Lattice study of finite size effect in the leading order of hadronic contribution to muon g-2

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1. Introduction
Target precision in lattice QCD

$$\boxed{\mathrm{Err}[a_\mu^{\mathrm{BNL}}] = 6.3 \times 10^{-10}}$$

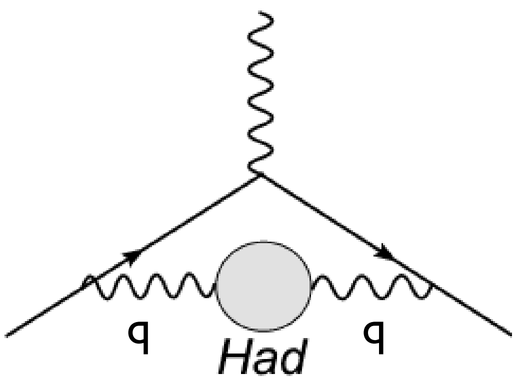
- Leading order of hadronic contribution (HLO)

  Integral of vacuum polarization from $q \in [0, \infty]$

  Target precision < 1% ~ $O(\mathrm{Err}[a_\mu^{\mathrm{BNL}}])$

  Dispersion theory (N_f=5) using R-ratio (e+e-) :
  $a_\mu^{\mathrm{HLO}} = 688.6(4.3) \times 10^{-10} \Rightarrow$ 0.6 % precision

  Jegerlehner, 1511.04473



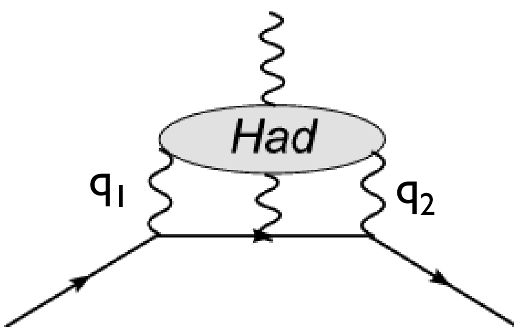
- Next-to-leading order (HNLO)

  Integral of hadronic light-by-light diagram
  from $q_1 \in [0, \infty]$, $q_2 \in [0, \infty]$

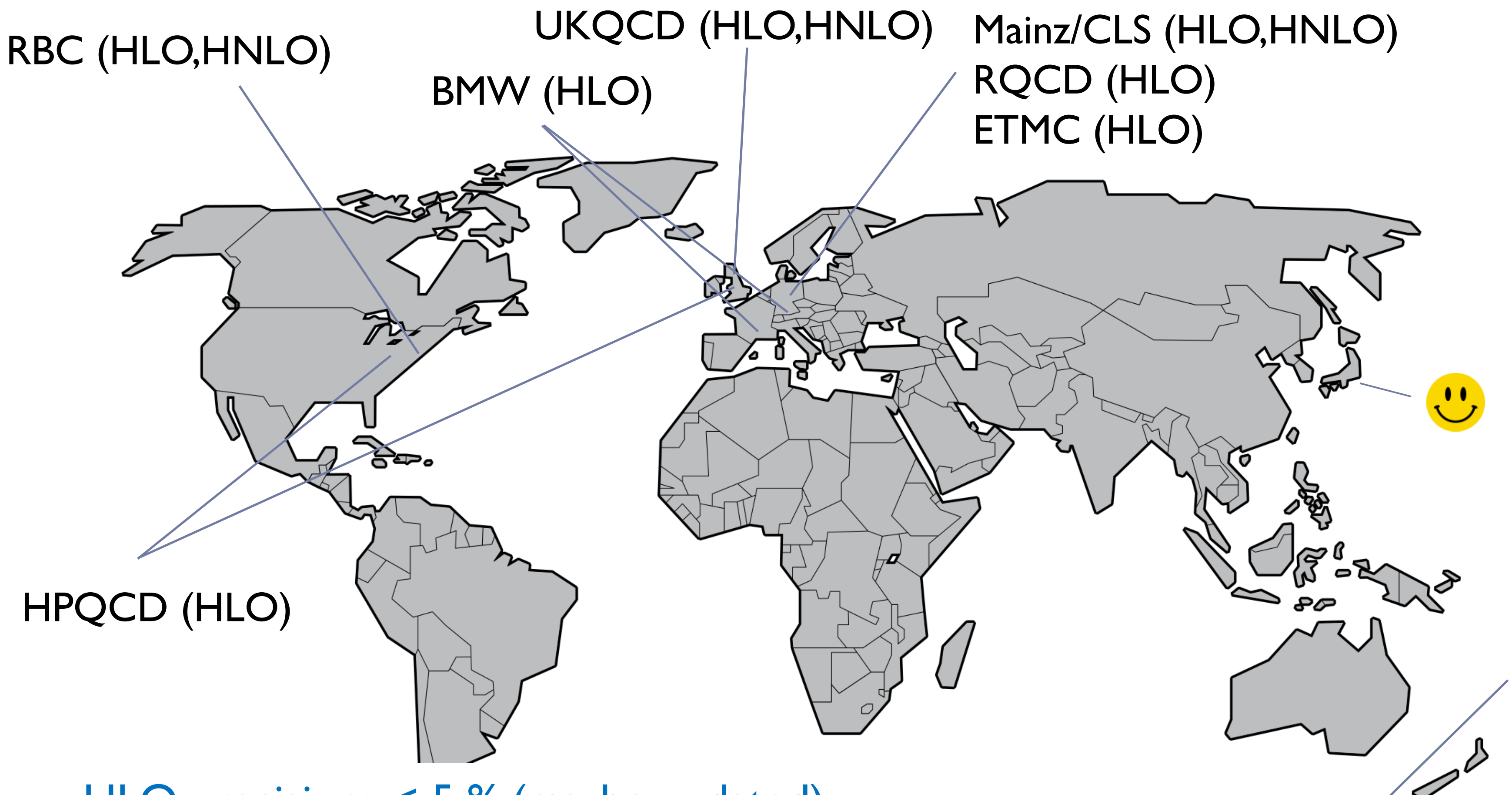
  Target precision ~ 10% ~ $O(\mathrm{Err}[a_\mu^{\mathrm{BNL}}])$

  Model: $a_\mu^{\mathrm{HLO}} = 10.6(0.3) \times 10^{-10} \Rightarrow$ ~3 %

  Prades et al., 0901.0306



Lattice works

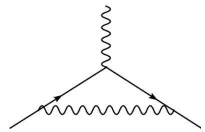


HLO precision: < 5 % (maybe updated)
HNLO: ~ 10 % statistical accuracy

Leading order of hadronic contribution

- Hadronic vacuum polarization (HVP)

$$a_\mu^{\mathrm{HLO}} = \int ds$$



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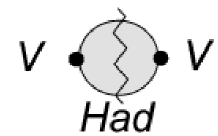
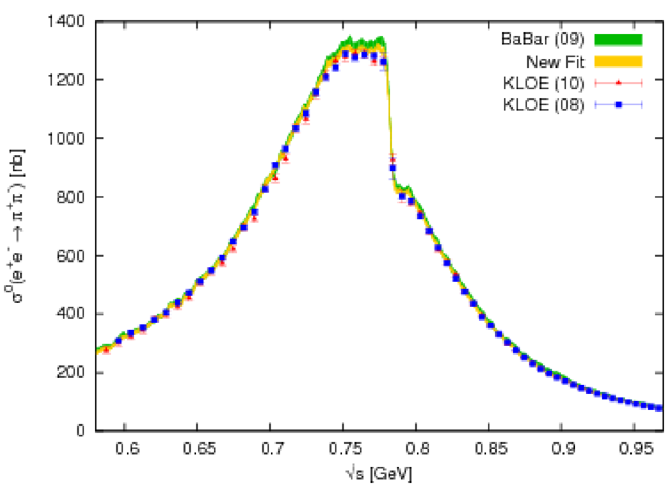
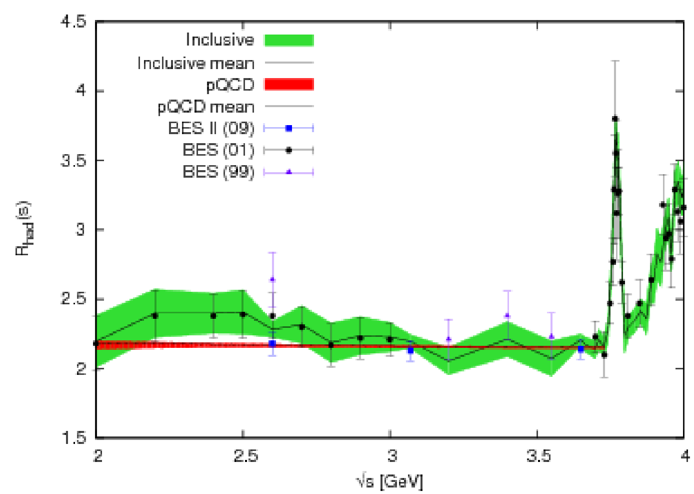


$$= \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \left[\int_{m_\pi^2}^{s_{\mathrm{cut}}} ds \frac{K(s)}{s} R_{\mathrm{had}}^{\mathrm{data}}(s) + \int_{s_{\mathrm{cut}}}^{\infty} ds \frac{K(s)}{s} R_{\mathrm{had}}^{\mathrm{pQCD}}(s)\right]$$

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (s/m_\mu^2)(1-x)}$$





Hagiwara, et al., J.Phys. G38,085003 (2011)

g-2 with Q integral

- Euclidean momentum integral

Lautrup et al., Phys. Rep. 3 (1972), Blum, PRL91(2003)

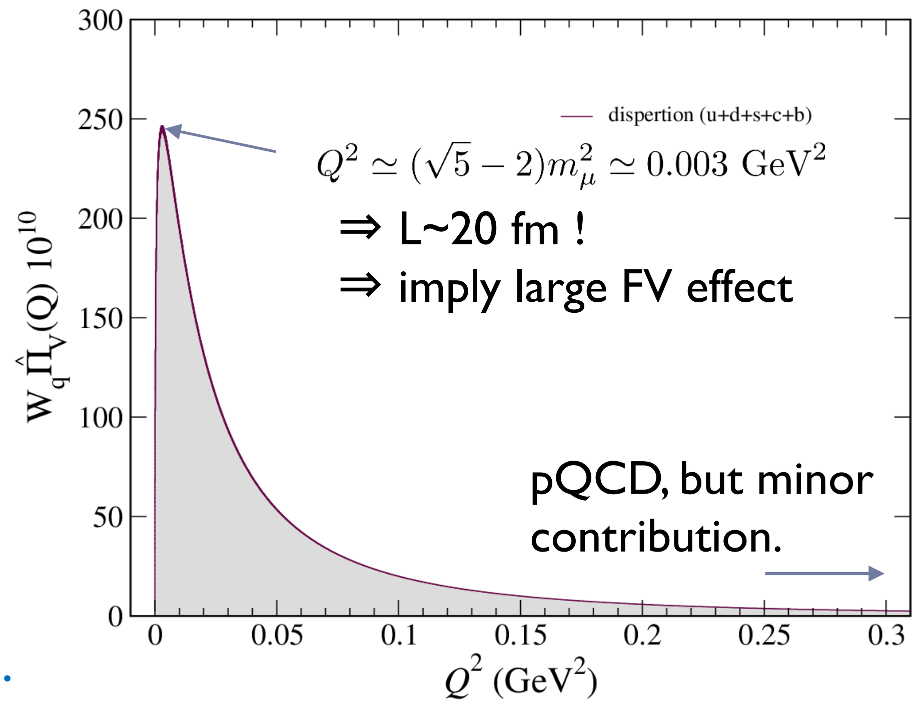
$$a_\mu^{\rm HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty ds K_E(s)\hat{\Pi}(s), \quad \hat{\Pi}(s) = 4\pi^2\big(\Pi(s) - \Pi(0)\big)$$

$$K_E(s) = \frac{1}{m_\mu^2} sZ(\hat{s})^3 \frac{1-\hat{s}Z(\hat{s})}{1+\hat{s}Z^2(\hat{s})}, \quad Z(s) = -\frac{\hat{s}-\sqrt{\hat{s}^2+4\hat{s}}}{2\hat{s}}, \quad \hat{s} = \frac{s}{m_\mu^2}$$

VPF tensor

$$\begin{aligned} \Pi_{\mu\nu} &= \int e^{iQx}\langle V_\mu(x)V_\nu(0)\rangle \\ &= (Q_\mu Q_\nu - Q^2)\Pi(Q) \\ V_\mu(x) &= \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d + \cdots \end{aligned}$$

- Pade function ($Q^2 < m_\rho^2$)
- Renormalization constant $\Pi(0)$ given from extrapolation.
- Q^2 integral from 0 -- ∞, but pQCD gives asymptotic function.



2. HVP on the lattice

g-2 with t integral

- Temporal integral

Bernecker, Meyer, EPL A47(2011)

$$a_\mu^{\mathrm{HLO}} = \int_0^\infty W_t(t) G(t), \quad G(t) = \int d^3x \langle V_i(x) V_i(0) \rangle$$

$$\hat{K}(t) = \frac{2}{m_\mu t^3} \int_0^\infty \frac{d\omega}{\omega} K_E(\omega^2) \left[\omega^2 t^2 - 4\sin^2(\omega t/2)\right]$$

$$W_t(t) = 4\alpha^2 m_\mu t^3 \hat{K}(t)$$

Pros

- On the lattice, <VV>(t) without momentum.
- Integral (summation) without extrapolation/interpolation.

Cons

- Temporal integral from 0 – ∞, we need to know asymptotic function
- Temporal boundary effect, backward propagation
- Discrete sum.

Possible to involve the large uncertainty due to FV effect and lattice artifact.

2. HVP on the lattice

Studies of finite volume

- ChPT

  Aubin et al., PRD93(2016)

  - Lowest-order SChPT gives VPF tensor: $\Pi_{\mu\nu}(q)$
  - 10% -- 15% discrepancy between $a_\mu^{HLO}[A_1]$ and $a_\mu^{HLO}[A_1^{44}]$

    consistent with lattice calculation (L=3.8 fm, 0.22 GeV pion, $m_\pi L=4.2$)

- Gounaris-Sakurai model

  Wittig (2016), Mainz 1705.01775

  - By using time-like pion form factor, g-2 can be described in infinite volume.
  - 5% FV effect in L=4 fm, 0.19 GeV pion, $m_\pi L=4$

- Anisotropic study

  Lehner (2016)

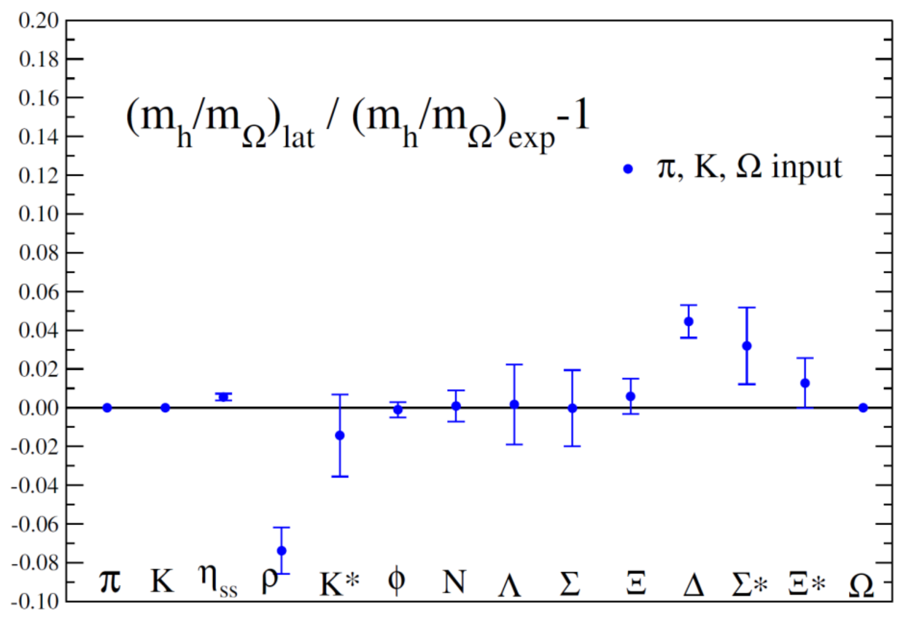
  - Coordinate space integral along temporal or spatial direction.
  - Discrepancy is a_μ^{HLO} [spatial] – a_μ^{HLO} [temporal] ~ 3%.

- **Direct lattice study (PACS)**
  - Comparison between two volumes in physical pion at fixed *a*
  - L > 5 fm, $m_\pi L \gtrsim 4$

PACS 96^4 and 64^4 at a=0.08 fm

PACS group recently generates two gauge ensembles:

- Nf=2+1 O(a) improved clover fermion + Stout smearing
- a=0.083 fm, and two lattice sizes 64^4 and 96^4
- (almost) physical pion,
  L=5.4 fm, 0.14 GeV ($m_\pi L$=3.8),
  L=8.1 fm, 0.145 GeV ($m_\pi L$=6.0)



PACS, 1511.09222

3. Strategy
Computation with AMA

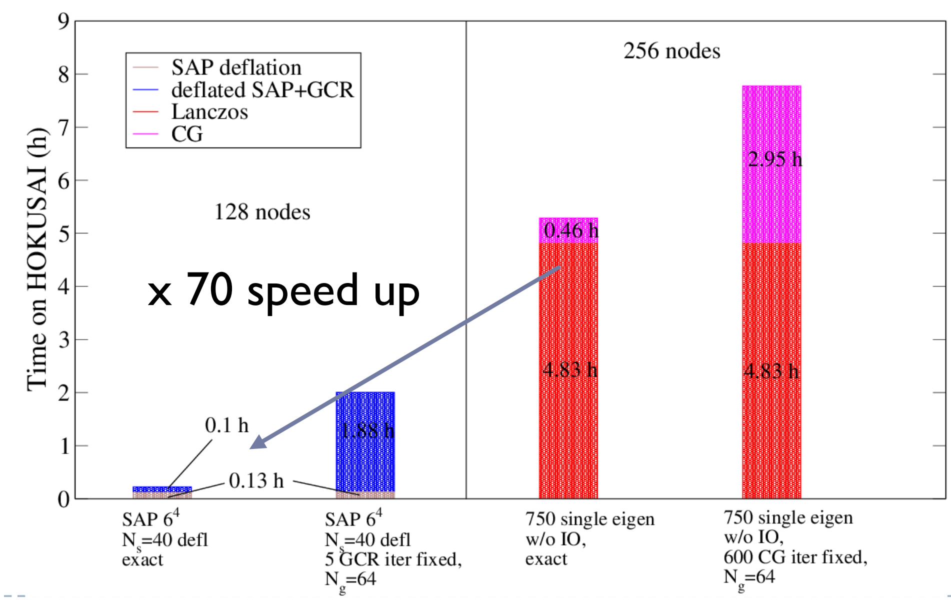
- Optimized AMA with SAP + deflation
  - Domain-decomposition, 6^4 domain size is chosen.
  - Deflation field, $N_s = 50$, 5 SAP cycles in single precision.
  - Deflated SAP + GCR for exact and approximation
    - Exact: ~30 GCR iteration (outer double precision loop)
    - Approximation: 5 fixed GCR iteration, $|r| \sim O(10^{-5})$

Blum et al., PRD88(2013), PRD91(2015), Mainz, NPB914 (2017)

Luscher, JHEP07 (2008)

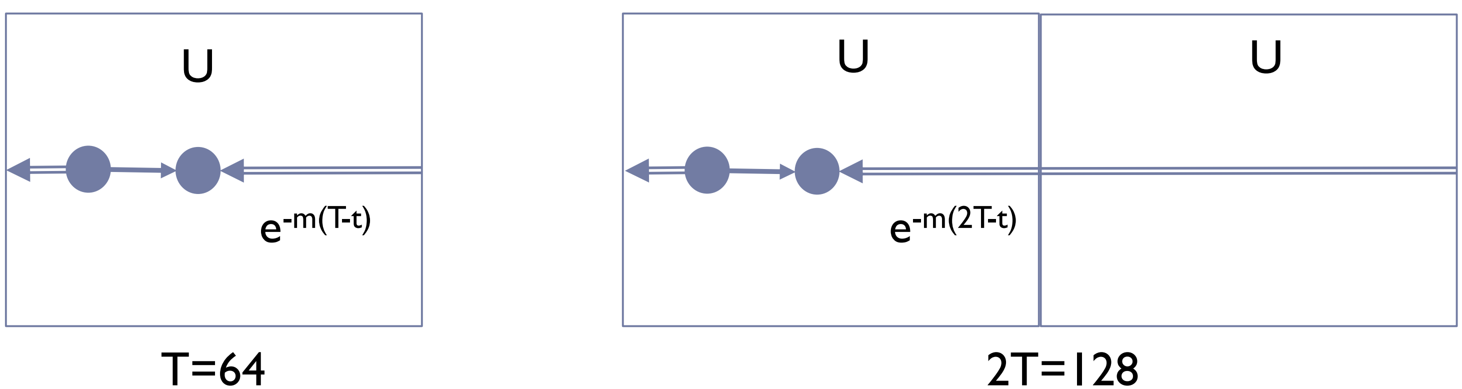
- Small cost for a generation of deflation field.
  $\Rightarrow$ no need **huge** storage (or memory) to store eigenvector
- 3x faster than lowmode deflated CG (using 750 modes)

96^4



Study of backward state propagation

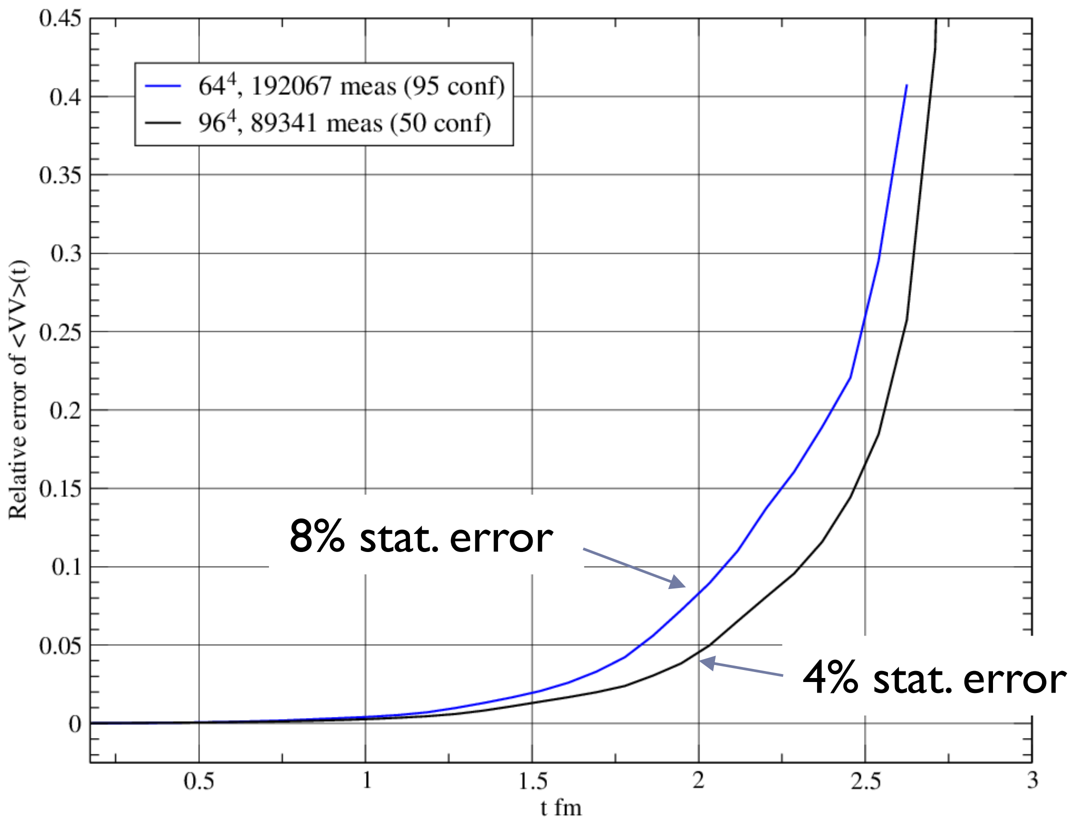
- Extension of temporal length
  - To study backward state effect, we extend temporal length.
  - Using duplicated gauge configurations for 64^4 lattice
  - Suppress the backward state effectively (consistently using periodic anti-periodic fermion)
  - Important check of finite t effect in t integral



High statistics in PACS configurations

96^4 : 50 configs., 89,341 meas (light) [m_π=0.145 GeV, L=8.1 fm]

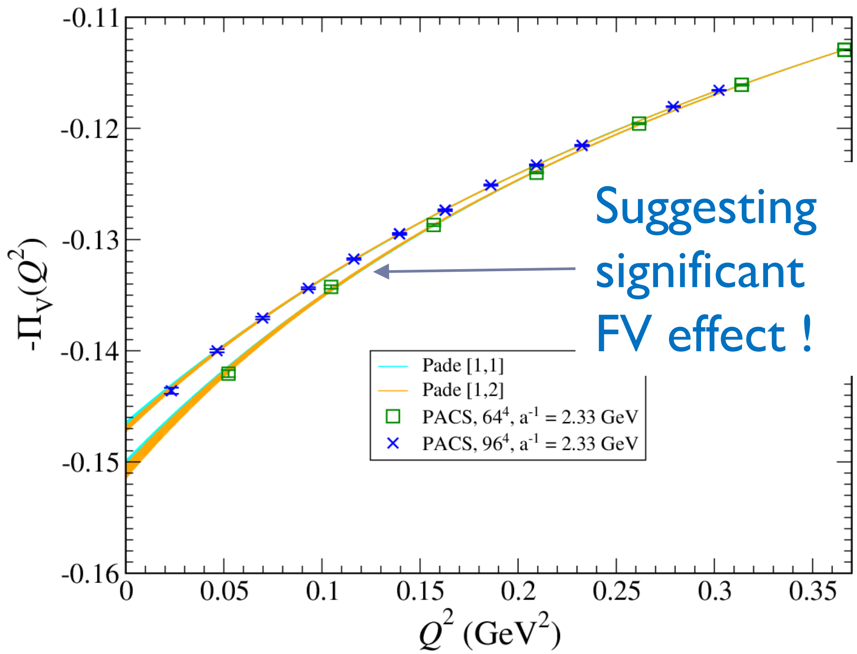
64^4: 95 configs., 192,067 meas (light) [m_π=0.14 GeV, L=5.4 fm]



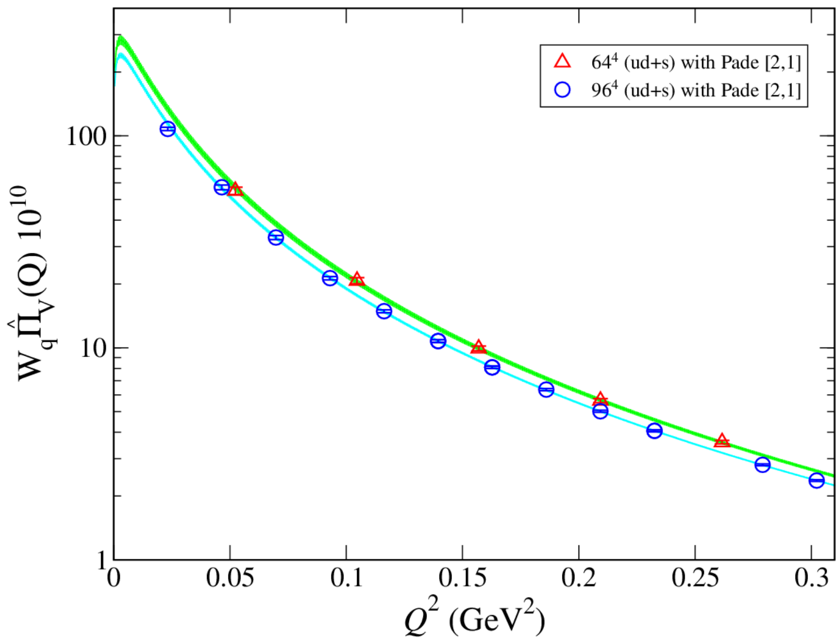
- Statistical error at large t is improved by large volume.

4. Preliminary result
Momentum dependence

➢ Π(Q)

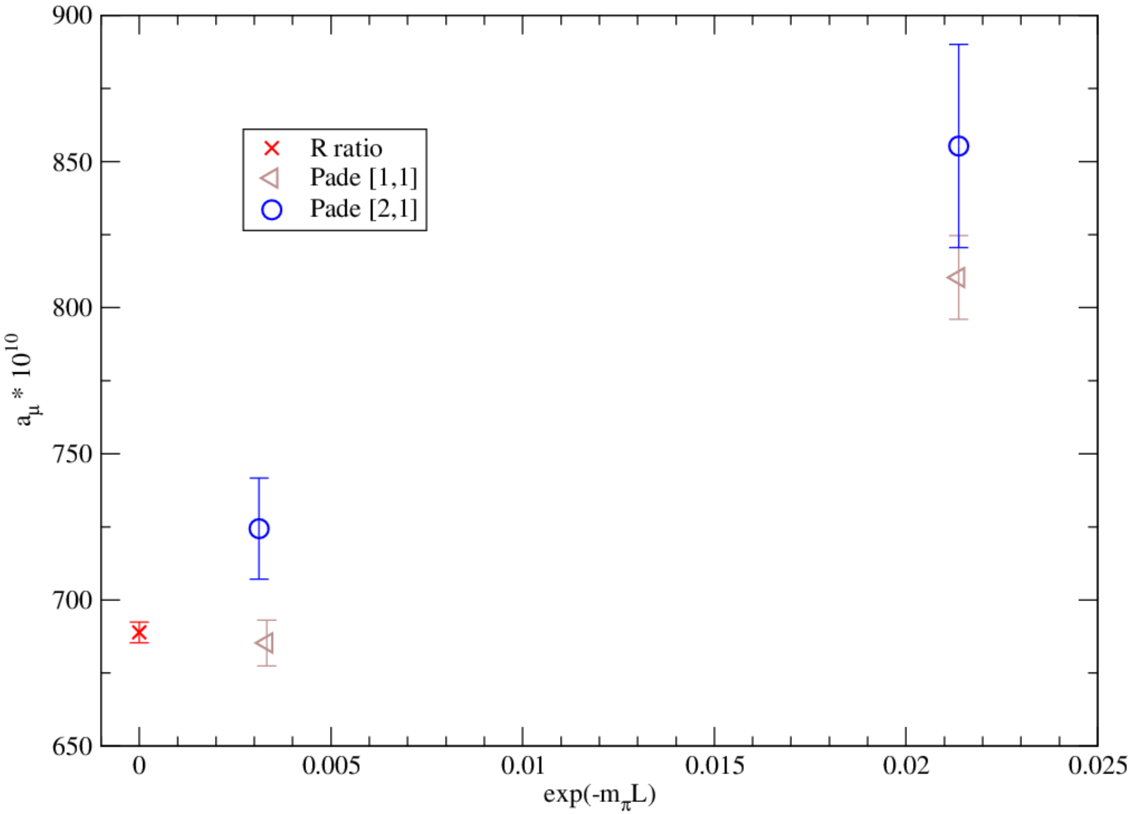


➢ Integrand



$$\Pi^{\mathrm{Pade}}(Q) = \Pi(0) + Q^2\left(A_0\delta_{n,m+1} + \sum_{k=1}^{m}\frac{A_k}{Q^2+B_k}\right)$$

4. Preliminary result
Q integral and volume dependence



64^4: Integral Q^2 = 0 – 0.39 GeV^2

96^4: Integral Q^2 = 0 – 0.39 GeV^2

Cf. Mainz 1705.01775

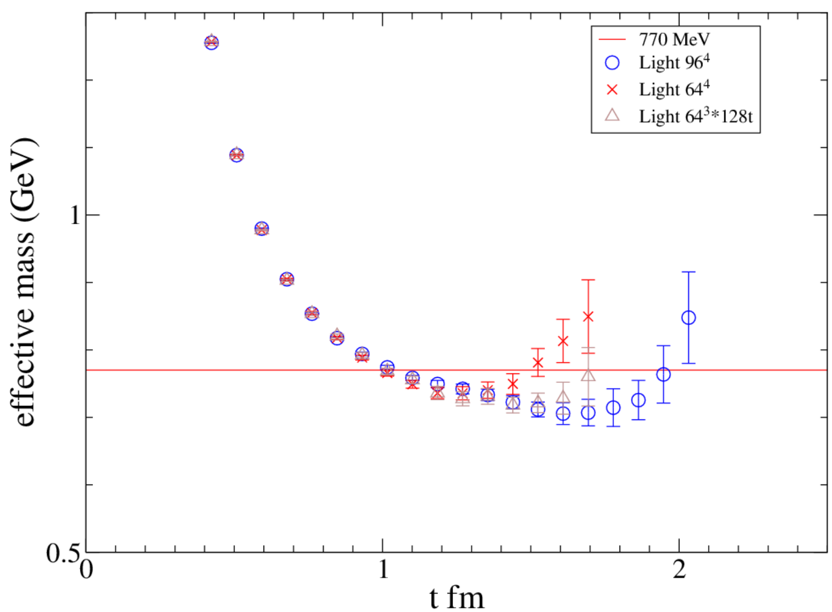
Q_{cut}^2=0.5 GeV^2 (0.19 GeV pion, L=4 fm)

$a_\mu \times 10^{10}$ [ud] = 504(10), $a_\mu \times 10^{10}$ [s] = 47.5(4)

4. Preliminary result

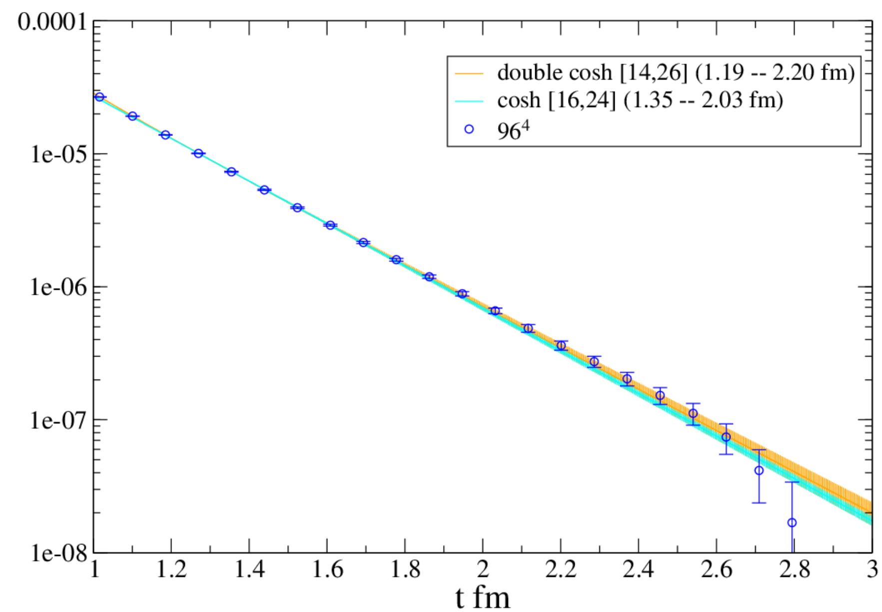
Vector-vector correlator

- Effective mass



Plateau appears above 1.3 fm, and its mass is below rho mass.

- Propagator

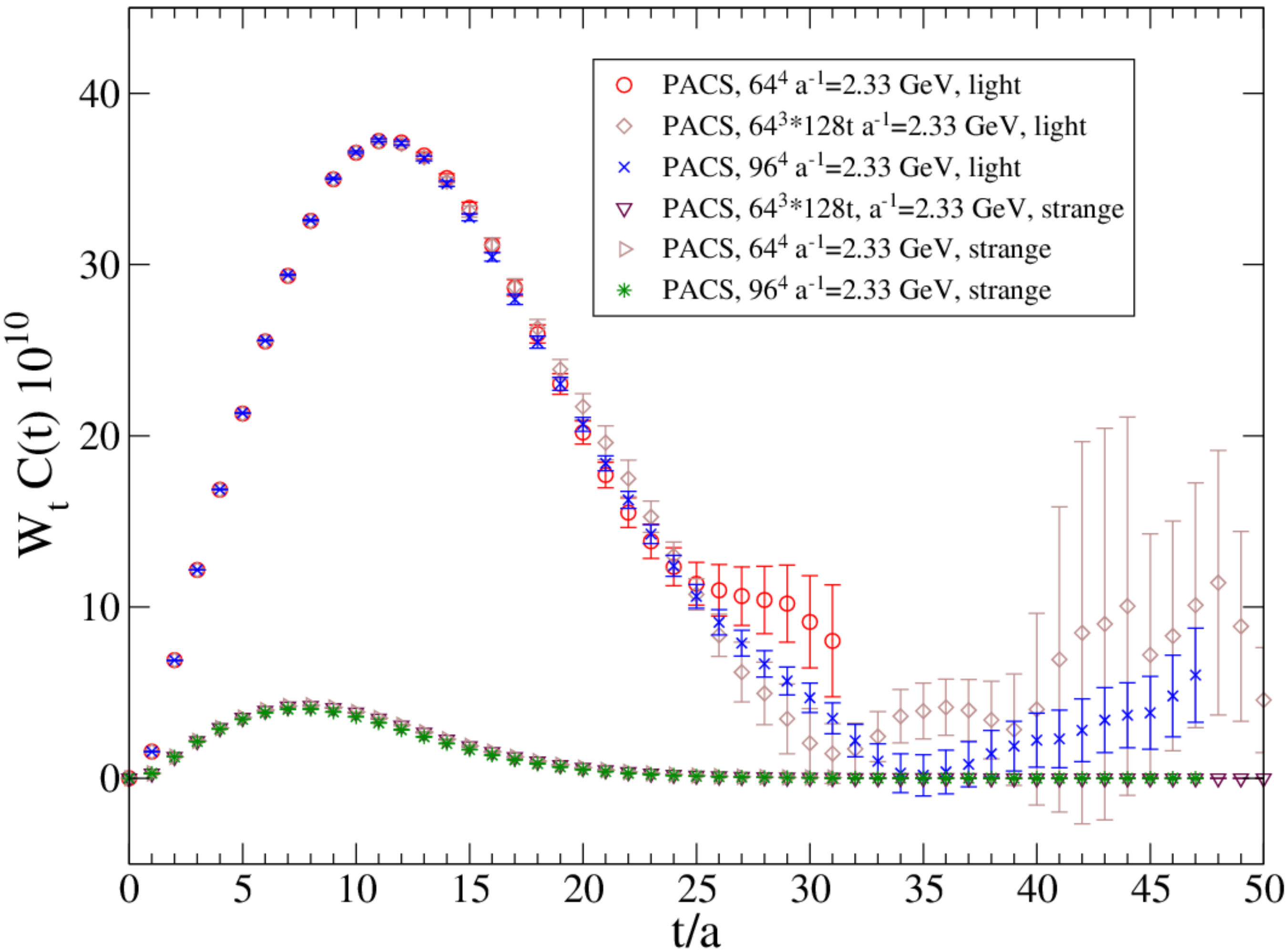


Separating from single exponential function.
⇒ Double exponential fitting

⇒lattice data implies that two-pion state appears above 1.3 fm.

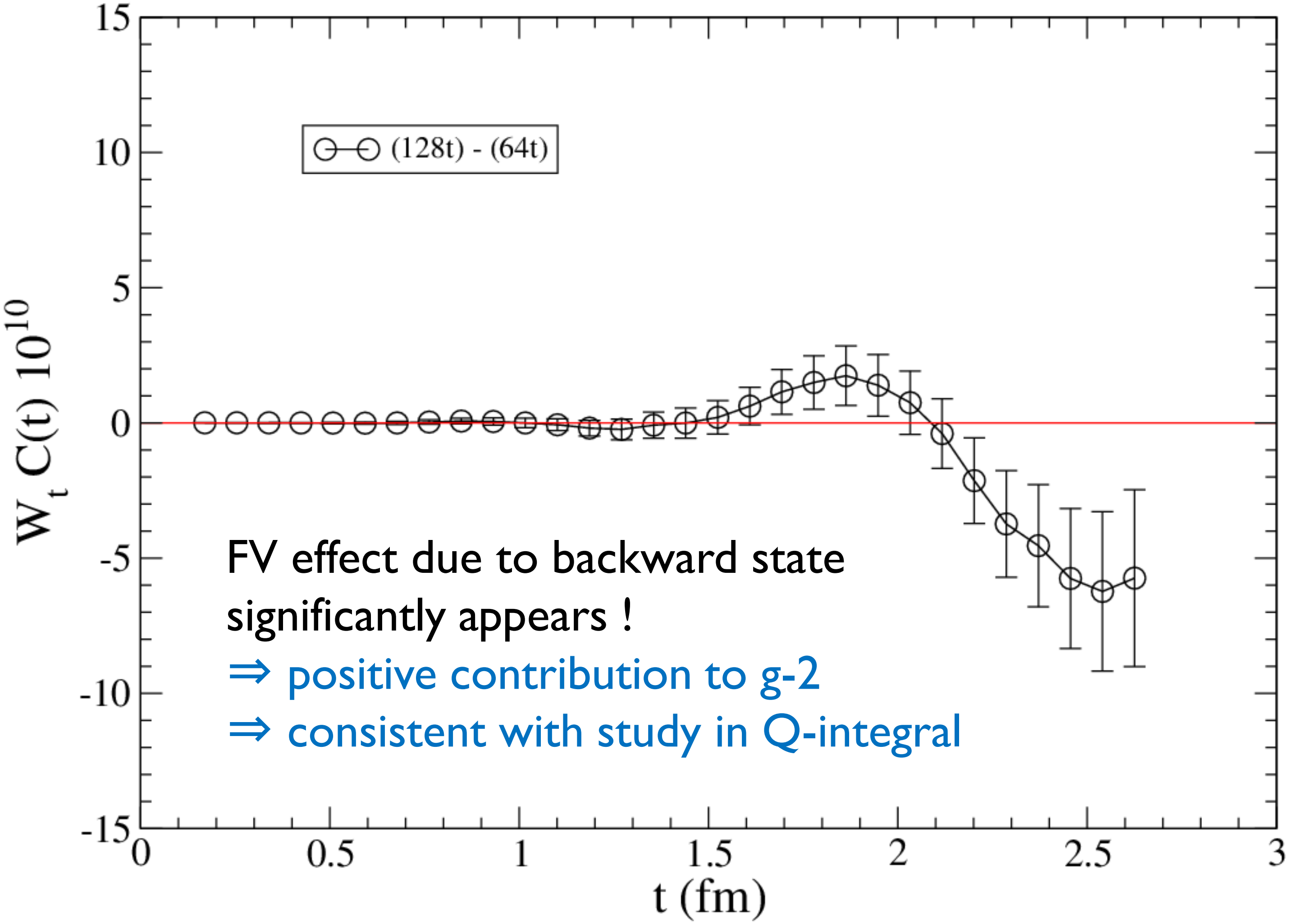
4. Preliminary result

Integrand along temporal direction

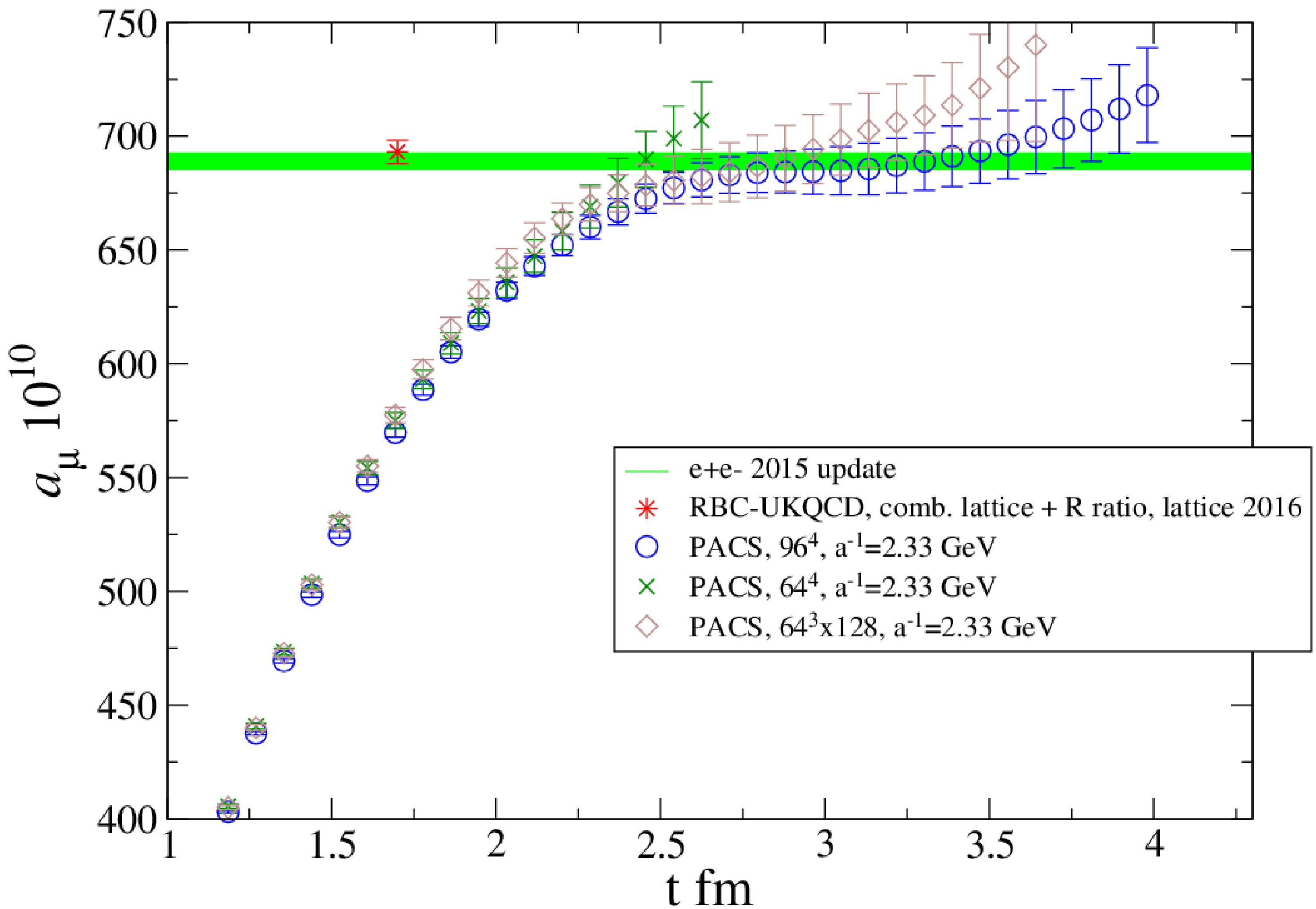


4. Preliminary result

Backward state contribution



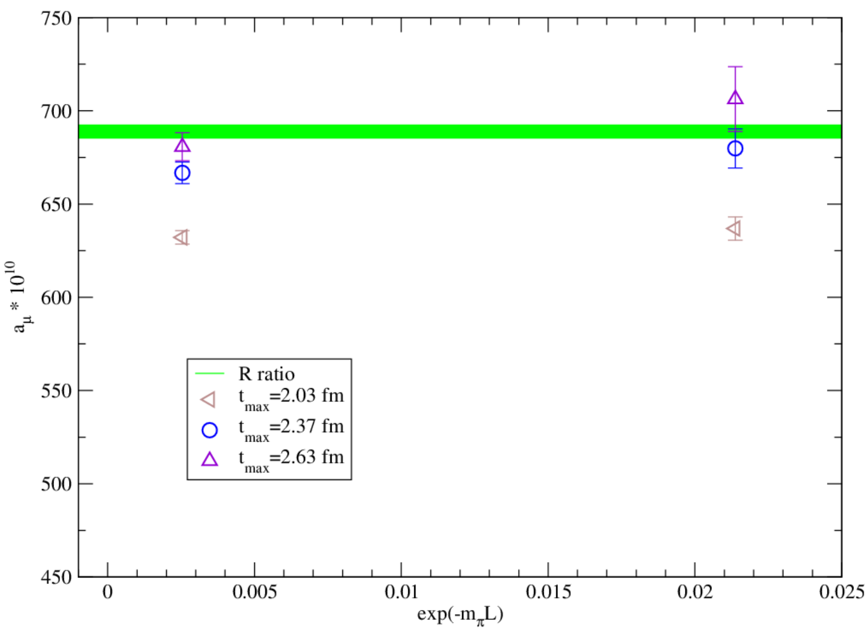
4. Preliminary result
t integral



4. Preliminary result

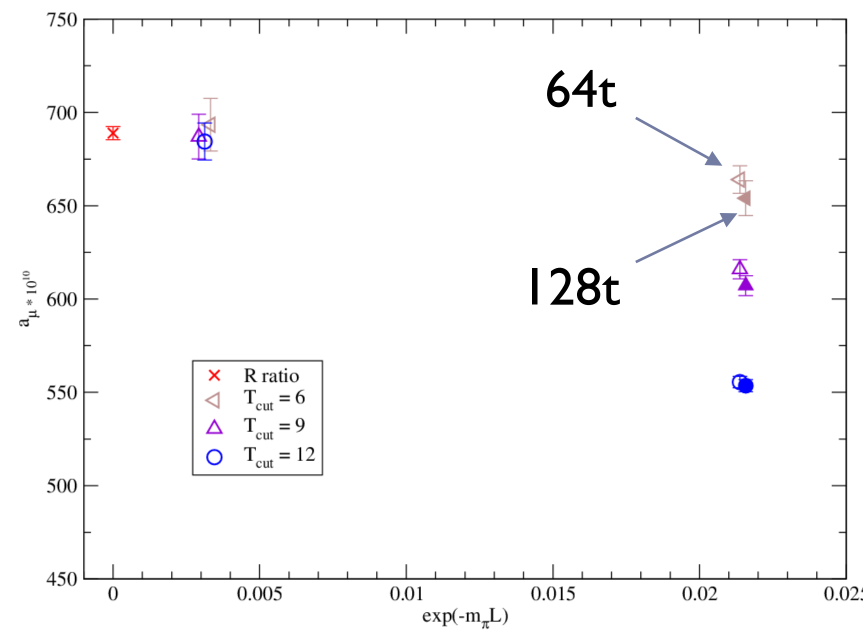
T_{max} and T_{cut} dependence

➢ T_{max} dependence



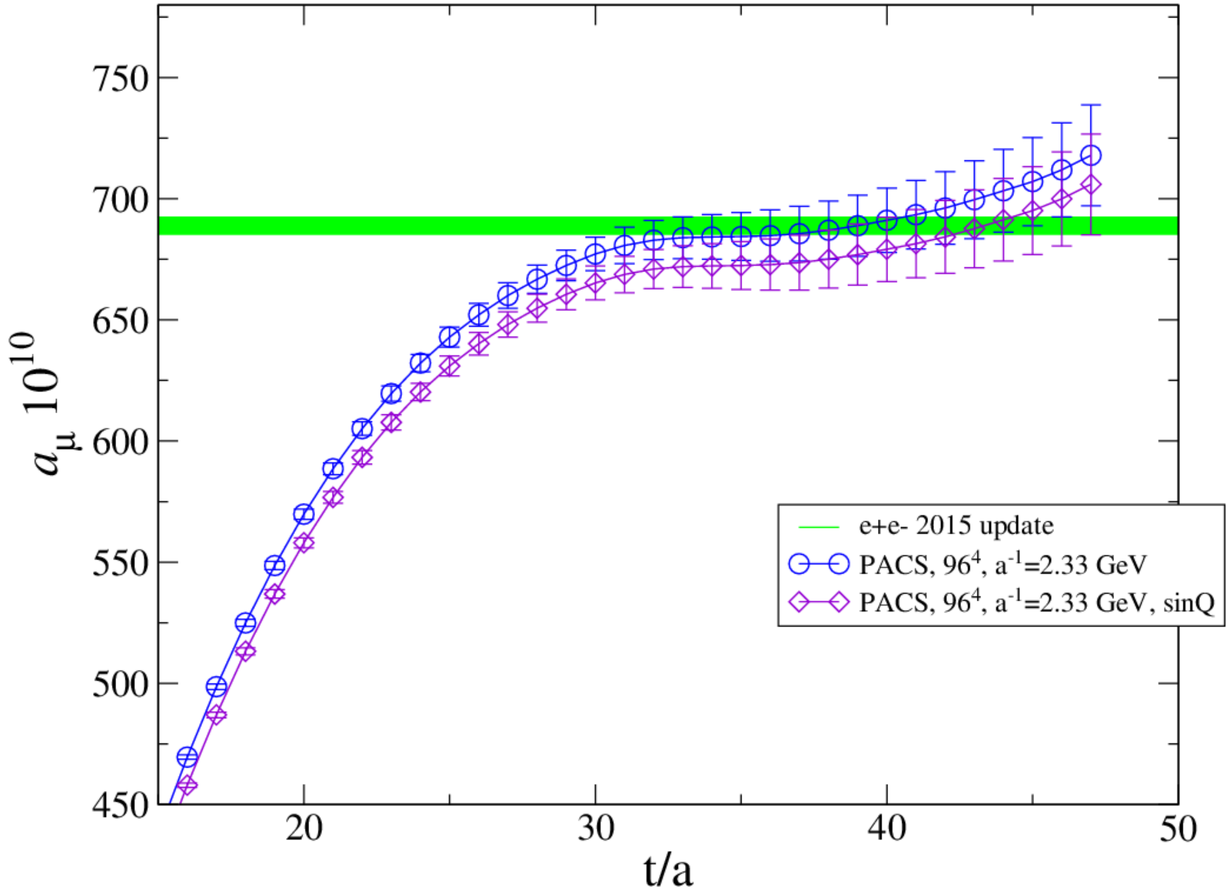
- The maximum t in the integral.

➢ T_{cut} dependence



- The t cut of tail in the integral
- The region of integral is changed depending on volume.

4. Preliminary result
Lattice artifact

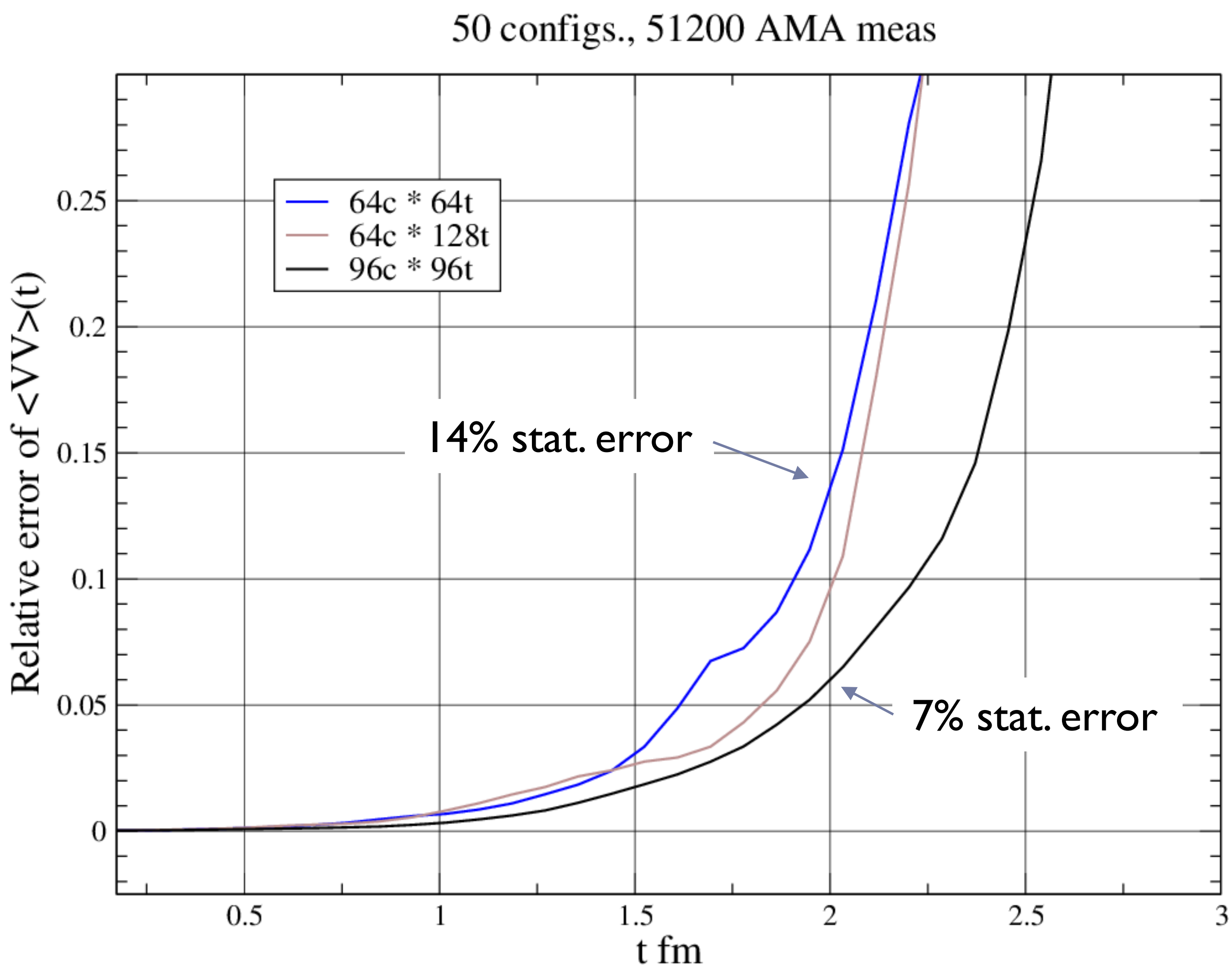


$$\hat{K}(t) = \frac{2}{m_\mu t^3}\int_0^\infty d\omega \frac{2\hat{\omega}}{\omega^2} K_E(\omega^2)\left[\hat{\omega}^2 t^2 - 4\sin^2(\omega t/2)\right], \quad \hat{\omega} = 2\sin\omega/2$$

Summary and future works

- Start HVP computation with two volumes in PACS
  - Direct lattice comparison for FV effect without models.
  - Analysis with both methods, Q-integral and t-integral
  - On 5 fm in physical pion, there is positive and large FV effect, especially due to backward state.
  - Study on physical pion is very important to correctly estimate uncertainties.
- Future
  - One more large volume and the infinite volume limit.
  - Continuum limit
  - Isospin breaking

Volume sum



50 configs., 51200 AMA meas
64c * 64t
64c * 128t
96c * 96t
14% stat. error
7% stat. error
Relative error of <VV>(t)
t fm
0
0.05
0.1
0.15
0.2
0.25
0.5
1
1.5
2
2.5
3

Domain decomposition

Luscher, Comp.Phys.Comm.156 (2004)

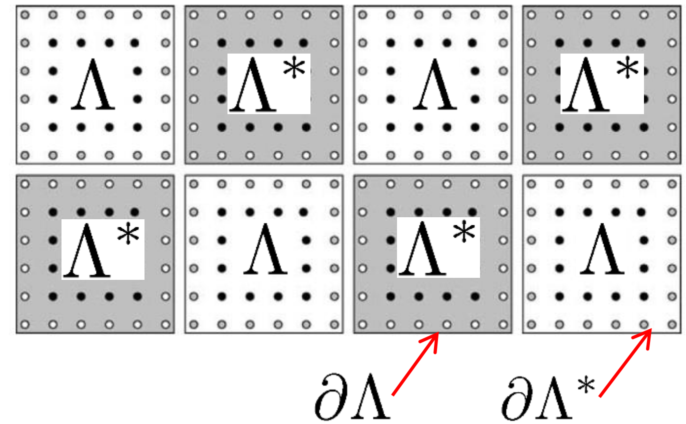
- Preconditioning for iterative solver

$$D_w = \begin{pmatrix} D_\Lambda & \partial D_\Lambda \\ \partial D_{\Lambda^*} & D_{\Lambda^*} \end{pmatrix} \qquad M_{\mathrm{sap}} = K \sum_{\nu=0}^{n_{\mathrm{cy}}-1} (1 - DK)^\nu$$

$$K = R_\Lambda^T D_\Lambda^{-1} R_\Lambda + R_{\Lambda^*}^T D_{\Lambda^*}^{-1} R_{\Lambda^*} - R_{\Lambda^*}^T D_{\Lambda^*}^{-1} D_{\partial\Lambda^*} D_\Lambda^{-1} R_\Lambda$$

$$R_\Lambda = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad R_{\Lambda^*} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

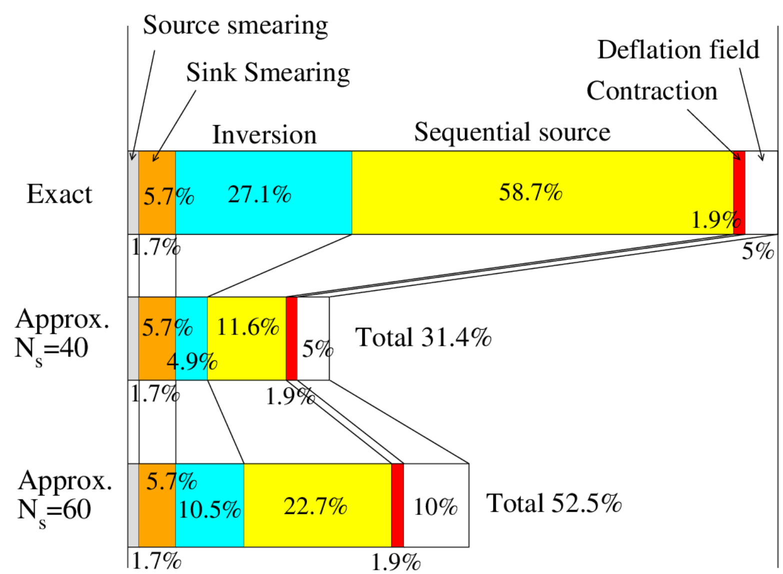
$$x = D^{-1} b \simeq M_{\mathrm{sap}} b : \ \text{preconditioner}$$



- Use in a generation of deflation field and projection
- Input parameters
  - Degree of SAP cycle: n_{cy}
  - SAP domain size: Λ_x , Λ_y , Λ_z , Λ_t
  - Precision of A^{-1} and D_Λ^{-1}
  - Number of deflation field: N_s

  Those can control the precision,

  e.g. small Λ and large n_{cy}



Mainz, NPB914 (2017)

FV study in Mainz

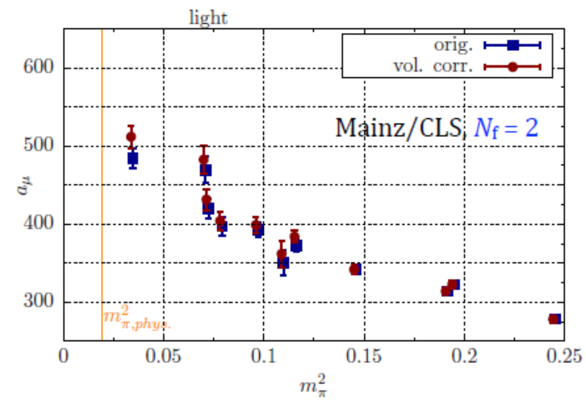
Finite-volume effects: TMR analysis

* Input quantity: timelike pion form factor $F_\pi(\omega) = |F_\pi(\omega)|\,e^{i\delta_{11}(k)}$
* Use Gounaris-Sakurai parameterisation and evaluate $|F_\pi(\omega)|$, $\delta_{11}(k)$ for given (m_π, m_ρ) of a given gauge ensemble
* Finite-volume effects in HVP dominated by long-distance contribution
* For $m_\pi = 190$ MeV, $L = 4.0$ fm, $m_\pi L = 4.0$:

$$a_\mu^{\mathrm{hvp}}(\infty) - a_\mu^{\mathrm{hvp}}(L) = 5.2\%$$

* Procedural variations: assign uncertainty of $\approx 10\%$

⇒ Dynamical theory of finite-volume effects in terms of m_ρ/m_π and $m_\pi L$



[H.B. Meyer et al., in preparation]

Hartmut Wittig Hadronic contributions to $(g-2)$ 34

4. Preliminary result

Size effect in one-loop ChPT