Hadronic light-by-light scattering in $(g-2)_\mu$ from Lattice QCD

Harvey Meyer

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PRiSMA
Cluster of Excellence

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Institute for Nuclear Physics
Helmholtz Institute Mainz

DFG Deutsche Forschungsgemeinschaft

References & list of coauthors

A) Analytic calculations: QED kernel, continuum $\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$ (π_0 with VMD TFF, lepton loop)

- N. Asmussen, J. Green, HM, A. Nyffeler

B) Implementation on the lattice:

- N. Asmussen, A. Gérardin, J. Green, G. von Hippel, HM, A. Nyffeler, H. Wittig.

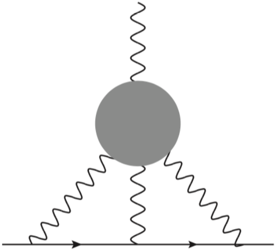
I *Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment on the Lattice*, Talk by N. Asmussen at the DPG meeting, Heidelberg, March 24, 2015

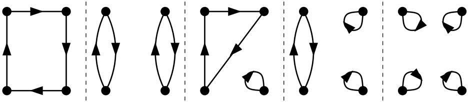
II Lattice 2015 proceedings of J. Green arXiv:1510.08384

III Lattice 2016 proceedings of N. Asmussen arXiv:1609.08454.

IV A. Nyffeler, talk at Workshop (KEK, Japan, Nov. 2016)

Euclidean coordinate-space approach to $a_\mu^{\rm HLbL}$





Quark Wick contraction topologies

$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\int d^4x \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm QCD}\Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\; z_\rho \Big\langle\, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0)\Big\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no finite-volume effects from the photons & affordable way (1d integral) to sample the integrand for the fully connected contribution.

HLbL Master Formula

master formula

$$a_\mu = F_2(0) = \frac{me^6}{3} \int_y \int_x \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$

four-point function

$$i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int_z z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

kernel function

$\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$

- weights the position-space vertex
- shall be precomputed and stored

From [I].

Notation: perturbation theory in Euclidean coordinate-space

Scalar propagators:

$$G_0(x) = \frac{1}{4\pi^2 x^2}, \qquad G_m(x) = \frac{m}{4\pi^2 |x|} K_1(m|x|).$$

Fermion propagator:

$$S(x) \equiv \int \frac{d^4p}{(2\pi)^4} \; \frac{-ip_\mu \gamma_\mu + m}{p^2 + m^2} \; e^{ipx} = \frac{m^2}{4\pi^2 |x|} \Big[\gamma_\mu x_\mu \, \frac{K_2(m|x|)}{|x|} + K_1(m|x|) \Big],$$

$U_n(z) =$ Chebyshev polynomials of the second kind:

$$U_0(z) = 1, \qquad U_1(z) = 2z, \qquad U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z) \qquad (n \geq 1),$$

Key property: orthogonal basis on S_3; if $\hat{\epsilon}$ is a unit vector

$$\Big\langle U_n(\hat{\epsilon} \cdot \hat{x}) \; U_m(\hat{\epsilon} \cdot \hat{y}) \Big\rangle_{\hat{\epsilon}} = \frac{\delta_{nm}}{n+1} \, U_n(\hat{x} \cdot \hat{y}).$$

Explicit form of the QED kernel

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} T^{(A)}_{\alpha\beta\delta}(x,y),$$

with e.g.

$$\mathcal{G}^{\mathrm{I}}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} \equiv \frac{1}{8}\mathrm{Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma] + 2(\delta_{\delta\sigma}\gamma_\rho - \delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},$$

$$\begin{aligned}
T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha(\partial^{(x)}_\beta + \partial^{(y)}_\beta)V_\delta(x,y),\\
T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\Big)\\
T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta)\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\Big),
\end{aligned}$$

$$\begin{aligned}
S(x,y) &= \bar{g}^{(0)}(|x|,\hat{x}\cdot\hat{y},|y|),\\
V_\delta(x,y) &= x_\delta\bar{\mathfrak{g}}^{(1)}(|x|,\hat{x}\cdot\hat{y},|y|) + y_\delta\bar{\mathfrak{g}}^{(2)}(|x|,\hat{x}\cdot\hat{y},|y|),\\
T_{\alpha\beta}(x,y) &= (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}.
\end{aligned}$$

The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six weight functions.

Sketch of the derivation

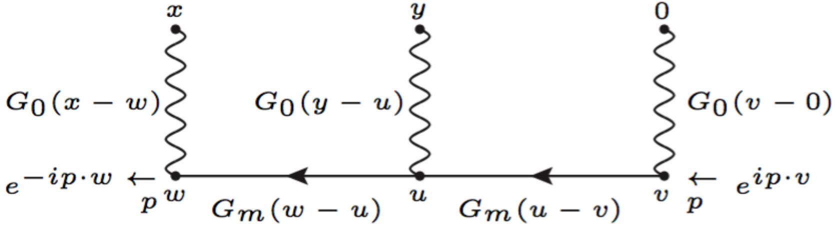
$$\hat{F}_2(0) = -\frac{i}{48m}\mathrm{Tr}\left\{[\gamma_\rho,\gamma_\tau]\,(-i\not{p}+m)\Gamma_{\rho\tau}(p,p)(-i\not{p}+m)\right\},$$

$$\Gamma_{\rho\sigma}(p,p) = -e^6\int_{x_1,x_2} K_{\mu\nu\lambda}(x_1,x_2,p)\,\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x_1,x_2),$$

$$K_{\mu\nu\lambda}(x_1,x_2,p) = \gamma_\mu(i\not{p}+\not{\partial}^{(x_1)}-m)\gamma_\nu(i\not{p}+\not{\partial}^{(x_1)}+\not{\partial}^{(x_2)}-m)\gamma_\lambda\,\mathcal{I}(\hat{\epsilon},x_1,x_2),$$

$$\mathcal{I}(\hat{\epsilon},x,y) = \int_{q,k}\frac{1}{q^2k^2(q+k)^2}\,\frac{1}{(p-q)^2+m^2}\,\frac{1}{(p-q-k)^2+m^2}e^{-i(qx+ky)}.$$

With $p = im\hat{\epsilon}$. From [III]. Diagrammatic representation of $\mathcal{I}(\hat{\epsilon},x,y)$:



The scalar function $\mathcal{I}(\hat{\epsilon}, x, y)$

$$\mathcal{I}(\hat{\epsilon}, x, y) = \int_u G_0(u-y)\; J(\hat{\epsilon}, u)\; J(\hat{\epsilon}, x-u),$$

$$J(\hat{\epsilon}, y) \quad \equiv \quad \int_u G_0(y-u)\, e^{m\hat{\epsilon}\cdot u} G_m(u) = \sum_{n\geq 0} z_n(y^2)\; U_n(\hat{\epsilon}\cdot\hat{y}),$$

$$\begin{aligned} S(x,y) &= \int_u G_0(u-y) s(x,u), \qquad\qquad \text{(IR regulated)} \\ V_\delta(x,y) &= \int_u G_0(u-y) v_\delta(x,u), \\ T_{\beta\delta}(x,y) &= \int_u G_0(u-y) t_{\beta\delta}(x,u), \end{aligned}$$

where

$$\begin{aligned} s(x,u) &= \Big\langle J(\hat{\epsilon},u) J(\hat{\epsilon}, x-u) \Big\rangle_{\hat{\epsilon}} = \sum_{n=0}^{\infty} z_n(u^2) z_n((x-u)^2)\; \frac{U_n(\hat{u}\cdot\widehat{x-u})}{n+1}, \\ v_\delta(x,u) &= \Big\langle \epsilon_\delta\; J(\hat{\epsilon},u) J(\hat{\epsilon}, x-u) \Big\rangle_{\hat{\epsilon}} = \dots \\ t_{\beta\delta}(x,u) &= \Big\langle \Big(\hat{\epsilon}_\delta \hat{\epsilon}_\beta - \frac{1}{4}\delta_{\beta\delta}\Big) J(\hat{\epsilon},u) J(\hat{\epsilon}, x-u) \Big\rangle_{\hat{\epsilon}} = \dots \end{aligned}$$

Example: Form Factor $\mathfrak{g}_2$

$$\bar{g}^{(2)}(x^2, x\cdot y, y^2) = \frac{1}{8\pi y^2 |x| \sin^3\beta} \int_0^\infty du\, u^2 \int_0^\pi d\phi_1$$

$$\left\{ 2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1 \right) \frac{Log}{\sin\phi_1} \right\} \sum_{n=0}^{\infty}$$

$$\Big\{ z_n(|u|) z_{n+1}(|x-u|) \left[|x-u|\cos\phi_1 \frac{C_n}{n+1} + (|u|\cos\phi_1 - |x|)\frac{C_{n+1}}{n+2} \right]$$

$$+ z_{n+1}(|u|) z_n(|x-u|) \left[(|u|\cos\phi_1 - |x|)\frac{C_n}{n+1} + |x-u|\cos\phi_1 \frac{C_{n+1}}{n+2} \right] \Big\}$$

where

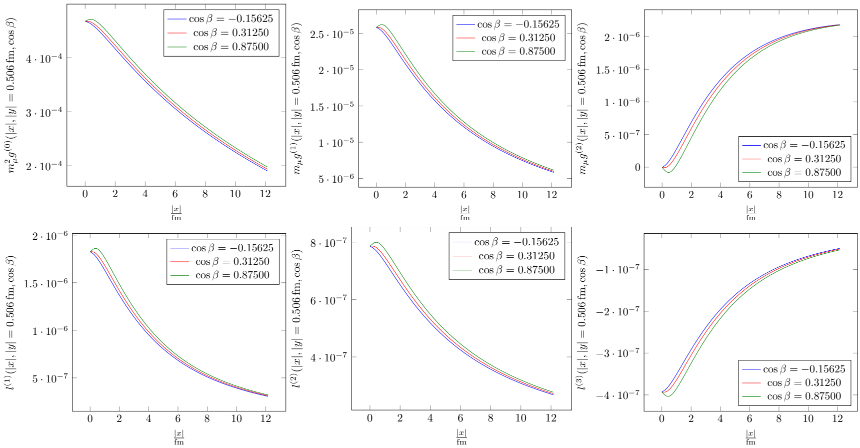
$$x\cdot y = |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1}$$

$$Log = \log\frac{y^2+u^2-2|u||y|\cos(\beta-\phi)}{y^2+u^2-2|u||y|\cos(\beta+\phi)}\,, \quad C_n = C_n^{(1)}\left(\frac{|x|\cos\phi_1 - |u|}{|u-x|}\right)$$

$z_n =$linear combination of products of two modified Bessel functions.

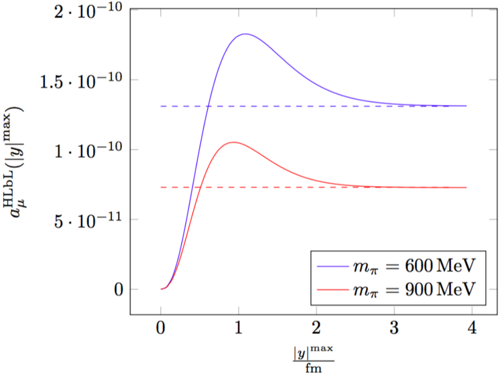
From [I]. Reminder: $V_\delta(x,y) = x_\delta \bar{\mathfrak{g}}^{(1)}(|x|, \hat{x}\cdot\hat{y}, |y|) + y_\delta \bar{\mathfrak{g}}^{(2)}(|x|, \hat{x}\cdot\hat{y}, |y|)$.

Complete set of weight functions: $|x|$ dependence



$\bar{\mathfrak{g}}^{(0)}(|x|, \hat{x} \cdot \hat{y}, |y|)$ contains an arbitrary additive constant (due to the IR divergence in $I(\hat{\epsilon}, x, y)$), which does not contribute to $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y)$.

Tests: contribution of the π^0 to $a_\mu^{\rm HLbL}$



From [III]. Dashed line = result from momentum-space integration (Knecht & Nyffeler PRD65, 073034 (2001)).

- Contribution is perhaps surprisingly long-range.
- The observation depends to some extent on the precise QED kernel used.
- The integrand can be roughly described by the form $(c_1|y|^4 + c_2|y|^2)e^{-m_\pi|y|}$.

Long-distance behaviour of the QED kernel

$$S(x,y) = \int_u G_{m_\gamma}(u-y) s(x,u), \qquad \text{(send } m_\gamma \to 0 \text{ later)}$$

i.e. $-\Delta_y S(x,y) = s(x,y)$. From here, now setting $m_\gamma = 0$,

$$\begin{aligned} -\Delta_y S(x\to\infty, y\to 0) &= -\frac{1}{32\pi^4 m|x|}(-1+\gamma_{\rm E}+\log(m|y|/2)), \\ -\Delta_y S(x=0, y\to\infty) &= \frac{1}{192\pi^2 m^2 y^2}, \end{aligned}$$

More explicitly: the π^0 pole contribution

Assume a vector-meson-dominance transition form factor (parameters: m_V, m_π and overall normalization)

$$\mathcal{F}(-q_1^2,-q_2^2) = \frac{c}{(q_1^2+m_V^2)(q_2^2+m_V^2)}, \qquad c = -\frac{N_c m_V^4}{12\pi^2 F_\pi}.$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \frac{c^2}{m_V^2(m_V^2-m_\pi^2)}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial y_\beta}\Big\{\epsilon_{\mu\nu\alpha\beta}\epsilon_{\sigma\lambda\rho\gamma}\Big(\frac{\partial}{\partial x_\gamma}+\frac{\partial}{\partial y_\gamma}\Big)K_\pi(x,y)$$
$$+\epsilon_{\mu\lambda\alpha\beta}\epsilon_{\nu\sigma\gamma\rho}\frac{\partial}{\partial y_\gamma}K_\pi(y-x,y)+\epsilon_{\mu\sigma\alpha\rho}\epsilon_{\nu\lambda\beta\gamma}\frac{\partial}{\partial x_\gamma}K_\pi(x,x-y)\Big\}.$$

where

$$K_\pi(x,y) \equiv \int d^4u\Big(G_{m_\pi}(u)-G_{m_V}(u)\Big)G_{m_V}(x-u)G_{m_V}(y-u) = K_\pi(y,x).$$

The lepton loop (mass m): fully analytic result for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$

$$\begin{aligned} i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \;=\;& \widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \\ &+\widehat{\Pi}^{(1)}_{\rho;\nu\lambda\mu\sigma}(y-x,-x) + x_\rho\,\Pi^{(r,1)}_{\nu\lambda\mu\sigma}(y-x,-x) \\ &+\widehat{\Pi}^{(1)}_{\rho;\lambda\nu\mu\sigma}(-x,y-x) + x_\rho\,\Pi^{(r,1)}_{\lambda\nu\mu\sigma}(-x,y-x). \end{aligned}$$

$$\begin{aligned} &\Pi^{(r,1)}_{\mu\nu\lambda\sigma}(x,y) \\ &= 2\Big(\frac{m}{2\pi}\Big)^8\Big[\frac{(-x_\alpha)(x-y)_\beta\;K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot l_{\gamma\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\ &+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot p(|y|)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\ &+\frac{(-x_\alpha)(x-y)_\beta\;K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot p(|y|)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\ &+\frac{(-x_\alpha)\;K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\cdot q_\gamma(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\ &+\frac{(x-y)_\beta\;K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\cdot q_\gamma(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\ &+\frac{(-x_\alpha)\;K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\cdot q_\delta(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\ &+\frac{(x-y)_\beta\;K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\cdot q_\delta(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\ &+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot l_{\gamma\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big] \end{aligned}$$

The lepton loop (continued)

$$
\begin{aligned}
&\widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y)\\
&= 2\Big(\frac{m}{2\pi}\Big)^8 \Big[\frac{(-x_\alpha)(x-y)_\beta K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\}\\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\}\\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\}\\
&+\frac{(-x_\alpha)(x-y)_\beta\; K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\}\\
&+\frac{(-x_\alpha)\; K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\, h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\}\\
&+\frac{(x-y)_\beta\; K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\, h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\}\\
&+\frac{(-x_\alpha)\; K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\, \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\}\\
&+\frac{(x-y)_\beta\; K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\, \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}
$$

$$
l_{\gamma\delta}(y) = \frac{2\pi^2}{m^2}\Big(\hat{y}_\gamma\hat{y}_\delta\; K_2(m|y|) - \delta_{\gamma\delta}\,\frac{K_1(m|y|)}{m|y|}\Big),\quad h_{\rho\gamma}(y) = \frac{\pi^2}{m^3}\Big(\hat{y}_\gamma\hat{y}_\rho\; m|y|\; K_1(m|y|) - \delta_{\gamma\rho}K_0(m|y|)\Big),
$$

$$
\hat{f}_{\rho\delta}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\rho\hat{y}_\delta\; m|y|K_1(m|y|) + \delta_{\rho\delta}K_0(m|y|)\Big\}\qquad q_\gamma(y) = \frac{2\pi^2}{m^2}\;\hat{y}_\gamma\; K_1(m|y|),
$$

$$
f_{\rho\delta\gamma}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\gamma\hat{y}_\delta\hat{y}_\rho\; m|y|K_2(m|y|) + (\delta_{\rho\delta}\hat{y}_\gamma - \delta_{\gamma\rho}\hat{y}_\delta - \delta_{\gamma\delta}\hat{y}_\rho)\,K_1(m|y|)\Big\},\quad p(|y|) = \frac{2\pi^2}{m^2}\,K_0(m|y|).
$$

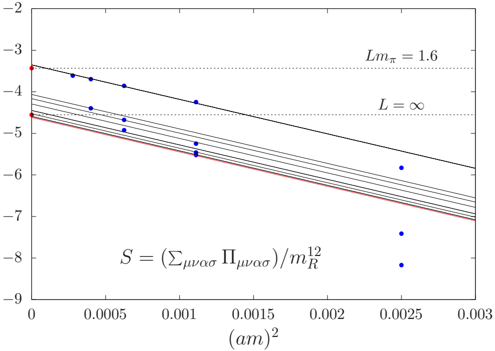
Part B) Implementation on the lattice

The vector four-point function for a free fermion:

- Wilson fermion action and local currents, $j_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$.
- Improvement and renormalization: $m = m_R = m_q\left(1 - \frac{1}{2}am_q\right)$, $Z_V = 1$, $b_V = 1$, $c_V = 0$. Discretization errors are $\mathrm{O}(a^2)$.
- Goal: compute $\Pi_{\mu\nu\alpha\sigma}(x, y, z)/m^{12}$ in the continuum, infinite-volume limit from the lattice.
- The fermion mass m sets the scale, tunable parameters are am and mL.
- Choice: L^4 lattices, $L/a = 32,\ 64,\ 80,\ 96$, $am = 0.017, \ldots, 0.050$, $mL = 1.6, \ldots, 4.8$.

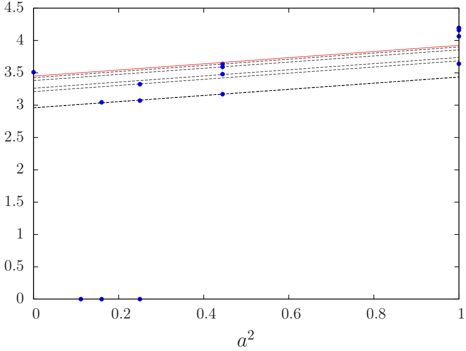
$$z = (0,0,0,d)\,,\ y = (0,0,d,d)\,,\ x = (0,0,d,0)\,, \quad d = \frac{1}{2m}\,.$$

The vector four-point function on the lattice



- $\Pi_{\mu\nu\alpha\sigma}(x, y, z) = Z_V^4 (1 + b_V\, am_q)^4\ \langle J_\mu(x) J_\nu(y) J_\alpha(z) J_\sigma(0)\rangle$
- Extrapolation used: $S(a, L) = S(0, \infty) + \alpha\, a^2 + \left(\beta + \frac{\gamma}{L}\right)\, e^{-2mL}$
- Lesson: $L/a = 32$ is insufficient to control both the infinite-volume and continuum extrapolation.

The integrated four-point function



$$-i\ \hat{\Pi}_{[\rho;\mu\nu\lambda\sigma]}(x,y) = \int \mathrm{d}^4 z\ z_{[\rho} \langle J_\mu(x) J_\nu(y) J_\lambda(0) J_{\sigma]}(z)\rangle\,,$$

- Goal: compute $-i\hat{\Pi}_{[\rho;\mu\nu\lambda\sigma]}(x,y)/m^7$ in the continuum and infinite-volume limit. Here: a linear combination of the components is taken.
- $x = (0, 0, d, -d)$ and $y = (0, 0, d, d)$, $d = \frac{1}{5m}$.

A variation on the strategy (in preparation)

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{B=1}^{36} \tau^B_{\rho\sigma\mu\nu\lambda}(x,y)\; \mathcal{F}^B(|x|, c_\beta, |y|)$$

$$a_\mu^{\rm HLbL} \quad = \quad \frac{me^6}{3} \sum_{B=1}^{36} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \qquad\qquad \underbrace{\int d^4x}_{=4\pi\int_0^\pi \sin^2(\beta)\int_0^\infty d|x|\,|x|^3}$$
$$\underbrace{\mathcal{F}^B(|x|, c_\beta, |y|)}_{\rm QED} \Big[\tau^B_{\rho\sigma\mu\nu\lambda}(x,y)\;\underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm QCD}\Big].$$

$$\begin{aligned}
\tau^1_{\rho\sigma\mu\nu\lambda}(x,y) &= \delta_{\mu\nu}(x_\rho\delta_{\sigma\lambda} - x_\sigma\delta_{\rho\lambda}),\\
\tau^2_{\rho\sigma\mu\nu\lambda}(x,y) &= \delta_{\mu\lambda}(x_\rho\delta_{\sigma\nu} - x_\sigma\delta_{\rho\nu}),\\
\tau^3_{\rho\sigma\mu\nu\lambda}(x,y) &= \delta_{\lambda\nu}(x_\rho\delta_{\sigma\mu} - x_\sigma\delta_{\rho\mu}),\\
\tau^4_{\rho\sigma\mu\nu\lambda}(x,y) &= (\delta_{\rho\nu}\delta_{\sigma\mu} - \delta_{\rho\mu}\delta_{\sigma\nu})x_\lambda,\\
\tau^5_{\rho\sigma\mu\nu\lambda}(x,y) &= (\delta_{\rho\lambda}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\lambda})x_\mu,\\
\tau^6_{\rho\sigma\mu\nu\lambda}(x,y) &= (\delta_{\rho\mu}\delta_{\sigma\lambda} - \delta_{\rho\lambda}\delta_{\sigma\mu})x_\nu, \qquad \ldots
\end{aligned}$$

Conclusion

- The covariant position-space method still looks like a promising approach. We plan to make the QED kernel publicly available.
- Tests of the QED kernel: π^0 pole, fermion loop.
- The π^0 contribution is very long-range, but with the π^0 contribution calculated, we hope to be able to correct for the finite-size effects on this contribution, by computing the transition form factor on the same ensemble (Antoine's talk).
- A parallel activity: analysis of the eight forward light-by-light scattering amplitudes, constraining the resonance transition form factors (with V. Pascalutsa; talk by A. Gérardin at Lattice 2017).