LO-HVP contribution to the muon $(g-2)$ from the Budapest-Marseille-Wuppertal collaboration

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HVP from LQCD: introduction

Consider in Euclidean spacetime (Blum '02)

$$\Pi_{\mu\nu}(Q) \quad =$$

$$= \quad \int d^4x \, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$$

$$= \quad \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2 \right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Then (Lautrup et al '69, Blum '02)

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\mu^2} w(Q^2/m_\mu^2) \hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$ & $w(Q^2/m_\mu^2)$ known fn that makes integrand peak for $Q^2 \sim (m_\mu/2)^2$

$\Rightarrow$ determine precisely

$\Pi_{\mu\nu}(Q)$ down to below $\sqrt{Q^2} \sim 50\,\text{MeV}$ $\longleftrightarrow$ $\langle J_\mu(x) J_\nu(0) \rangle$ up to above $\sqrt{x^2} \sim 4\,\text{fm}$

Low-Q^2 challenges in finite volume (FV)

A. In L^4, $Q_\mu \Pi_{\mu\nu}(Q) = 0$ does not imply $\Pi_{\mu\nu}(Q=0) = 0$

$$\Pi_{\mu\nu}(Q=0) = \int_\Omega d^4x \langle J_\mu(x) J_\nu(0)\rangle = \int_\Omega d^4x \partial_\rho [x_\mu \langle J_\rho(x) J_\nu(0)\rangle]$$

$$\int_{\partial\Omega} d^3x_\rho [x_\mu \langle J_\rho(x) J_\nu(0)\rangle] \propto L^4 \exp(-EL/2)$$

$\Rightarrow$ as $Q_\mu \to 0$, $\Pi(Q^2) = \Pi_{\mu\nu}(Q)/(Q_\mu Q_\nu - Q^2\delta_{\mu\nu})$ receives $1/Q^2$ enhanced FV effect

B. Particularly problematic, as need $\Pi(0)$ renormalization

C. Need $\hat{\Pi}(Q^2)$ interpolation because in $T \times L^3$, w/ $T \geq L$ and periodic BCs, have $Q_{\text{min}} = \frac{2\pi}{T} \sim 135\,\text{MeV} > \frac{m_\mu}{2} \sim 50\,\text{MeV}$ for $T \sim 9\,\text{fm}$

Dealing with low-Q^2 problems: ad A, B & C

- Compute on lattice

$$C(t) = \frac{1}{3}\sum_{i=1}^{3}\sum_{\vec{x}}\langle J_i(x)J_i(0)\rangle$$

- Decompose

$$\begin{aligned} C(t) &= C^{ud}(t) + C^{s}(t) + C^{c}(t) + C^{\text{disc}}(t) \\ &= C^{I=1}(t) + C^{I=0}(t) \end{aligned}$$

w/ $C^{I=1} = \frac{9}{10}C^{ud}$

- Define (Bernecker et al '11, BMWc '13, Lehner '14, ...) (ad A, B)

$$\hat{\Pi}^f(Q^2) \equiv \Pi^f(Q^2) - \Pi^f(0) = \frac{1}{3}\sum_{i=1}^{3}\frac{\Pi^f_{ii}(0)-\Pi^f_{ii}(Q)}{Q^2} - \Pi^f(0) = 2\sum_{t=0}^{T/2}\mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2}+\frac{t^2}{2}\right]\mathrm{Re}\,C^f(t)$$

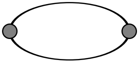
- Consider also for $Q \in \mathbb{R} \neq n\frac{2\pi}{T}$, $n \in \mathbb{Z}$ (RBC/UKQCD '15, ...)(ad C)

$\rightarrow$ gives $a_\mu^{\text{LO-HVP}}$ up to exponentially suppressed FV corrections

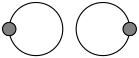
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ must have physically light π

E. Two contributions



quark-connected (qc)



quark-disconnected (qd)

where qd contributions are $SU(3)_f$ and Zweig suppressed but very challenging

F. $\langle J_\mu^{ud}(x) J_\nu^{ud}(0)\rangle_{\text{qc}}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ very high statistics + tricks

G. To control $\langle J_\mu(x) J_\nu(0)\rangle$ at $\sqrt{x^2} \gtrsim 3\,\mathrm{fm}$ $\rightarrow$ w/ periodic BCs need L and/or $T \gtrsim 6\,\mathrm{fm}$

H. Need controlled continuum limit

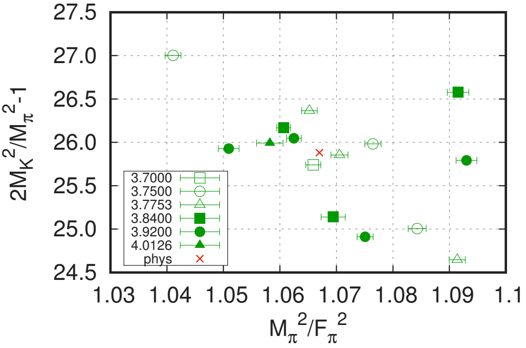
I. Include c quark for higher precision and good matching onto perturbation theory

Simulation details: ad D - I

15 high-statistics simulations w/ $N_f = 2+1+1$ flavors of 4-stout staggered quarks:

- Bracketing physical m_{ud}, m_s, m_c
- 6 a's: $0.134 \rightarrow 0.064\,\mathrm{fm}$
- $L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$
- Conserved EM current
- Close to 9M / 39M conn./disc. measurements

β	a [fm]	$T \times L$	#conf-conn	#conf-disc
3.7000	0.134	64 × 48	1000	1000
3.7500	0.118	96 × 56	1500	1500
3.7753	0.111	84 × 56	1500	1500
3.8400	0.095	96 × 64	2500	1500
3.9200	0.078	128 × 80	3500	1000
4.0126	0.064	144 × 96	450	-

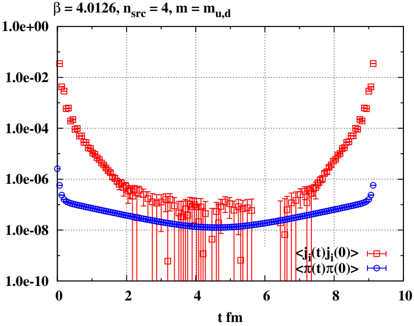


Light pions and statistics: ad D, E, F

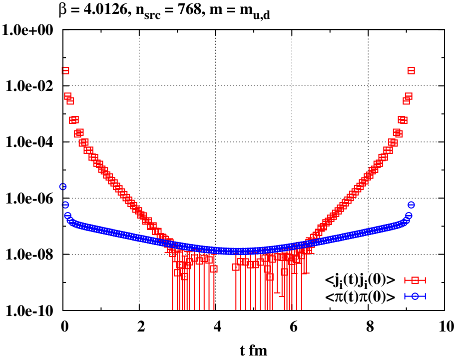
$\langle\pi(t)\pi(0)\rangle$ vs $\frac{81}{25}C^{ud}(t)$ as a function of t

m_{ud}, m_s, m_c physical, $a \simeq 0.064\,\mathrm{fm}$, $L = 96a \simeq 6.1\,\mathrm{fm}$, $T = 144a \simeq 9.2\,\mathrm{fm}$

Good stats: 4×441 meas.



For $\delta_{\mathrm{stat}} a_{\mu,\,ud}^{\text{LO-HVP}} \sim 1\%$: 768×441 meas.



→ noise/signal in $C^{ud/\mathrm{disc}}(t)$ grows exponentially w/ t

→ 768/64/4/6000 sources for $ud/s/c$/disc. w/ AMA (Blum et al '13)

→ Use approximate $SU(3)_f$ symmetry for noise cancellation in $C^{\mathrm{disc}}(t)$ (Francis et al '14)

Statistics and upper/lower bounds on $C^{ud/\text{disc}}(t)$: ad F

Signal lost for $t \gtrsim 3\,\text{fm}$ for $C^{ud/\text{disc}}(t)$

$\Rightarrow$ to control statistical error, consider strict upper and lower bounds for $t > t_c$:

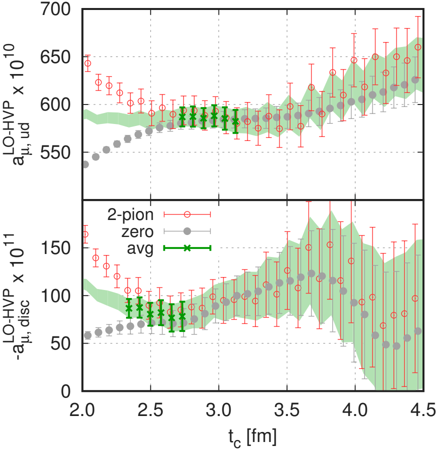
Connected $(I = 1)$

$$0 \leq C^{ud}(t) \leq C^{ud}(t_c)\,\frac{\varphi(t)}{\varphi(t_c)}$$

Disconnected $(I = 0,\ t_c$ large enough$)$

$$0 \leq -C^{\text{disc}}(t) \leq \frac{1}{10}\,C^{ud}(t_c)\,\frac{\varphi(t)}{\varphi(t_c)}$$

with $\varphi(t) = \cosh\left[E_{2\pi}(T/2 - t)\right]$, $E_{2\pi} \simeq 2\sqrt{M_\pi^2 + (2\pi/L)^2}$



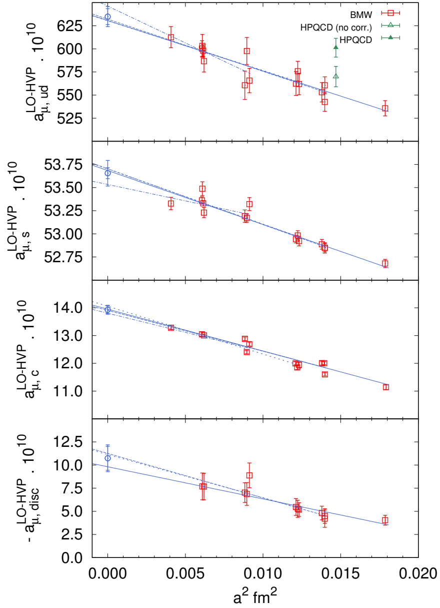
$\rightarrow$ for $t \geq t_c$ where bounds meet, replace $C^{ud/\text{disc}}(t)$ by average of bounds

$\rightarrow$ obtain $a_{\mu,\,ud/\text{disc}}^{\text{LO-HVP}}(Q \leq Q_{\text{max}})$ for each simulation & for $Q^2_{\text{max}} = 1, \cdots, 5$

$\rightarrow$ vary t_c for systematic

$\rightarrow$ $a_{\mu,\,s/c}^{\text{LO-HVP}}(Q \leq Q_{\text{max}})$ obtained directly w/out bounds

Continuum limit of $a_{\mu,f}^{\text{LO-HVP}}(Q^2 \leq 5\,\text{GeV}^2)$: ad H



- With 6 a's, have full control over continuum limit
- Get good χ^2/dof w/ extrapolation linear in a^2 and interpolations, linear in M_π^2 and M_K^2
- Strong continuum extrapolation for $a_{\mu,ud/\text{disc}}^{\text{LO-HVP}}$ due to taste violations and for $a_{\mu,c}^{\text{LO-HVP}}$ due to large m_c
- Get continuum systematic from all results and by cutting results with $a \geq 0.134,\ 0.111,\ 0.095\,\text{fm}$
- Obtain other $a_{\mu,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}})$ and $\hat{\Pi}(Q_{\text{max}}^2)$, $Q_{\text{max}}^2 = 1, \cdots, 5\,\text{GeV}^2$, in entirely analogous fashion

Hi Q^2 & matching challenges

J. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0,+\infty[$, but $\frac{\pi}{a} \sim 9.7\,\mathrm{GeV}$ for $a \sim 0.064\,\mathrm{fm}$

I. Include c quark for higher precision and good matching onto perturbation theory

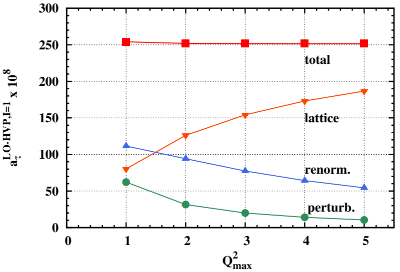
Matching to perturbation theory: ad I & J

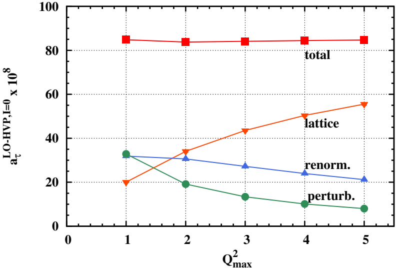
Consider separation ($\ell = e,\ \mu,\ \tau$)

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\ \hat{\Pi}^f(Q_{\text{max}}^2) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

- Compute $\Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$ using $R_{\text{pert}}(s)$ to $O(\alpha_s^4)$ from Harlander et al '03
- Not relevant for $\ell = e, \mu$ but important for τ
- Perfect matching of continuum lattice results for $Q_{\text{max}}^2 \geq 2\,\text{GeV}^2$

  $\rightarrow$ control $\hat{\Pi}(Q^2)$ up to $Q^2 \rightarrow \infty$
- Get matching systematic from considering $Q_{\text{max}}^2 = 2$ and $5\,\text{GeV}^2$





Finite-volume challenges

K. Even in our large volumes w/ $L \gtrsim 6.1\,\mathrm{fm}$ & $T \geq 8.7\,\mathrm{fm}$, finite-volume (FV) effects can be significant (Aubin et al '16)

Finite-volume effects from χPT: ad K

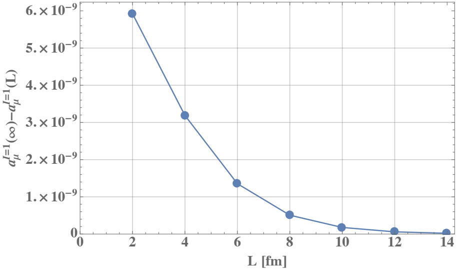
- HVP contribution to a_μ comes from Euclidean momenta
  $\Rightarrow$ FV effects are exponentially suppressed in L, T

- Because $L \gtrsim 6.1\,\mathrm{fm}$ and $T \geq 8.7\,\mathrm{fm}$ (i.e $LM_\pi \gtrsim 4.2$), expect them to be small

- However, work with $L \sim$ fixed
  $\Rightarrow$ FV effects cannot be estimated from simulations and need model

- Long-distance $I = 1$ contribution dominated by 2π and $I = 0$, by 3π
  $\Rightarrow$ dominant FV effects in $I = 1$ channel
  $\rightarrow$ these could be well described by $\pi^+\pi^-$ loop
  (Aubin et al '16)

- Plot: $\pi^+\pi^-$ loop contribution to $a_{\mu,\,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,\,I=1}^{\text{LO-HVP}}(L)$ computed numerically vs L w/ $T = 3L/2$



- Actually obtain $a_{\mu,\,I=1}^{\text{LO-HVP}}$ from $C^{I=1}(t)$ in χPT exactly as in lattice computation w/ bounds, t_c procedure, interpolation in Q^2 etc.

- That procedure gives for $L = 6\,\mathrm{fm}$, result very similar to above:
  $a_{\mu,\,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,\,I=1}^{\text{LO-HVP}}(L) = 13.4 \times 10^{-10}$
  $\Rightarrow +1.9\%$ correction to $a_\mu^{\text{LO-HVP}}(6\,\mathrm{fm})$

- Assign 100% error to this correction

QED & isospin breaking challenges

L. Our $N_f = 2+1+1$ calculation has $m_u = m_d$ and $\alpha = 0$

$\Rightarrow$ missing effects compared to HVP from dispersion relations that are relevant at %-level precision

Isospin breaking effects: ad L

Get missing effects from phenomenology

Effect	corr. to $a_\mu^{\text{LO-HVP}} \times 10^{10}$
$\rho-\omega$ mix.	2.71
$\rho-\gamma$ mix.	-2.74
FSR	4.22
EM in M_π, M_ρ, Γ_ρ	-11.17
$\pi^0\gamma$	4.64(4)
$\eta\gamma$	0.65(1)
Total	$-1.69(20)$

- Thanks to F.Jegerlehner (& M. Benayoun) for correspondance and numbers
- Results based on Gounaris-Sakurai fit to e^+e^-, from $2M_\pi$ to $1\,\text{GeV}$
- EM modes from M. Benayoun et al '12
- F.J. estimates error to $\sim 10\%$ of total (i.e. 0.2×10^{-10}), we take 50% of largest contribution (i.e. 5.5×10^{-10} or 300% of total)
- Thus: $\Delta_{\text{IB}} a_\mu^{\text{LO-HVP}} = (-1.7 \pm 5.5) \times 10^{-10}$

Systematic errors and preliminary results

- Stat. error: jackknife
- $a \to 0$: from 4 (3) cuts on a for conn. (disc.)
- bounds: from $t_c = 3.100(2.600) \pm 0.134\,\mathrm{fm}$ vs $t_c = 2.966(2.466) \pm 0.134\,\mathrm{fm}$ for conn. (disc.)
- PT match: from $Q^2_{\text{max}} = 2\,\mathrm{GeV}^2$ vs $Q^2_{\text{max}} = 5\,\mathrm{GeV}^2$
- FV: 100% of χPT FV correction
- IB: 50% of largest phenomenological IB correction

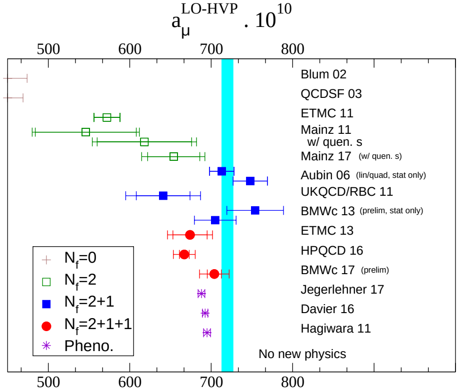
Contrib.	$a_\mu^{\text{LO-HVP}} \times 10^{10}$
$I = 1$	585(8)(6)(7)
$I = 0$	120(4)(3)
Total	704(9)(7)(13)(6)

Error on total:

- Stat. = 1.2%
- LQCD syst. = 0.9%
- FV = 1.9%
- IB = 0.8%
- Total = 2.6%

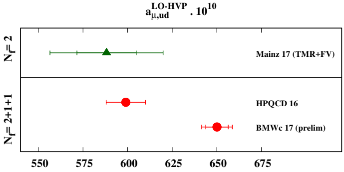
Compare w/ upper bound (Bell et al '69) using Π_1 from 1612.02364 [hep-lat] $= 792(24)$

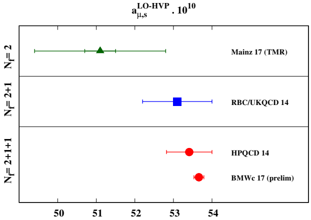
Comparison

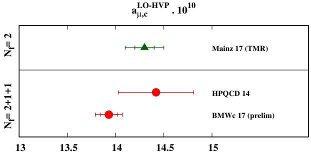


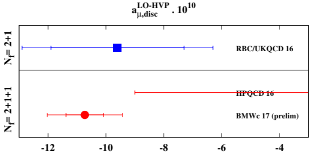
- "No New Physics" $= (720 \pm 7) \times 10^{-10}$ obtained from Davier '16
- BMWc '17 consistent w/ "No new physics" & pheno.
- Total uncertainty of 2.6% is $\sim (6 \div 7) x$ pheno. error
- BMWc '17 is larger than other $N_f = 2 + 1 + 1$ results
  $\rightarrow$ difference w/ HPQCD '16/ETM '13 is $\sim 1.6/0.9\sigma$

More detailed comparison









- BMWc '17 *ud* contribution is signficantly larger than other $N_f = 2+1+1$ results
  $\rightarrow$ difference w/ HPQCD '14/Mainz '17 is $\sim 2.4/1.5\sigma$
- BMWc '17 *c* contribution is slightly smaller than other $N_f = 2+1+1$ results
- BMWc '17 is only calculation performed directly at physical quark masses w/ 6 *a*'s to fully control continuum extrapolation
- BMWc '17 $\delta a_{\mu,\,\text{disc}}^{\text{LO-HVP}} = 1.5 \times 10^{-10}$
  $\rightarrow$ contributes only 0.2% to error on $a_\mu^{\text{LO-HVP}}$

Conclusions and outlook

- Calculation of all relevant contributions to $a_\mu^{\text{LO-HVP}}$ directly at physical m_{ud} (also have slope and curvature of $\hat{\Pi}(Q^2)$ at $Q^2 = 0$, see 1612.02364)
- Fully controlled continuum limit and matching to perturbation theory
- Only model/pheno. assumptions for small FV, QED and $m_u \neq m_d$ corrections
- Consistent with "no new physics" and dispersive methods, but error $\sim (6 \div 7)\times$ larger; some tension with HPQCD 16 on $a_{\mu,\,ud}^{\text{LO-HVP}}$
- Total error is 2.6%, dominated by poorly controlled FV effects
- Need $\sim$ 0.2% to match upcoming experiments !
  - $\Rightarrow$ increase statistics by $\times 50 \div 100$
  - $\Rightarrow$ control FV effects directly w/ simulations
  - $\Rightarrow$ compute QED and $m_d \neq m_u$ correction to relevant observables

Now the real fun begins!