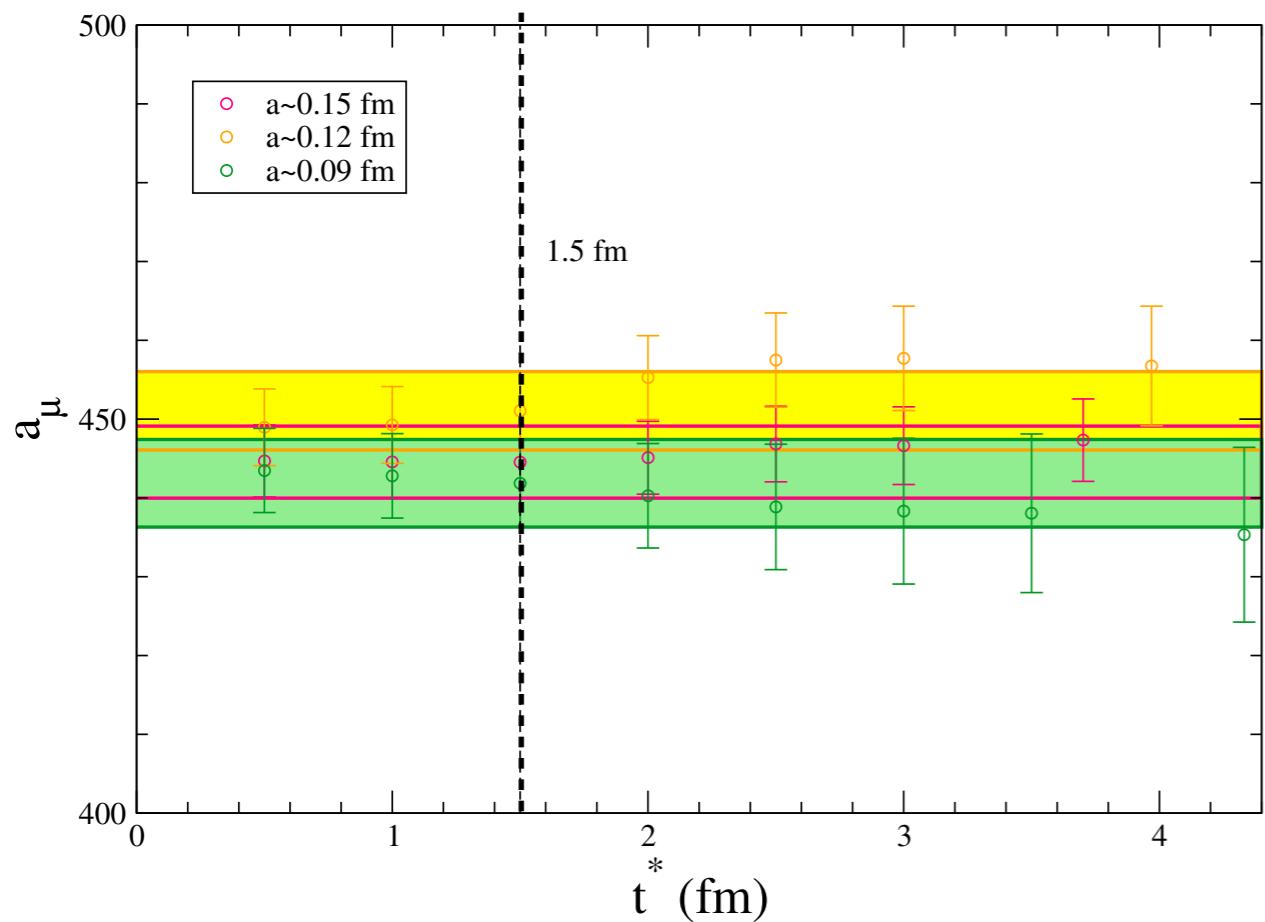
- ◆ Choose t^* such that value of a_μ^{HVP} comes primarily from data region ($t < t^*$), but before errors in a_μ^{HVP} begin increasing rapidly
 - ❖ With $t^* = 1.5 \text{ fm}$, the data contribution is $\gtrsim 80\%$ for all ensembles



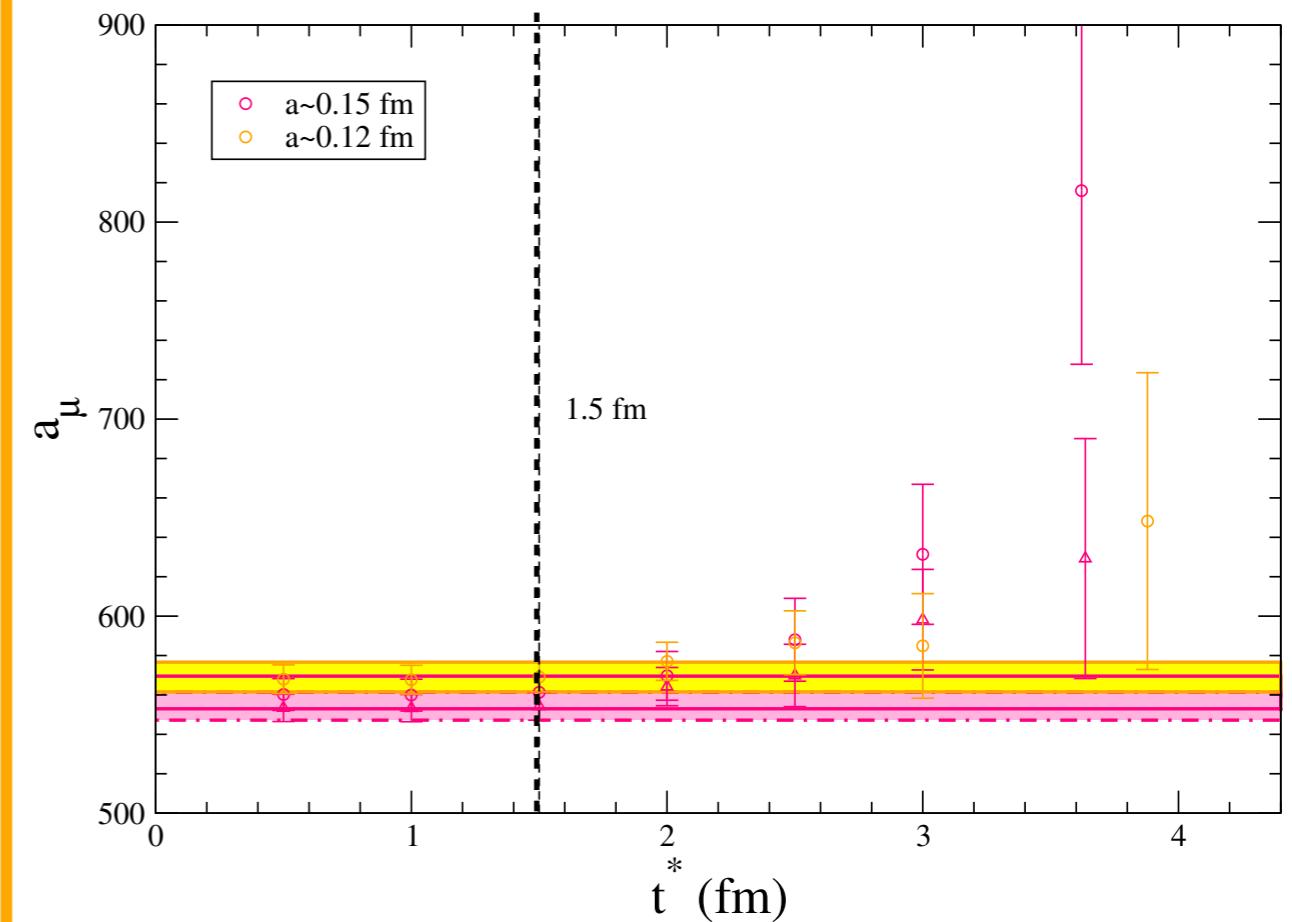
Noise-reduction check: a_μ vs. t^*

- ♦ a_μ^{HVP} independent of t^* from $t^* = 0.5 \text{ fm}$ (<20% data) to $t^* \sim 2.0 \text{ fm}$ (~95% data)

$m_l/m_s = 1/5$

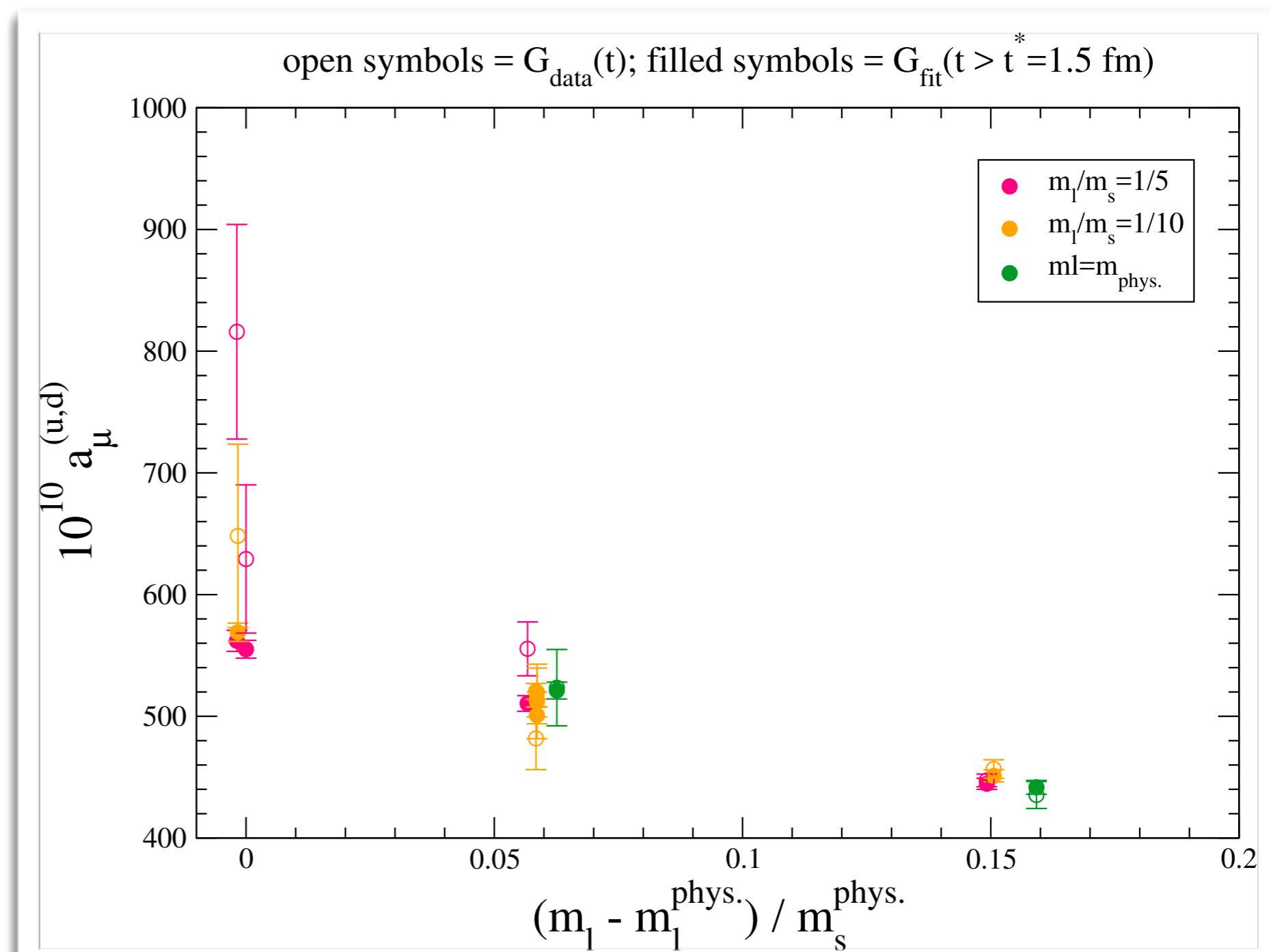


$m_l = m_{\text{phys.}}$



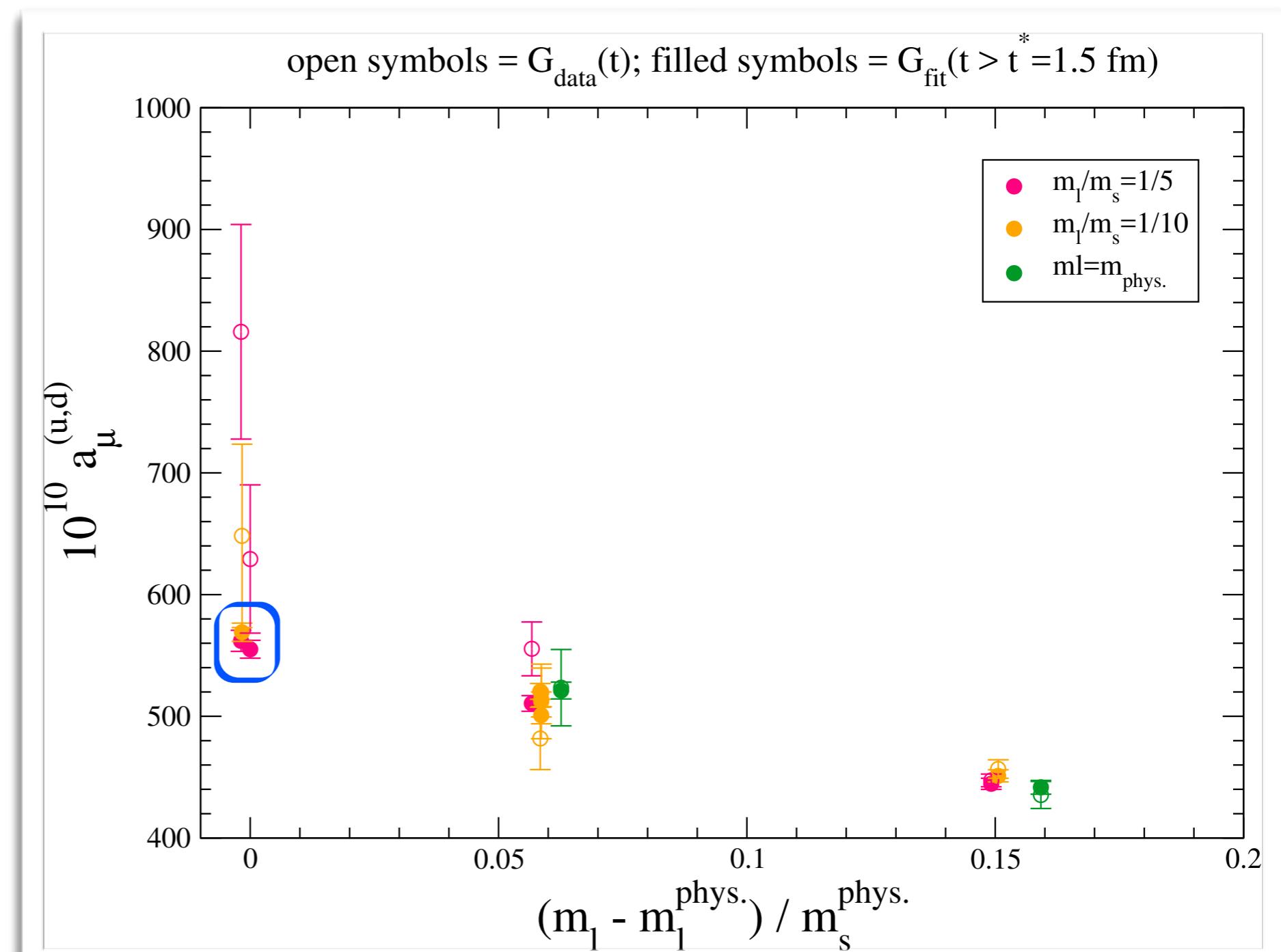
Check 2: comparison with a_μ from data

- ◆ a_μ^{HVP} computed with $G_{\text{fit}}(t)$ for $t > 1.5 \text{ fm}$ consistent with data within 1σ on 8/11 ensembles
- ◆ Use of fitted correlator above $t^* = 1.5 \text{ fm}$:



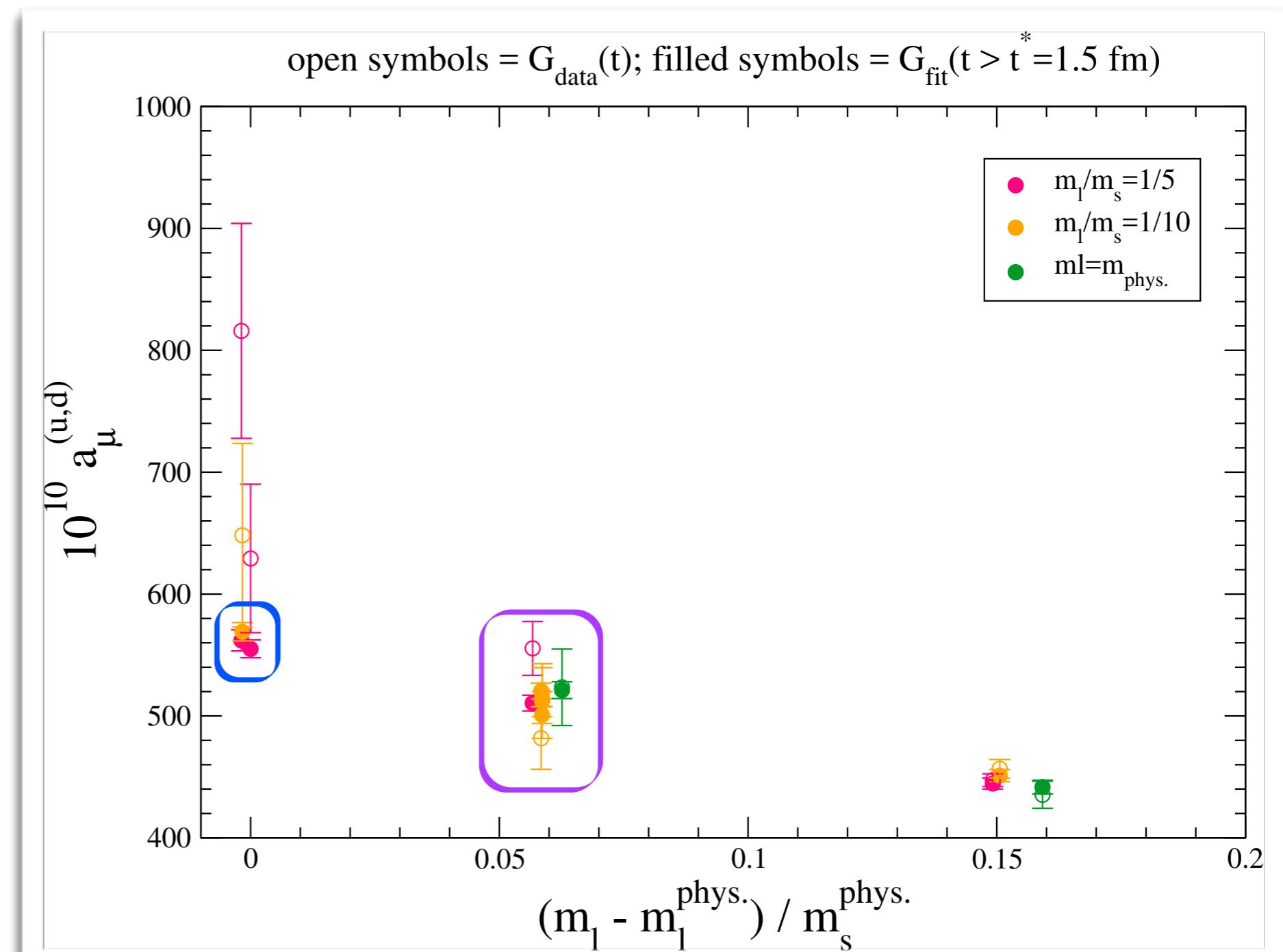
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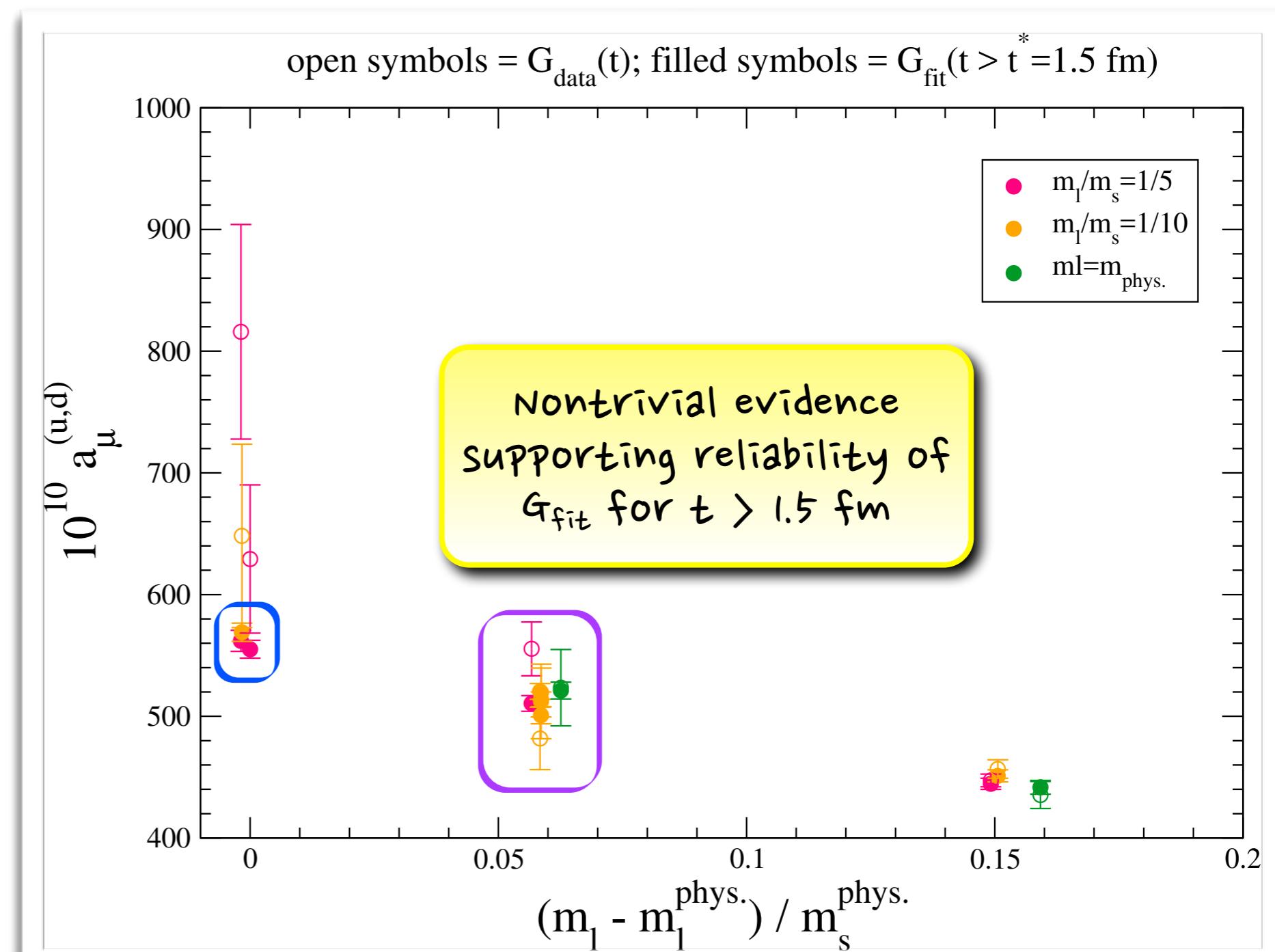
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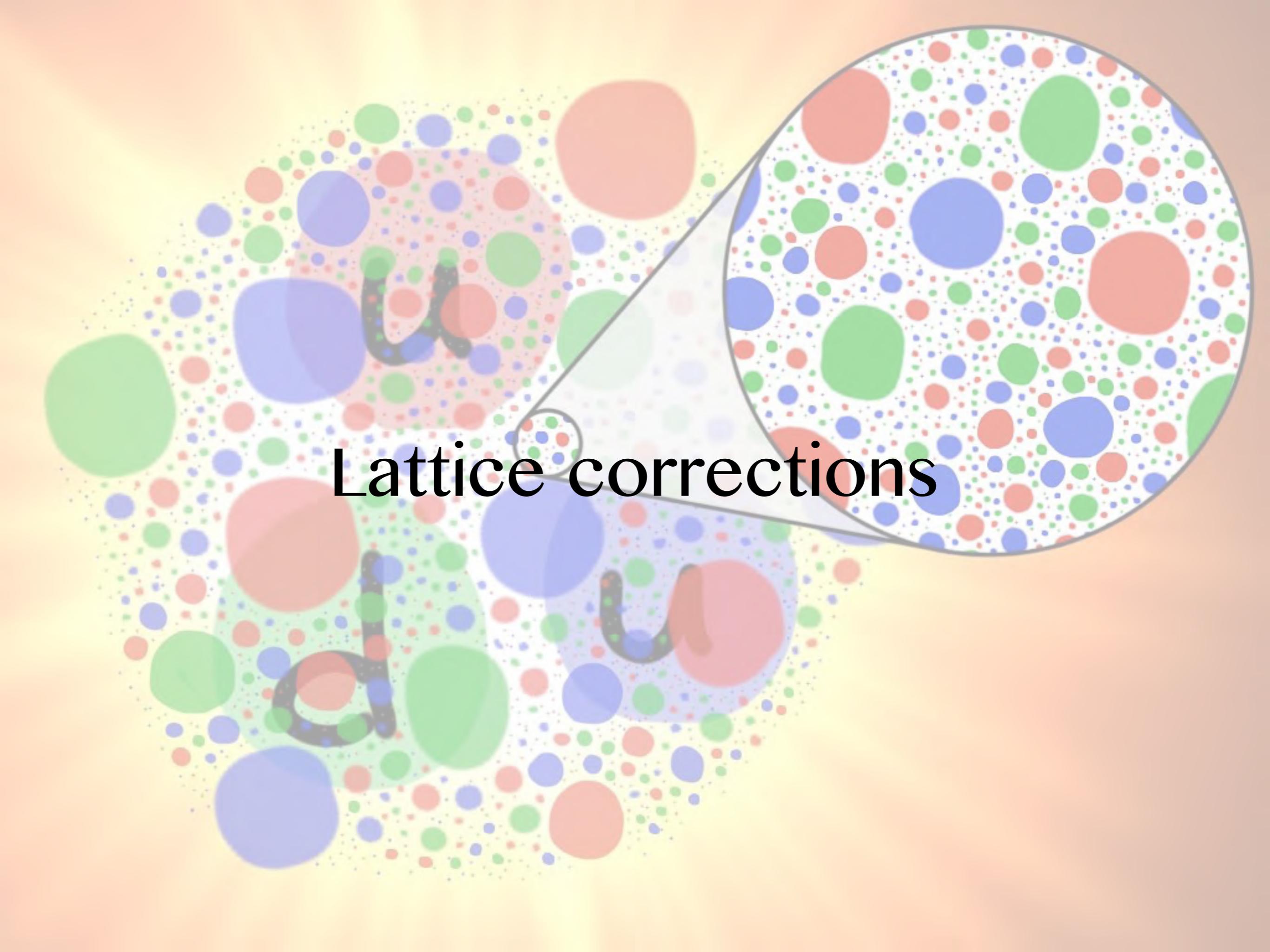
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Lattice corrections

Lattice corrections to a_μ

(1) Rescale Π_{ij} s by $(m_\rho^{\text{lat.}}/m_\rho^{\text{exp.}})^{2i}$ and calculate a_μ^{HVP}

- ❖ Use E_0 values from 2-point correlator fits, which are consistent with experiment and with estimated m_ρ values from NNLO χ PT fit of other lattice-QCD results based on dedicated calculations that include better operators, sources, etc...
- ❖ **Substantially reduces light-quark-mass dependence** (and chiral-continuum extrapolation error)

(2) Correct a_μ^{HVP} for finite-volume and taste-breaking discretization effects

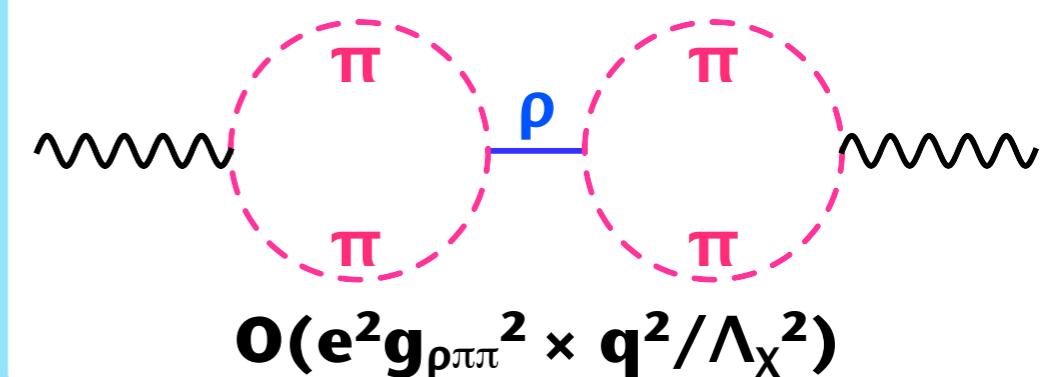
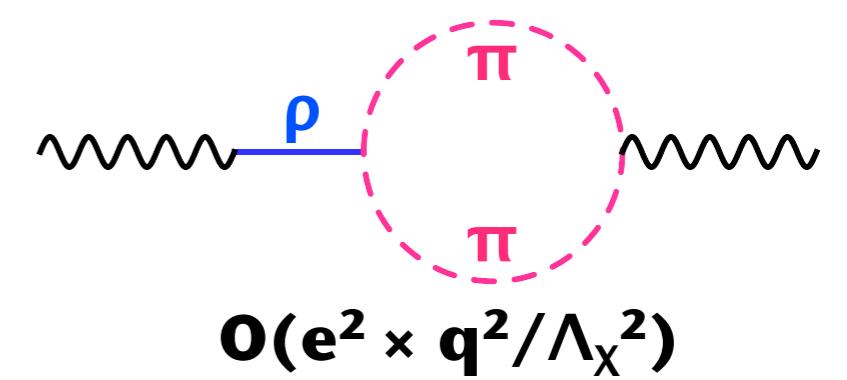
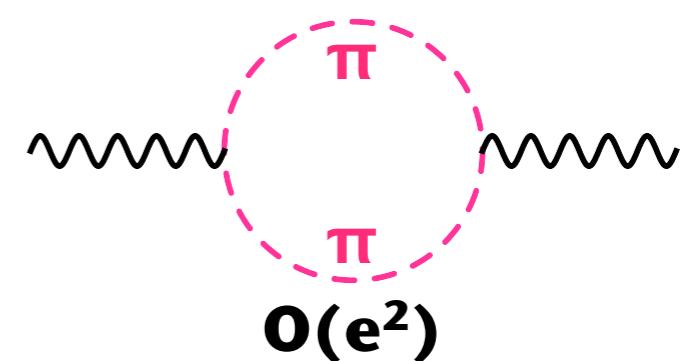
- ❖ Calculate corrections to 1-pion-loop order and $O(g^2)$ in scalar QED
- ❖ **Account for finite lattice spatial volume** by replacing momentum integral with sum (assume infinite temporal extent) and **account for taste breaking** by averaging contributions from all pion-taste pairings

(3) Subtract lattice (infinite-volume) $\gamma \rightarrow \pi^+ \pi^- \rightarrow \gamma$ vacuum polarization contribution and add back physical $\pi^+ \pi^-$ contribution to a_μ^{HVP} (also estimated in 1-loop scalar QED)

Calculation of lattice corrections

- ♦ Use extended chiral perturbation theory that includes π 's, ρ 's, and γ 's [Jegerlehner & Szafron, EPJC71 1632 (2011)]
 - ❖ Focus on pion-loop diagrams sensitive to spatial volume & sea-pion masses
 - ❖ Calculate contributions to a_μ^{HVP} to all orders in leading interactions that couple ρ^0 - γ - $\pi^+\pi^-$ channels
- ♦ Corrections given by difference between results in infinite volume / continuum, and in finite volume with lattice artifacts
 - ❖ Contribution from leading $\pi\pi$ bubble about 5× larger than from diagrams with ρ meson
 - ❖ Corrections largely from taste splittings, and decrease with lattice spacing
 - ❖ Take 10% uncertainty on corrections to account for higher-order terms in chiral expansion suppressed by $m_q/\Lambda_\chi, q^2/\Lambda_\chi^2$

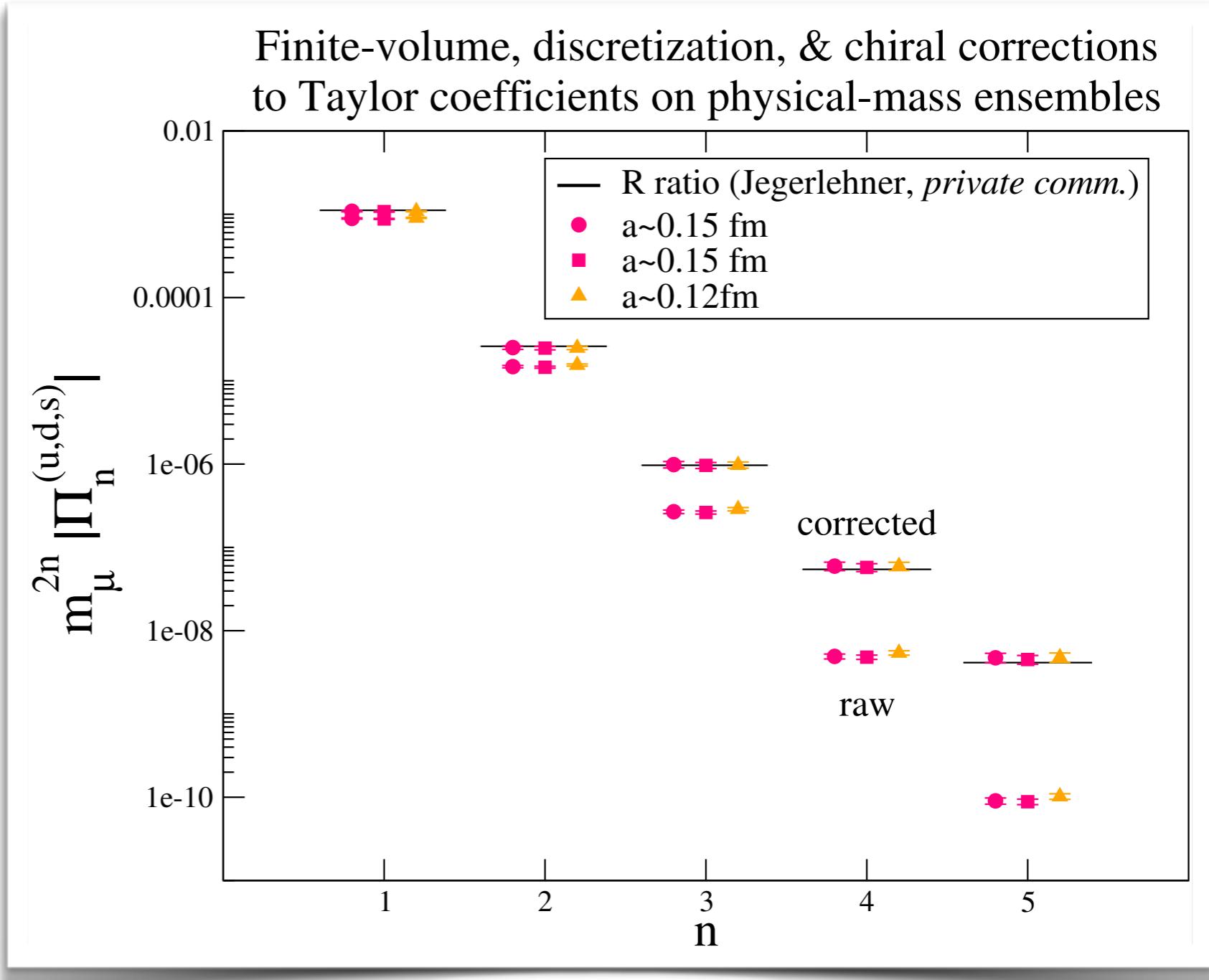
(diagrams below + all iterations of these diagrams)



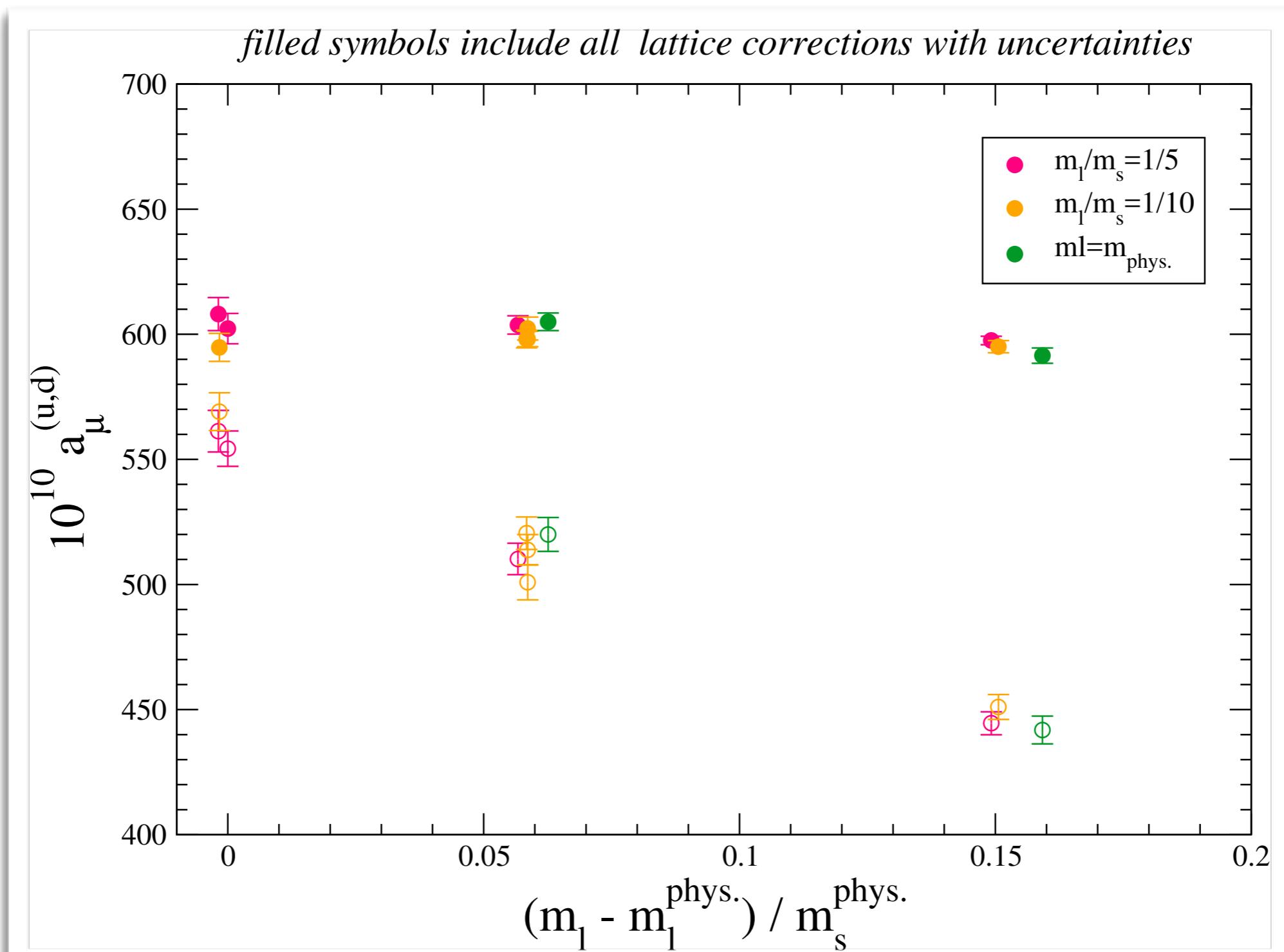
Test of lattice corrections

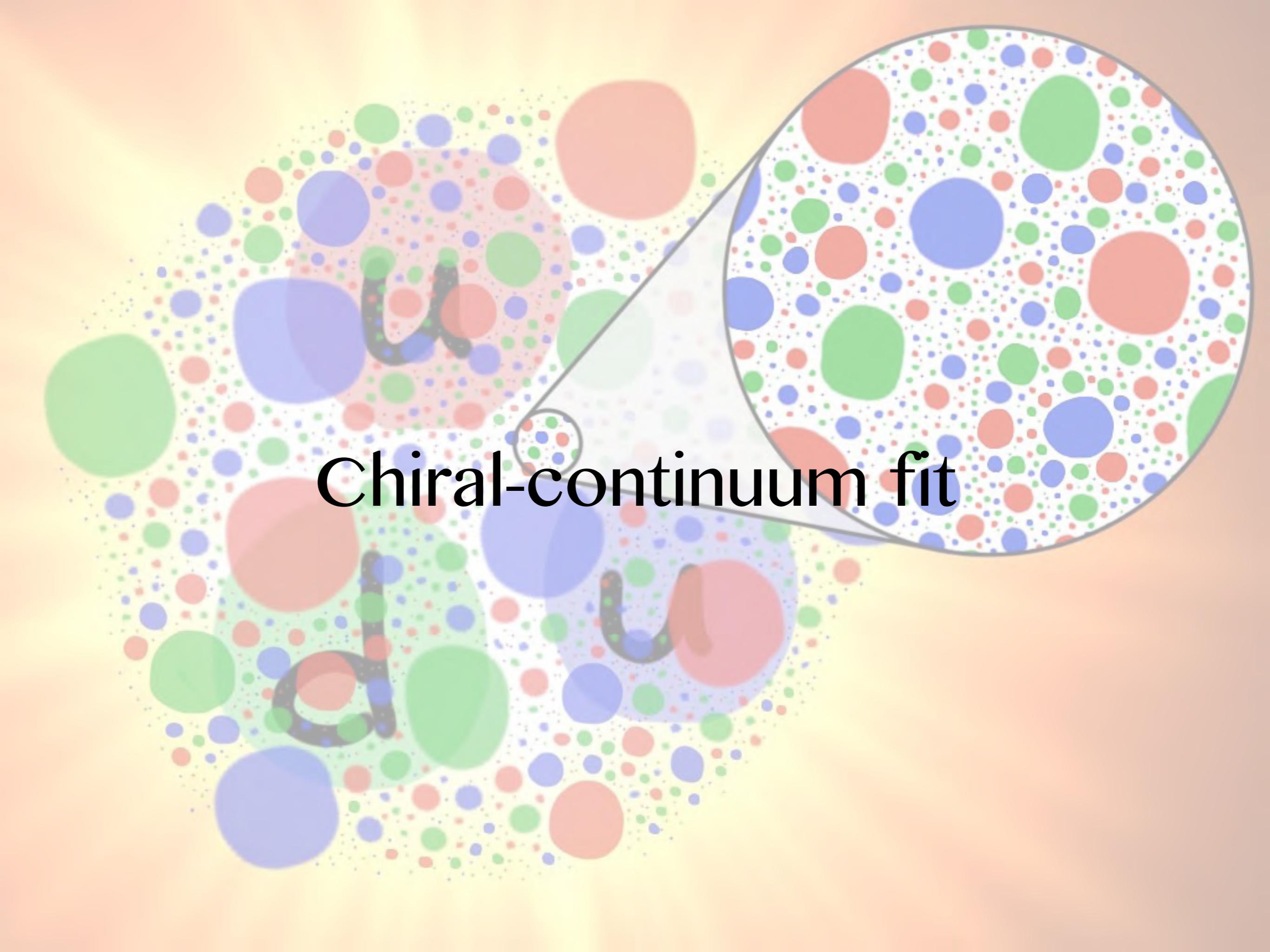
- ◆ Compare Taylor coefficients on physical-mass ensembles with values obtained from experimental $e^+e^- \rightarrow$ hadrons data
- ◆ Corrections bring sum of u/d/s/c-quark Π s into agreement with experiment

→ Evidence that scalar-QED calculation of finite-volume + discretization corrections is reliable ✓



m_l dependence after corrections





Chiral-continuum fit

Chiral-continuum fit

$$a_\mu = a_\mu^{\text{LO}} \times \left(1 + c_\ell \frac{\delta m_\ell}{\Lambda} + c_s \frac{\delta m_s}{\Lambda} + \tilde{c}_\ell \frac{\delta m_\ell}{m_\ell} + c_{a^2} \frac{(a\Lambda)^2}{\pi^2} \right)$$
$$\delta m_f \equiv m_f - m_f^{\text{phys.}}, \quad \Lambda = 500 \text{ MeV}$$

Constraints

$$a_\mu^{\text{LO}} = 600(200) \times 10^{-10}$$

$$c_\ell, c_{a^2} = 0(1), \quad c_s = 0.0(0.3), \quad \tilde{c}_\ell = 0.00(0.03)$$

Chiral-continuum fit

$$a_\mu = a_\mu^{\text{LO}} \times \left(1 + c_\ell \frac{\delta m_\ell}{\Lambda} + c_s \frac{\delta m_s}{\Lambda} + \tilde{c}_\ell \frac{\delta m_\ell}{m_\ell} + c_{a^2} \frac{(a\Lambda)^2}{\pi^2} \right)$$
$$\delta m_f \equiv m_f - m_f^{\text{phys.}}, \quad \Lambda = 500 \text{ MeV}$$

- ◆ Correct for quark-mass mistuning
- ◆ c_s small because only enters through sea, and m_s well tuned

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$\delta m_f \equiv m_f - m_f^{\text{phys.}}, \quad \Lambda = 500 \text{ MeV}$

- ◆ Correct for quark-mass mistuning
- ◆ c_s small because only enters through sea, and m_s well tuned

- ◆ Correct for small residual quark-mass dependence after $\pi^+\pi^-$ correction

Constraints

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Chiral-continuum fit

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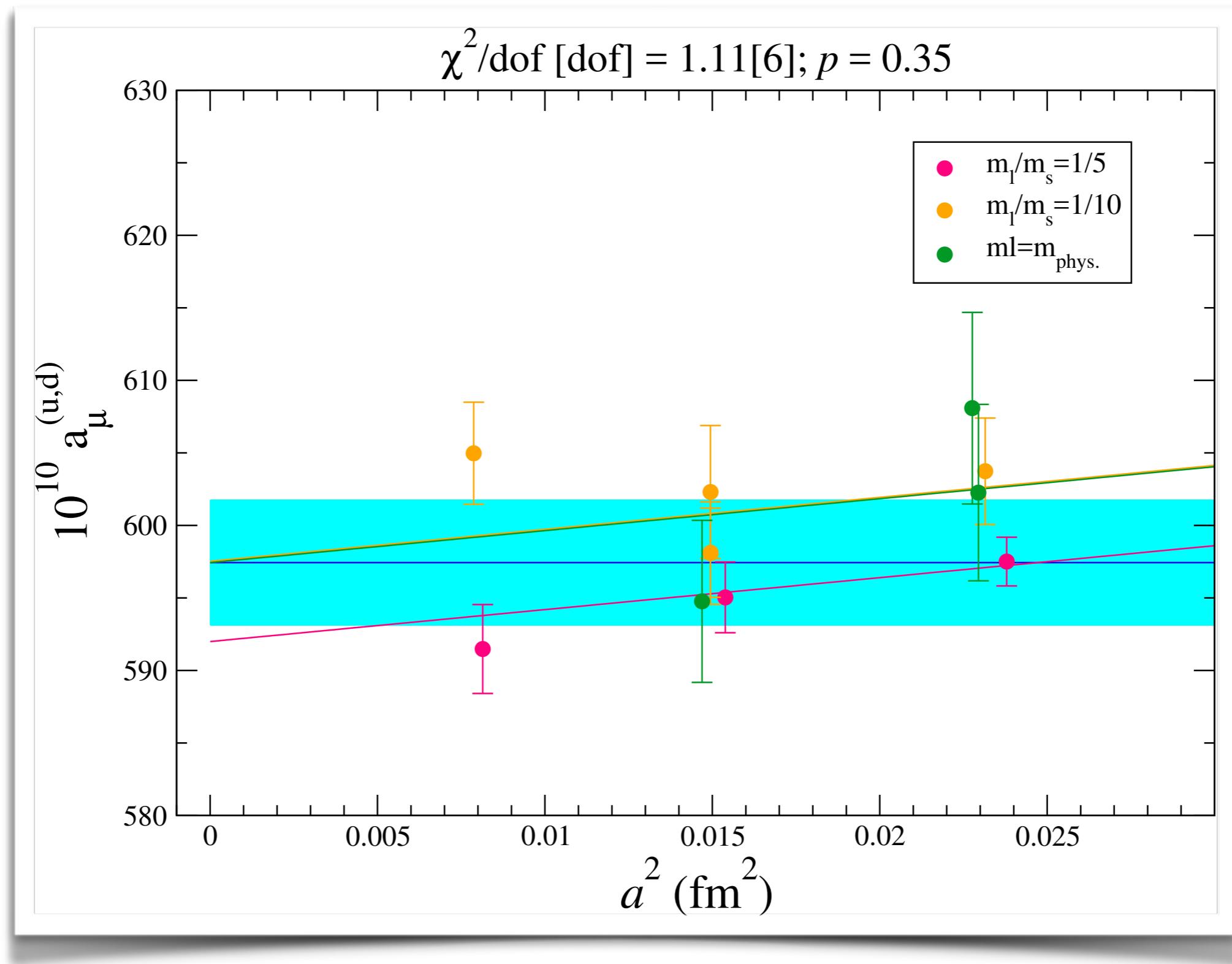
- ◆ Account for residual generic and taste-breaking discretization errors

Constraints

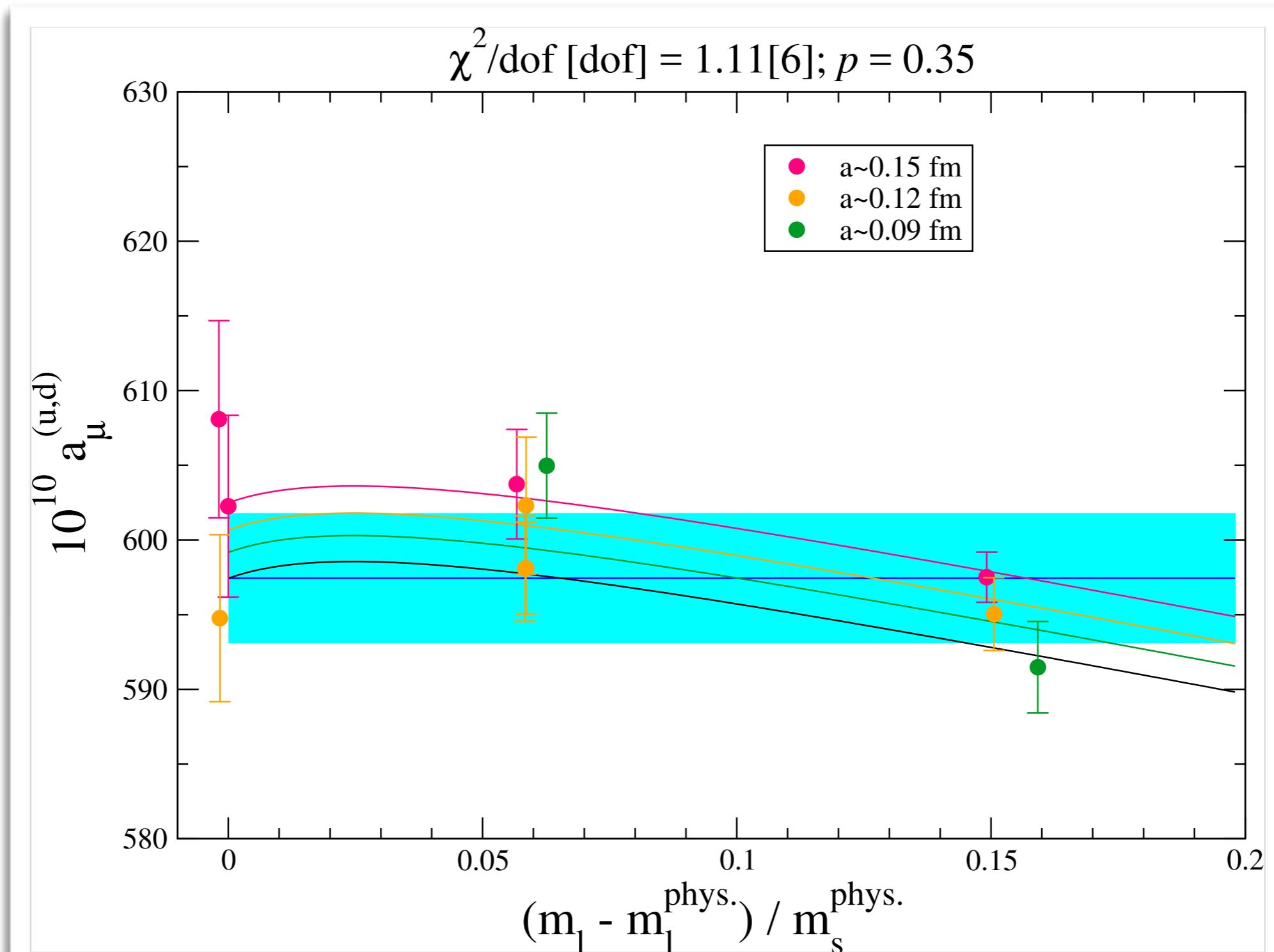
$$a_\mu^{\text{LO}} = 600(200) \times 10^{-10}$$

$$c_\ell, c_{a^2} = 0(1), \quad c_s = 0.0(0.3), \quad \tilde{c}_\ell = 0.00(0.03)$$

Continuum extrapolation

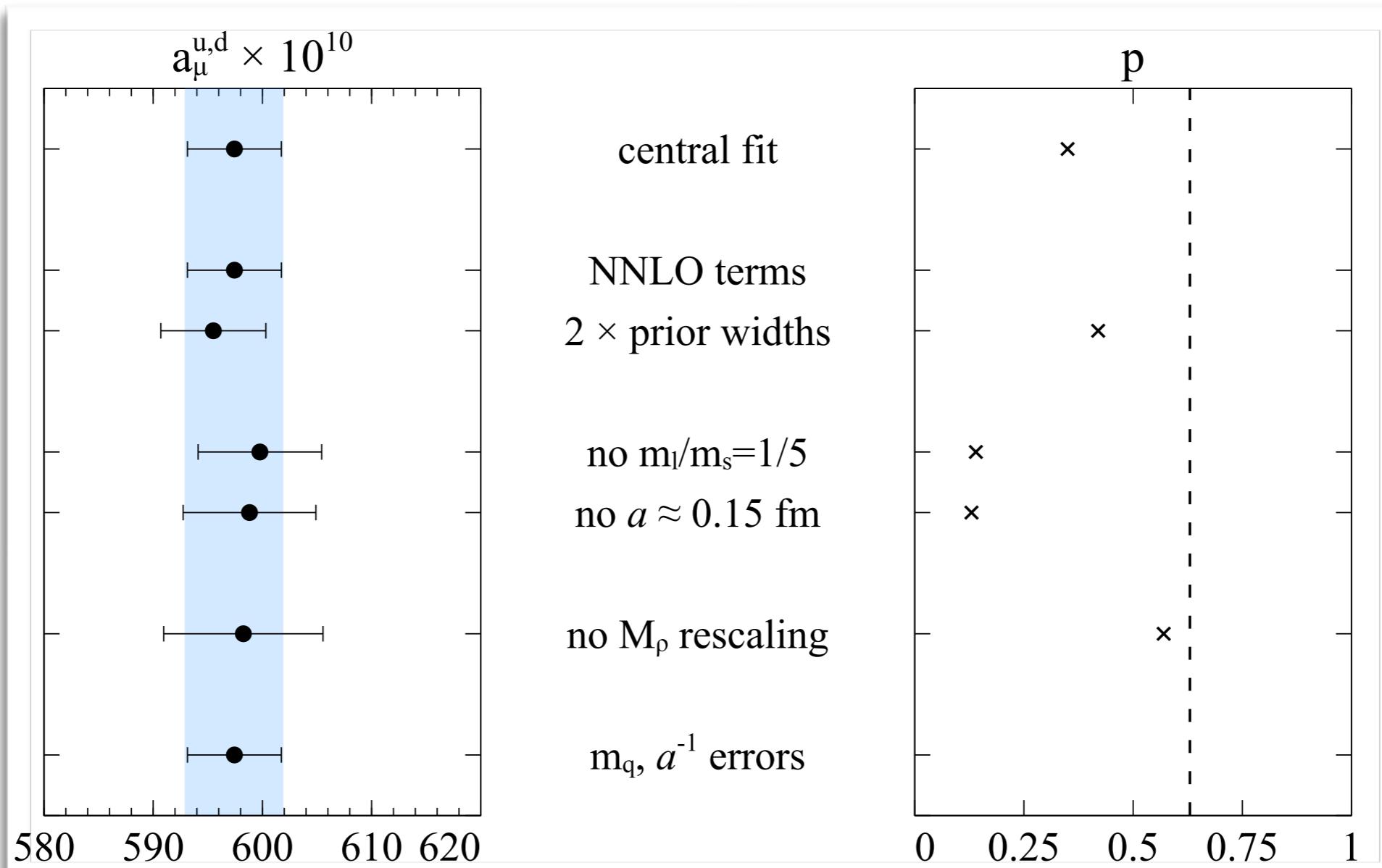


Light-quark-mass interpolation



Fit stability

★ Fit result stable with the addition of higher-order terms,
increased prior widths, omission of data, M_ρ rescaling, ...





Error budget & final result

Preliminary error budget

- ♦ Data-driven analyses & model calculations suggest QED & $m_u \neq m_d$ contributions are at or below $\sim 1\%$ [Cirigliano et al., JHEP 0208 (2002) 002, Hagiwara et al., PRD69, 093003 (2004), Wolfe & Maltman, PRD83 (2011) 077301] → take 1% for each

$a_\mu^{ud,\text{HVP}}$ uncertainty contribution	HPQCD 1601.03071	Fermilab/MILC + HPQCD 2017 preliminary
QED & isospin-breaking corrections	1.4	1.4
$\pi\pi$ states (t^*)	0.5	0.5
Statistics + 2pt fit	0.4	0.5
Finite-volume & discretization corrections	0.7	0.4
Chiral (m_l) extrapolation/interpolation	0.4	0.2
Continuum ($a \rightarrow 0$) extrapolation	0.2	0.2
Current renormalization (Z_V)	0.2	0.2
Pion mass ($M_{\pi,5}$) uncertainty	—	0.09
Sea (m_s) adjustment	0.2	0.08
Experimental M_ρ	—	0.06
Padé approximants	0.4	0.0
Lattice-spacing (a^{-1}) uncertainty	< 0.05	< 0.00
Total	1.8%	1.7%

Preliminary error budget

- ♦ Data-driven analyses & model calculations suggest QED & $m_u \neq m_d$ contributions are at or below ~1% [Cirigliano et al., JHEP 0208 (2002) 002, Hagiwara et al., PRD69, 093003 (2004), Wolfe & Maltman, PRD83 (2011) 077301] → take 1% for each

$a_{\mu}^{ud, \text{HVP}}$ uncertainty contribution	HPQCD 1601.03071	Fermilab/MILC + HPQCD 2017 preliminary
Apply FV correction as total shift to a_{μ} rather than individual moments	1.4 0.5 0.4	1.4 0.5 0.5
Statistics + ZPT fit	0.4	0.5
Finite-volume & discretization corrections	0.7	0.4
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Statistics + zpt fit	0.4	0.4
Finite-volume & discretization corrections	0.7	0.4
Chiral (m_l) extrapolation/interpolation	0.4	0.2
Continuum ($\kappa \rightarrow 0$) extrapolation	0.2	0.2
Additional physical-mass ensemble	0.2	0.2
Pion mass ($M_{\pi,5}$) uncertainty	—	0.09
Sea (m_s) adjustment	0.2	0.08
Experimental M_{ρ}	—	0.06
Padé approximants	0.4	0.0
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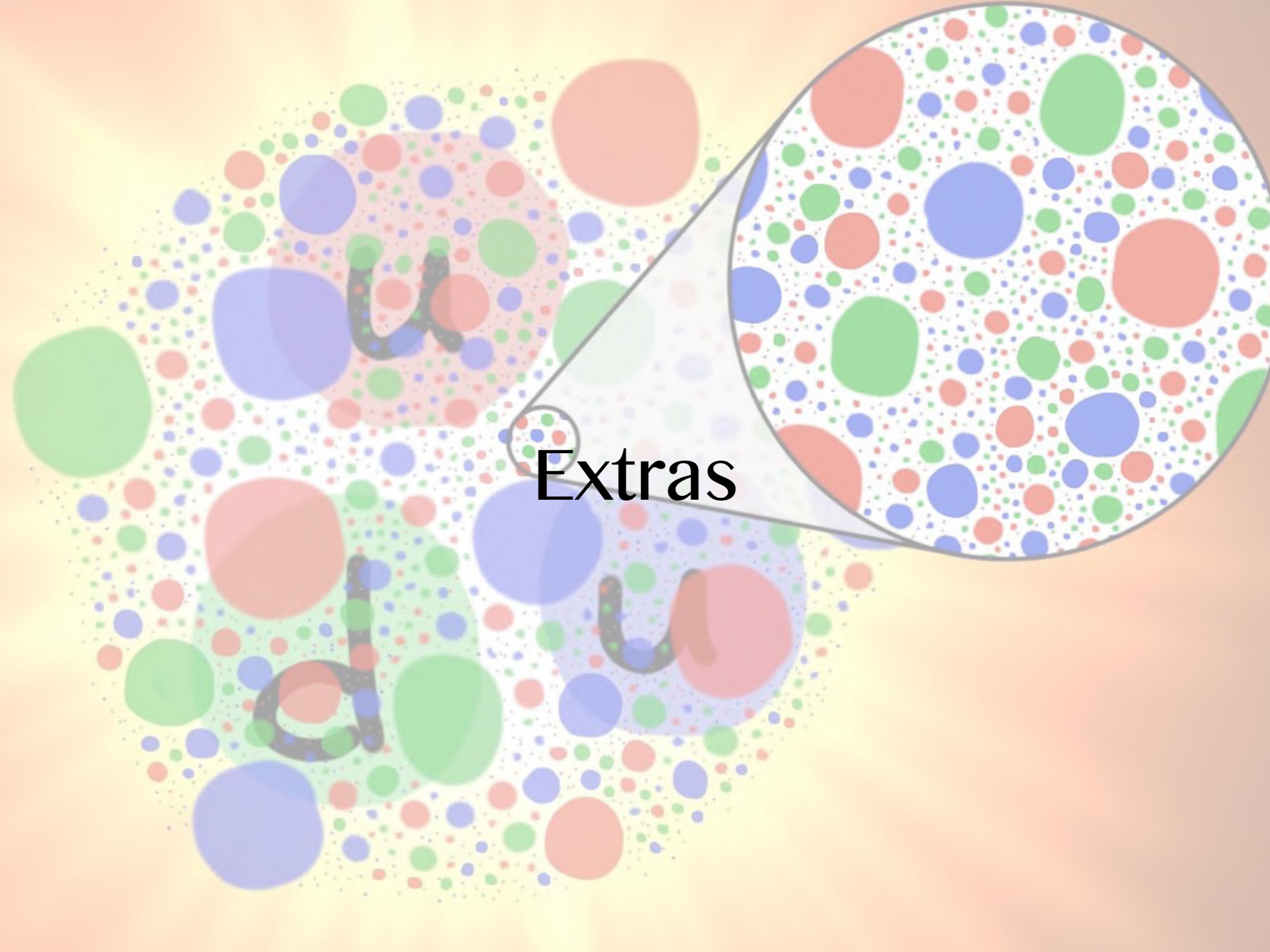
$a_{\mu}^{ud, \text{HVP}}$ uncertainty contribution	HPQCD 1601.03071	Fermilab/MILC + HPQCD 2017 preliminary
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Statistics + zpt fit	0.4	0.4
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Chiral (m_l) extrapolation/interpolation	0.4	0.2
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Pion mass ($M_{\pi,5}$) uncertainty	—	0.09
Sea (m_s) adjustment	0.2	0.08
Experimental M_{ρ}	—	0.06
Padé approximants	0.4	0.0
Total relative error (-1)	< 0.05	< 0.00
Use "time-momentum rep." / [3,3] Padés	1.8%	1.7%

Summary and outlook

- ♦ First preliminary analysis from joint Fermilab Lattice / HPQCD / MILC effort with additional ensemble and analysis improvements yields determination of light-quark connected contribution to a_μ^{HVP} with 1.7% precision
 - ❖ Tests for stability of results, comparison with experimental $e^+e^- \rightarrow \text{hadrons}$ data, and other consistency checks substantiate methodology and error estimates
- ♦ Current and proposed running focusing on reducing the leading sources of error in our result for $a_\mu^{\text{HVP,LO}}$ from
 - (1) Omission of isospin-breaking and electromagnetism,
 - (2) Omission of the quark-disconnected contribution, and
 - (3) Finite spatial volume and staggered discretization effects



With direct determinations of these corrections / contributions in hand, plus data at finer lattice spacings, expect to obtain sub-per cent precision in the coming 1 or 2 years!



Extras

Moments + Padé approximants

- ◆ “Time moments” method introduced by HPQCD in PRD89 (2014) no.11, 114501

(1) Calculate Taylor coefficients of vacuum polarization function

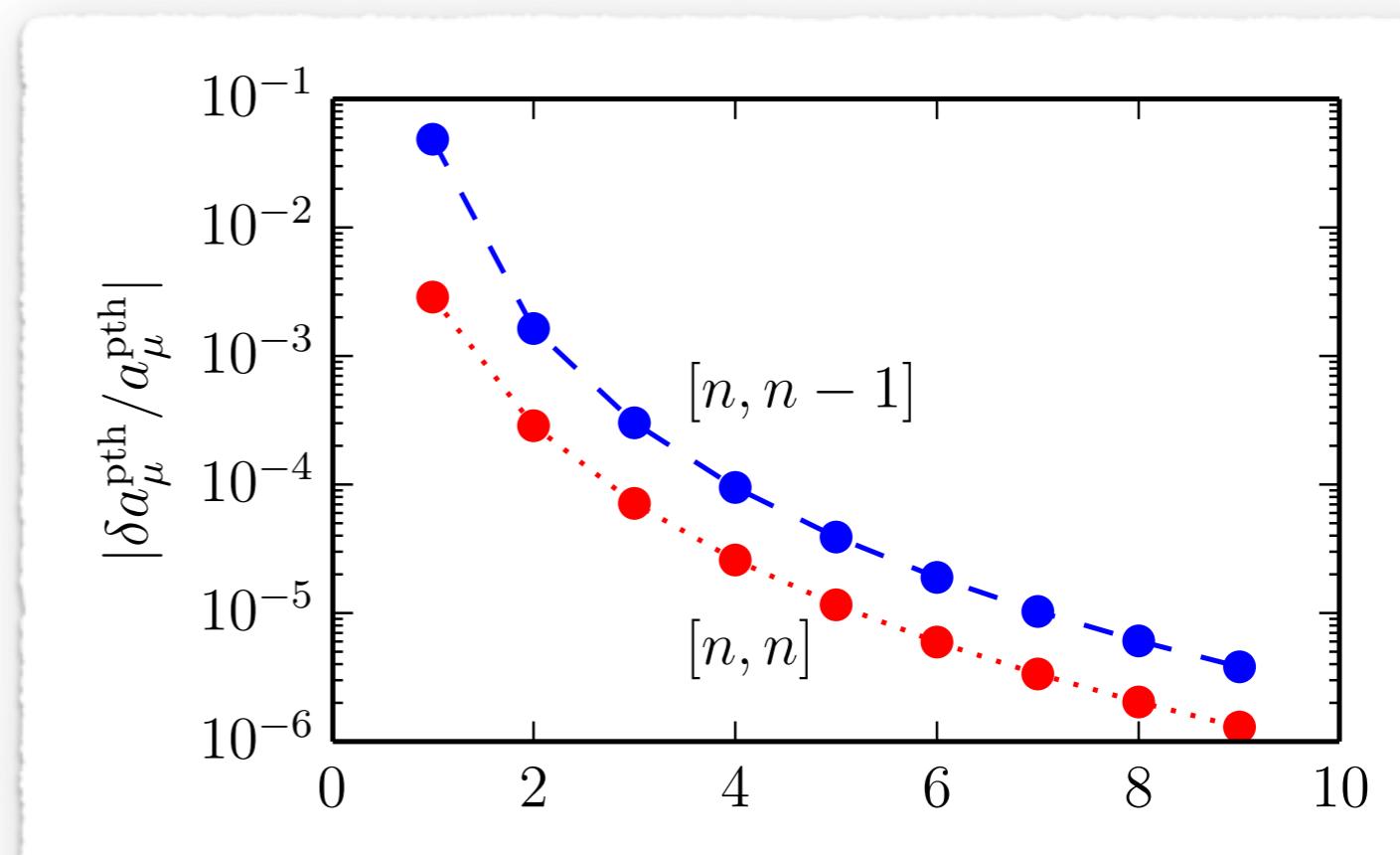
$\Pi(q^2)$ from time moments of vector current-current correlators

(2) Replace Taylor series for $\Pi(q^2)$ by its $[n,n]$ and $[n,n-1]$ Padé approximants to obtain the correct high- q^2 behavior

❖ Exact result always between $[n,n]$ and $[n,n-1]$ Padé

❖ $[2, 2]$ approximant sufficient to obtain $\sim 0.5\%$ precision

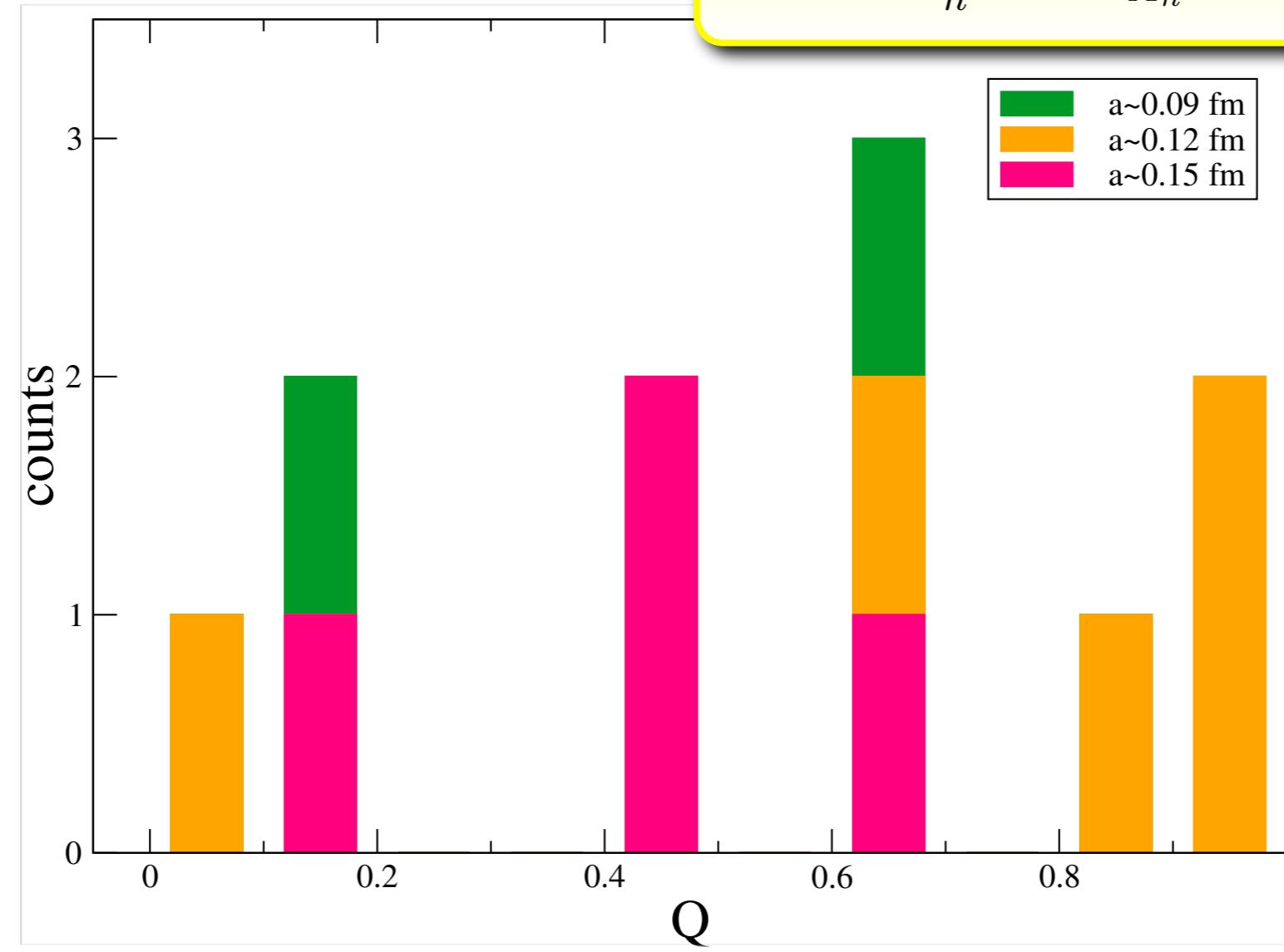
(3) Plug $\Pi(q^2)$ into standard 1-loop QED integral to obtain a_μ^{HVP}



Q-value distribution

- ◆ Constrained 2-point-correlator fits
minimize augmented $\chi^2_{\text{aug}} = \chi^2_{\text{data}} + \chi^2_{\text{prior}}$:

$$\chi^2_{\text{data}} = \sum_{t,t'} \Delta G(t) \sigma_{t,t'}^{-2} \Delta G(t')$$
$$\chi^2_{\text{prior}} = \sum_n \frac{(A_n - \tilde{A}_n)^2}{\tilde{\sigma}_{A_n}^2} + \sum_n \frac{(E_n - \tilde{E}_n)^2}{\tilde{\sigma}_{E_n}^2}$$



Summary of lattice corrections

Ensemble		$10^{10} a_\mu^{\text{HVP}}$	$+\pi^+\pi^-$	$+ \text{FV,disc}$
	raw	$+\text{rescaling}$		
l1648f211b580m01300m0650m838	444.5(4.6)	536.9(1.5)	590.6(1.5)	597.5(1.7)
l2448f211b580m0064m0640m828	510.2(6.3)	549.5(3.3)	588.3(3.3)	603.7(3.7)
l3248f211b580m00235m0647m831	561.3(8.3)	566.1(4.2)	559.8(4.3)	608.1(6.5)
l3248f211b580m002426m06730m8447	554.3(7.1)	559.7(3.5)	555.4(3.7)	602.3(6.0)
l2464f211b600m01020m0509m635	451.1(5.0)	536.5(2.4)	590.1(2.4)	595.0(2.4)
l2464f211b600m00507m0507m628	500.9(7.0)	549.3(4.4)	588.5(4.4)	602.3(4.6)
l3264f211b600m00507m0507m628	513.9(6.1)	547.1(2.8)	586.1(2.8)	598.1(3.1)
l4064f211b600m00507m0507m628	520.5(6.5)	547.9(3.3)	586.6(3.3)	598.1(3.5)
l4864f211b600m00184m0507m628	569.1(7.5)	560.2(3.4)	553.9(3.5)	594.8(5.5)
l3296f211b630m00740m0370m440	441.9(5.6)	533.9(3.0)	588.2(3.0)	591.5(3.1)
l4896f211b630m00363m0363m430	520.0(6.8)	558.4(3.4)	598.0(3.4)	605.0(3.5)

Finite-volume + discretization error

Ensemble	$10^{10} \times \Delta a_\mu$		
	$\pi\pi$	ρ	total
l1648f211b580m01300m0650m838	8.24	-1.33	6.92
l2448f211b580m0064m0640m828	18.61	-3.13	15.48
l3248f211b580m00235m0647m831	57.76	-9.46	48.29
l3248f211b580m002426m06730m8447	55.89	-9.04	46.85
l2464f211b600m01020m0509m635	5.93	-0.95	4.98
l2464f211b600m00507m0507m628	16.46	-2.7	13.76
l3264f211b600m00507m0507m628	14.46	-2.41	12.04
l4064f211b600m00507m0507m628	13.88	-2.36	11.51
l4864f211b600m00184m0507m628	48.8	-7.92	40.87
l3296f211b630m00740m0370m440	3.83	-0.6	3.23
l4896f211b630m00363m0363m430	8.42	-1.39	7.02

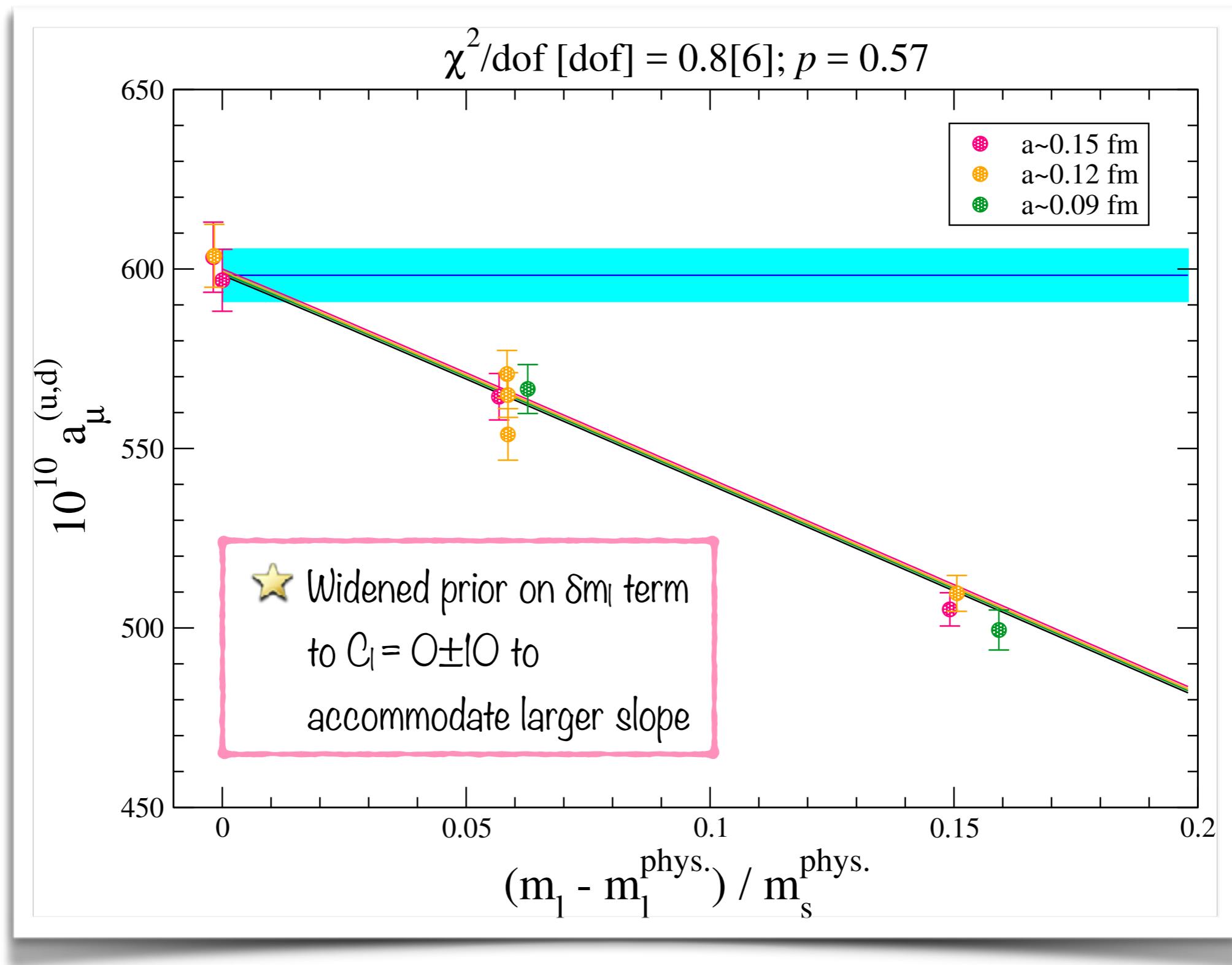
- ◆ Contributions from leading $\pi\pi$ bubble about 5× larger than from diagrams with ρ meson
- ◆ Finite-volume + discretization corrections largely from taste splittings between staggered pions in the sea, and become smaller as continuum limit is approached

$\pi\pi$ states/ t^* uncertainty

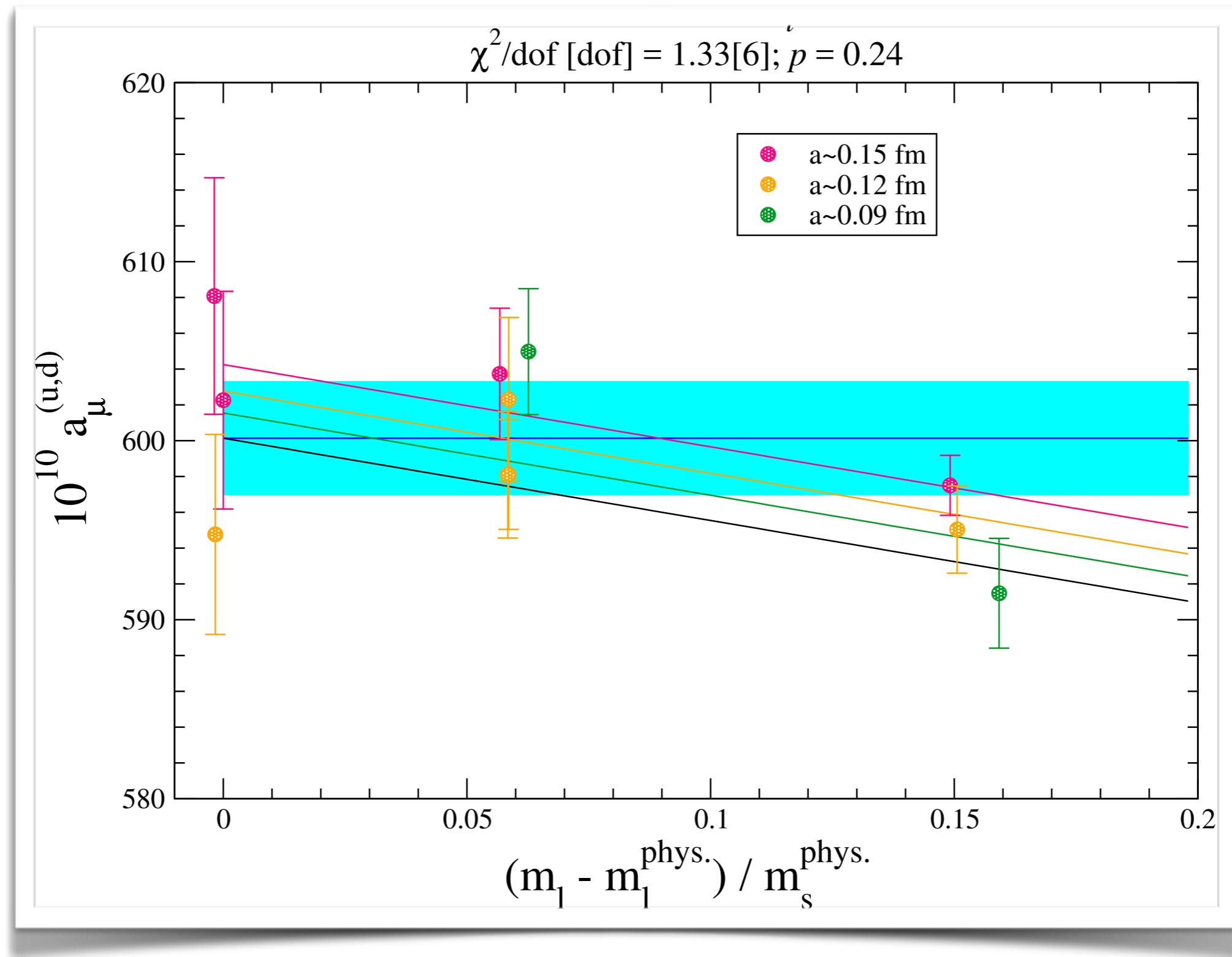
- ◆ Correlator on physical-mass ensembles has not decayed to asymptotic $t \rightarrow \infty$ $\pi\pi$ ground state by the lattice temporal extent
- ◆ Calculate low-energy $\pi\pi$ contribution to a_μ^{HVP} from $t > t^* = 1.5$ fm within chiral theory to be 3×10^{-10} and take as bound systematic error associated with $\pi\pi$ states below the p mass
 - ◆ Add estimated error in quadrature to physical a_μ^{HVP} after chiral-continuum fit
- ◆ Check estimate with data by changing t^* to 0.5 fm on physical-mass ensembles, which doubles the estimated $\pi\pi$ contribution in the chiral theory
- ◆ Observed shifts consistent with systematic error estimate

Ensemble	$t^* = 1.5$ fm	$t^* = 0.5$ fm	$ \Delta $
l3248f211b580m00235m0647m831	561.3(8.3)	560.2(8.1)	1.0(1.9)
l3248f211b580m002426m06730m8447	554.3(7.1)	553.4(6.9)	0.9(1.4)
l4896f211b630m00363m0363m430	520.0(6.8)	514.9(6.4)	5.1(2.0)

Fit without M_Q rescaling



Fit without $(m_\ell - m_\ell^{\text{phys}})/m_\ell$ term



QED & isospin-breaking errors

- ♦ Data-driven phenomenological analyses & model calculations suggest that contributions from QED & $m_u \neq m_d$ to the connected u/d contribution to leading-order a_μ^{HVP} are at or below the ~1% level → take 1% for each and add in quadrature

(1) QED

- Dominant EM effect from $\pi^0\gamma$ vacuum polarization bubbles estimated to be $\Delta(a_\mu^{\text{HVP}}) = 4.6(2) \times 10^{-10}$ from fit of $e^+e^- \rightarrow \pi^0\gamma$ data gives [[Hagiwara et al., PRD69, 093003 \(2004\)](#)]

(2) Isospin breaking

- Dominant isospin-breaking contribution from $\rho\omega$ mixing estimated to be $\Delta(a_\mu^{\text{HVP}}) \sim 2-5 \times 10^{-10}$ from fit of $e^+e^- \rightarrow \pi^+\pi^-$ data, where range comes from spread of models [[Wolfe & Maltman, Phys.Rev. D83 \(2011\) 077301](#)]
- χ PT estimate for contributions from $\rho\omega$ mixing and $\rho^+ - \rho^0$ width difference $\Delta(a_\mu^{\text{HVP}}) \sim 6 \times 10^{-10}$ is consistent [[Cirigliano, Ecker, Neufeld, JHEP 0208 \(2002\) 002](#)]