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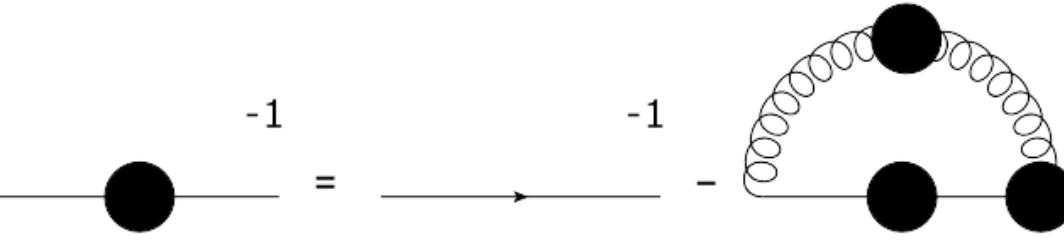
Can Dyson-Schwinger Equations offer a model-independent estimate?



Pablo Roig



Ongoing work done in collaboration with ***Adnan Bashir & Khépani Raya***



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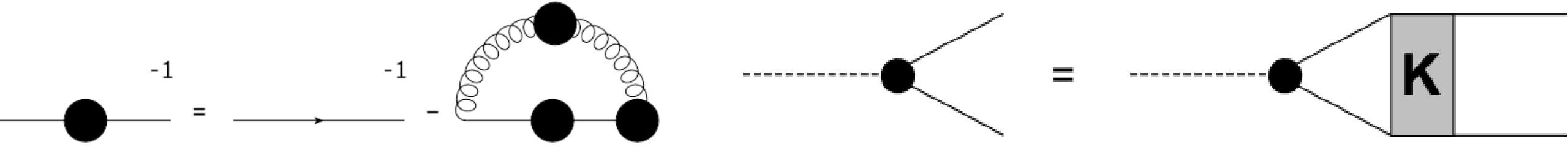
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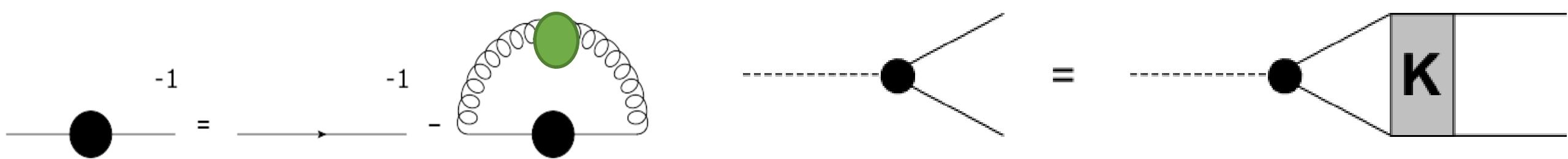
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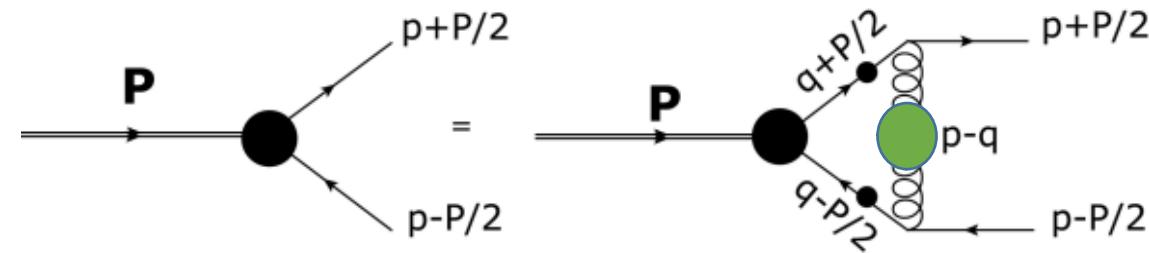
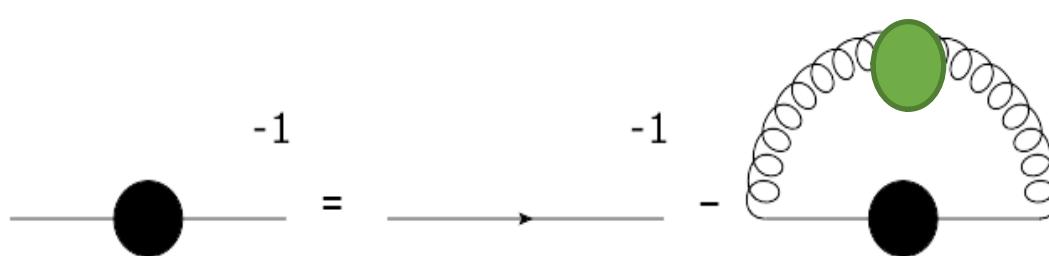
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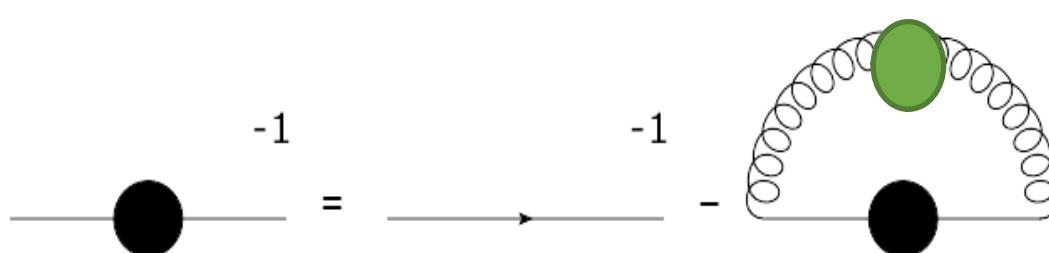
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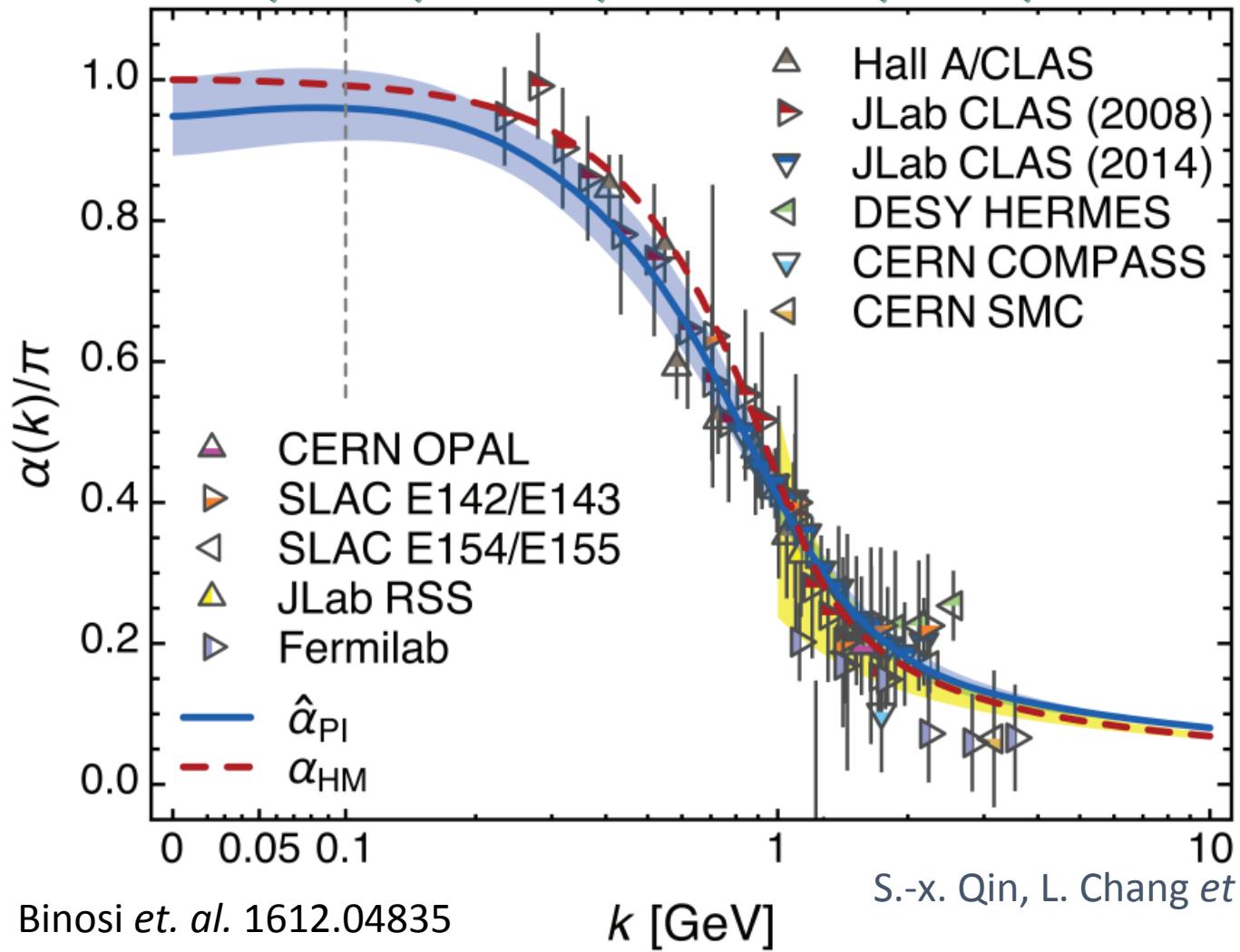
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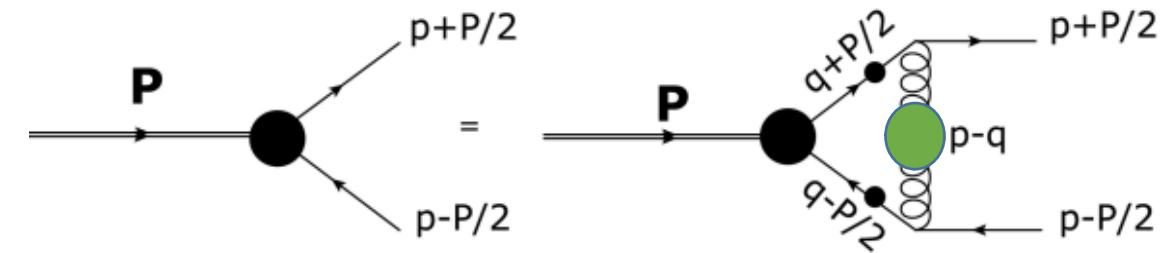
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Binosi et. al. 1612.04835

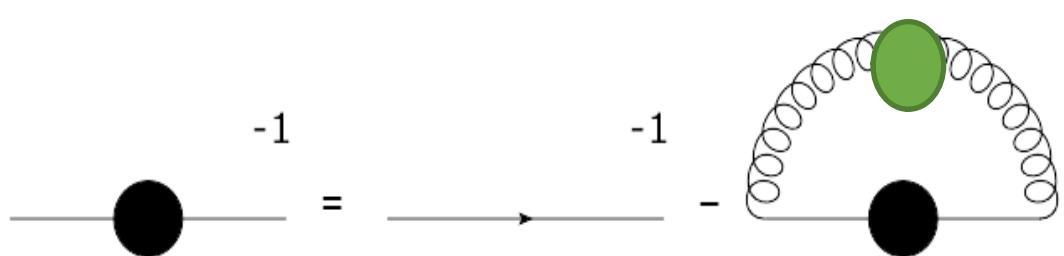
k [GeV]

S.-x. Qin, L. Chang et al. PRC84, 042202(R) (2011)

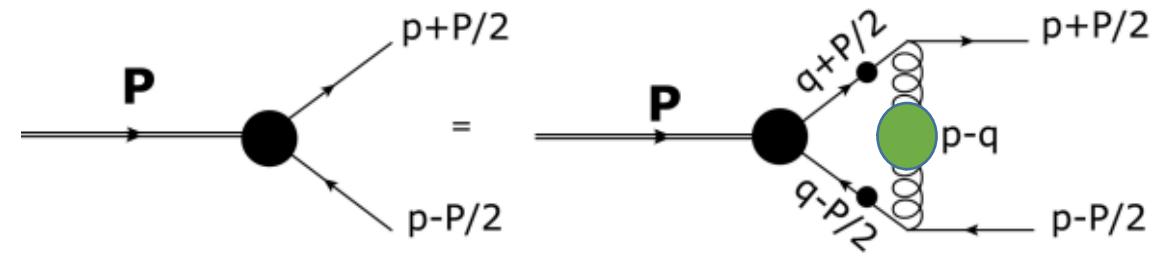
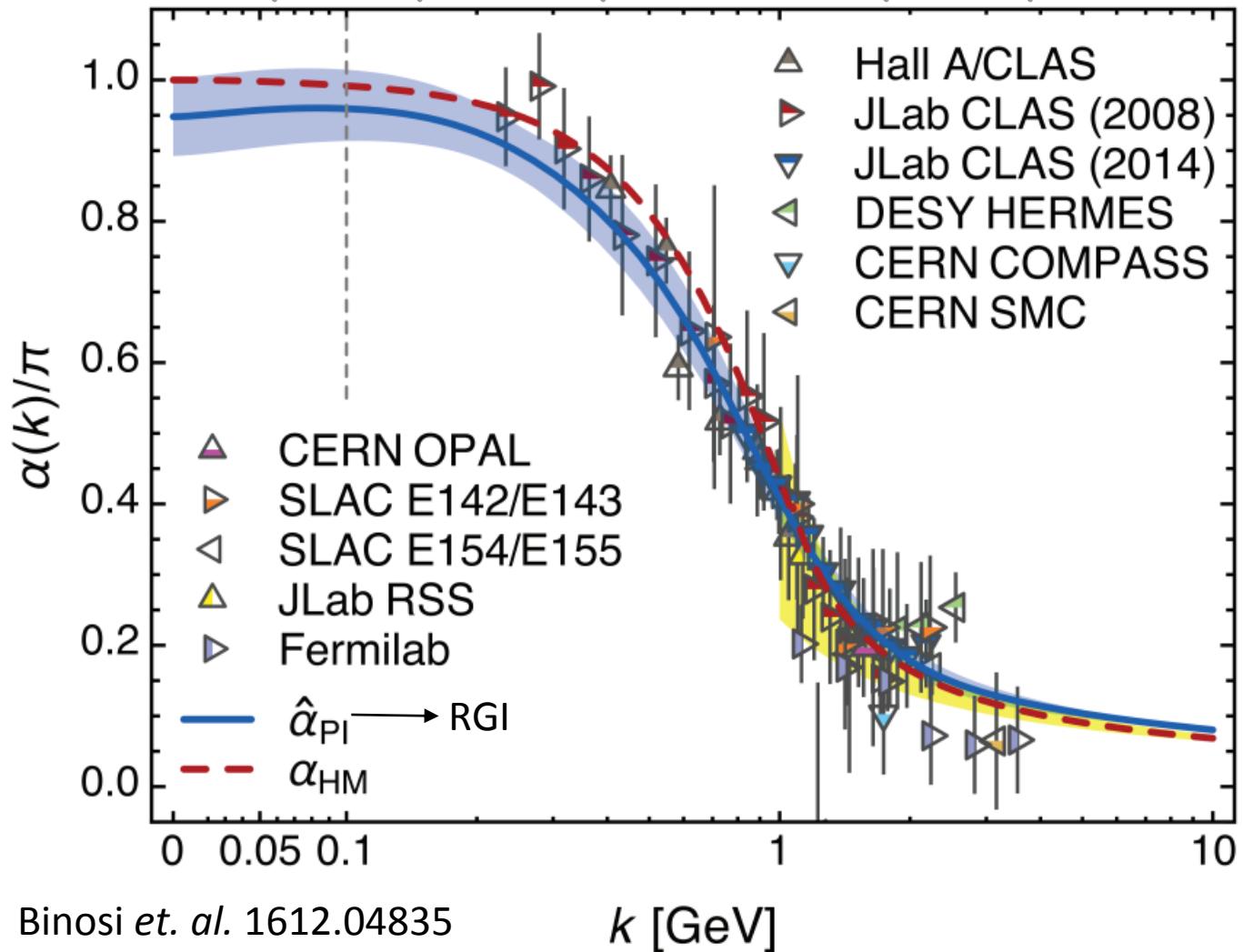


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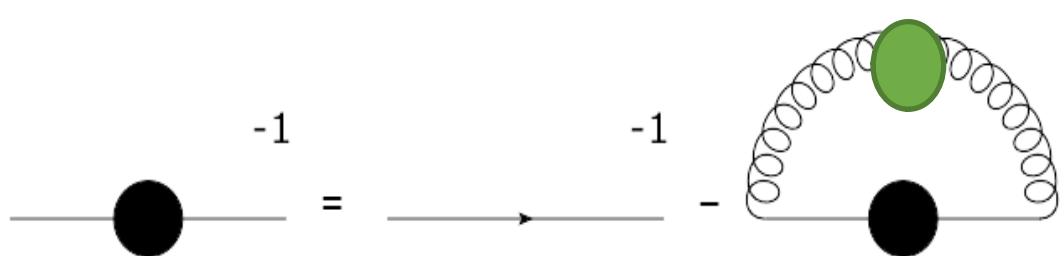
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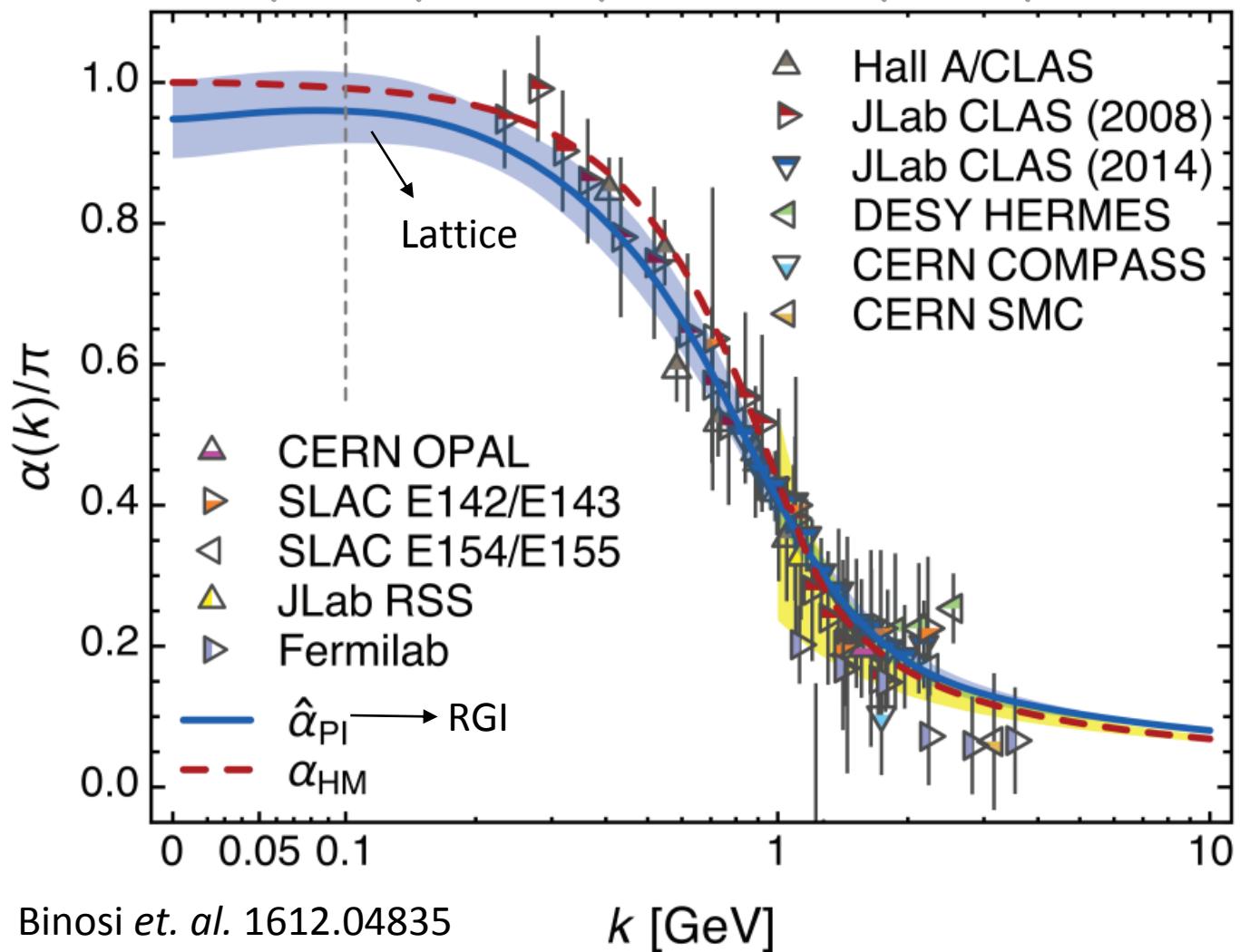
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A. Aguilar, D. Binosi, J. Papavassiliou and J. Rodríguez-Quintero, Phys. Rev. D **80**, 085018 (2009).

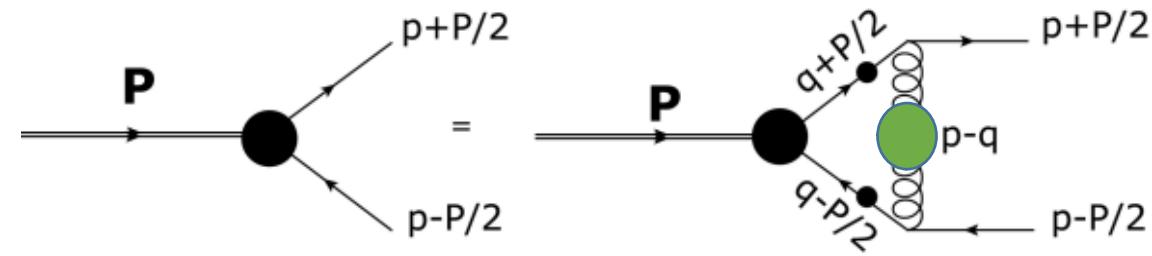


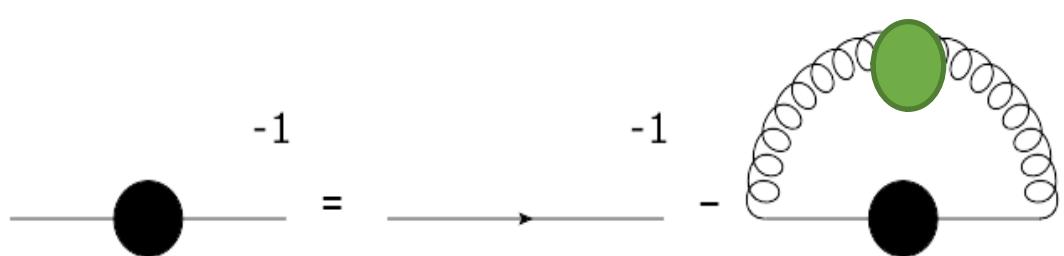
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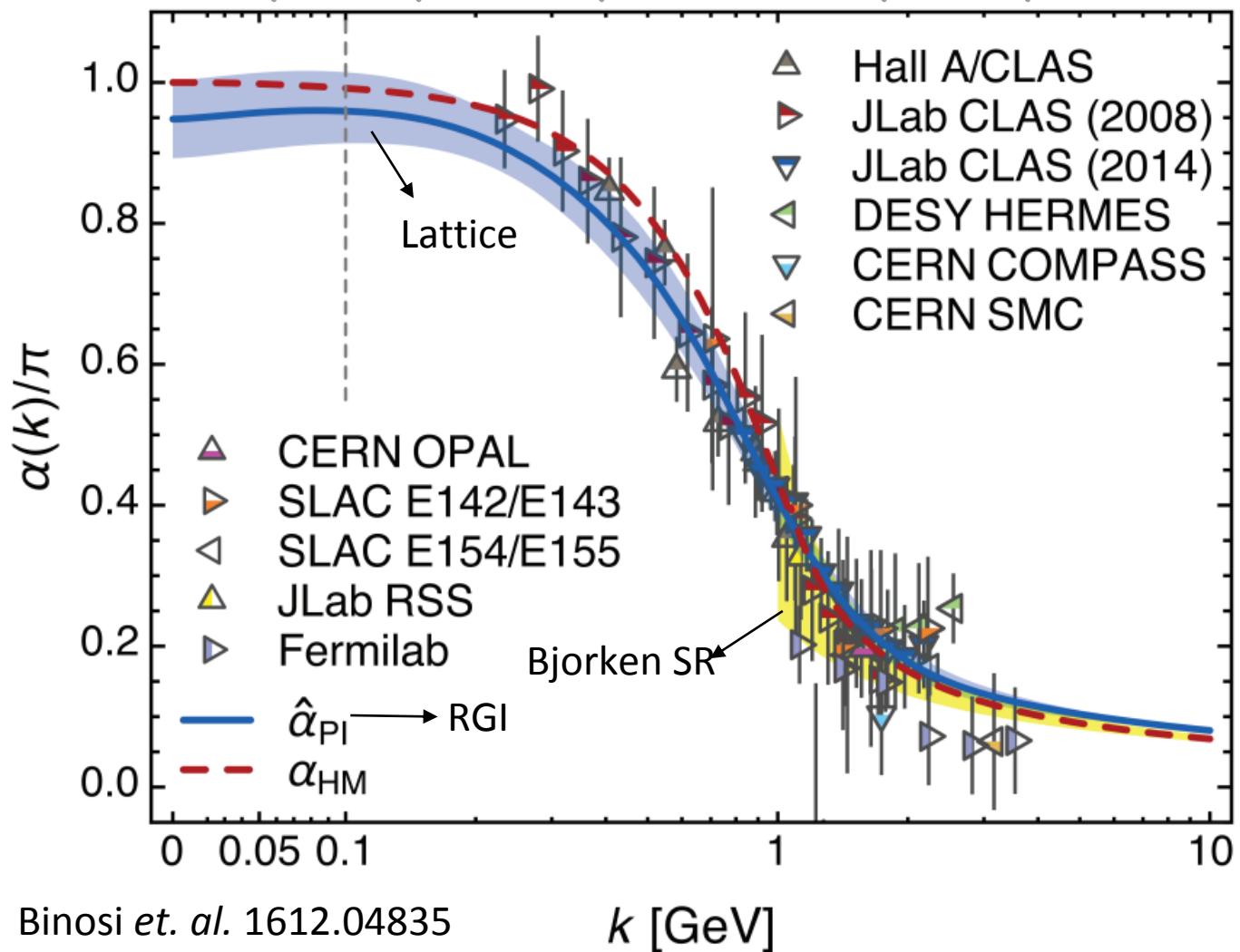
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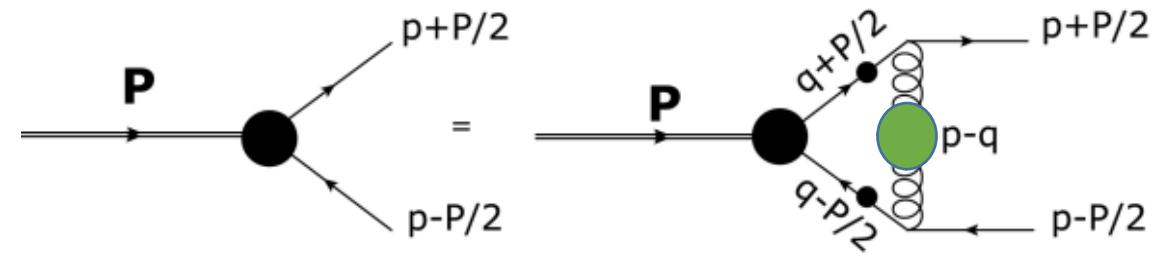
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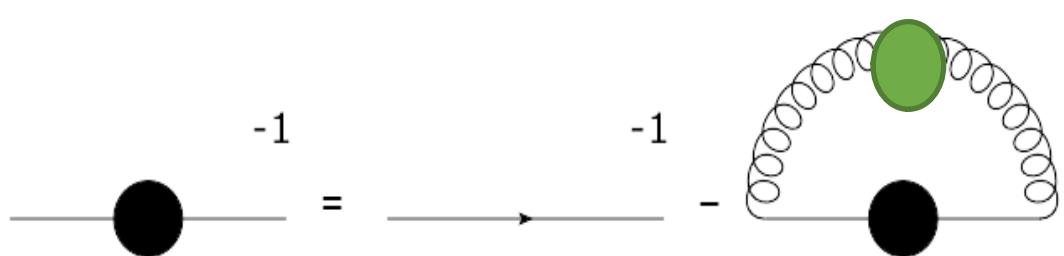
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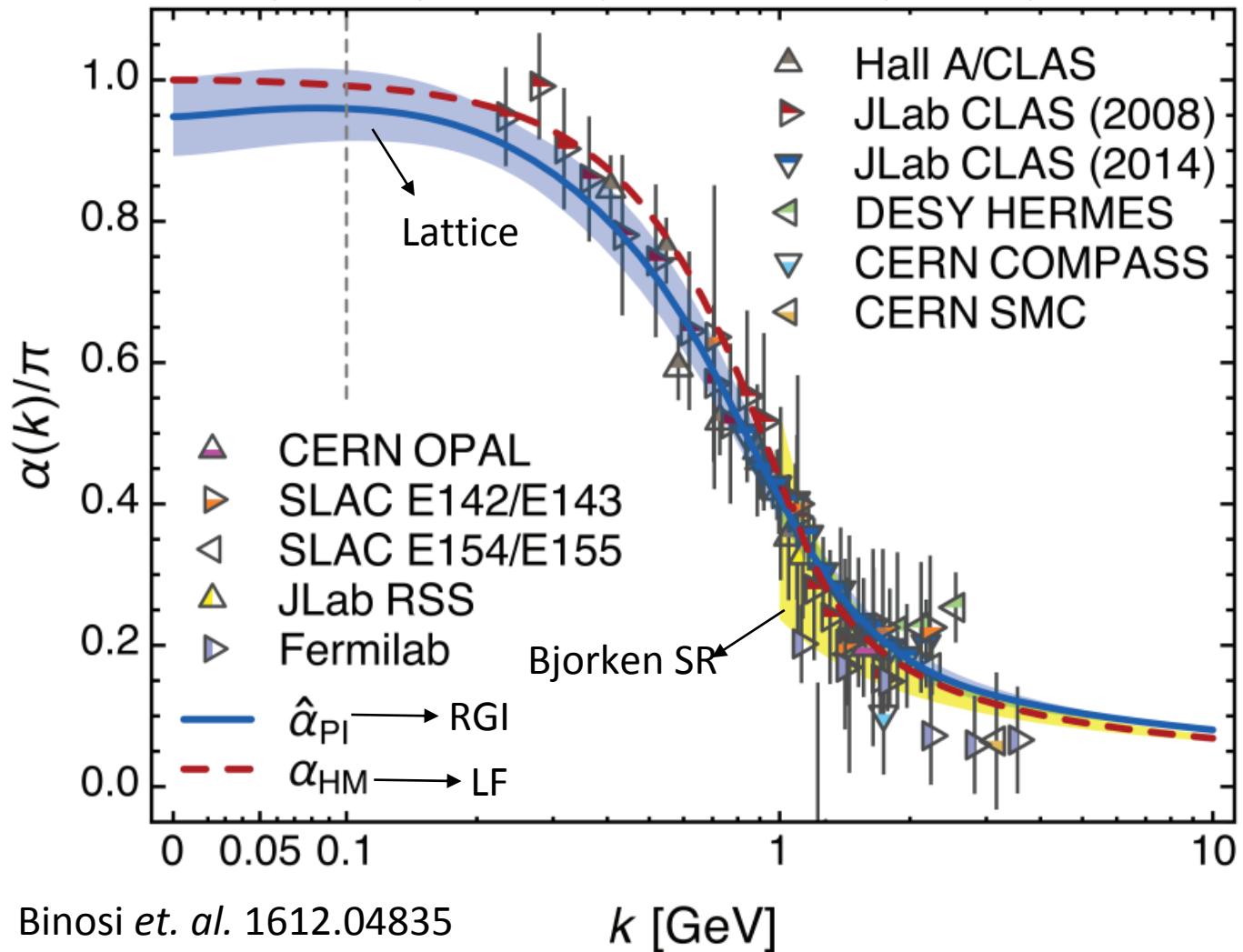


$$\int_0^1 dx [g_1^p(x, k^2) - g_1^n(x, k^2)] = \frac{g_A}{6} \left[1 - \frac{1}{\pi} \alpha_{g_1}(k^2) \right]$$

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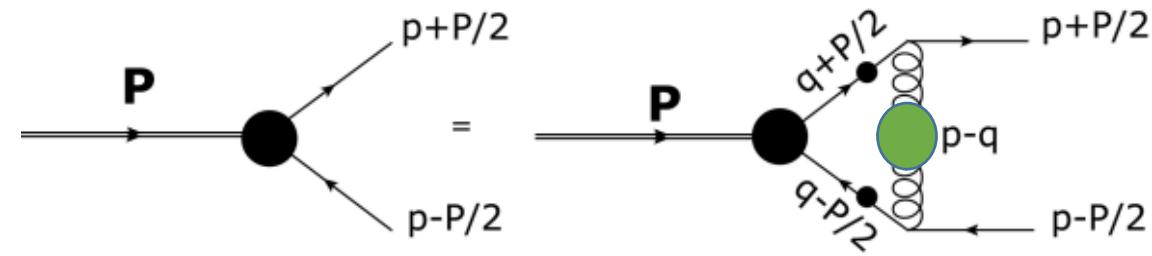
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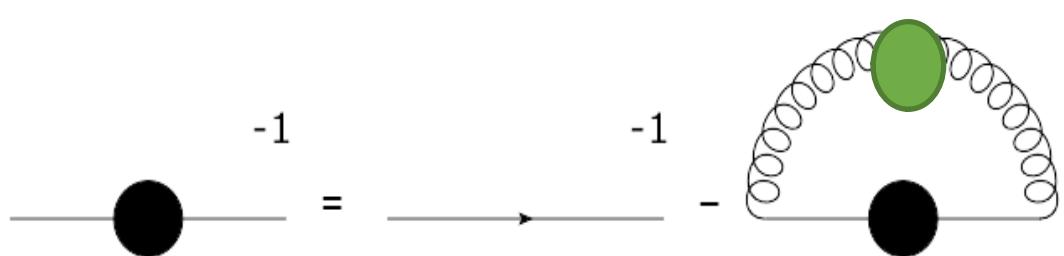
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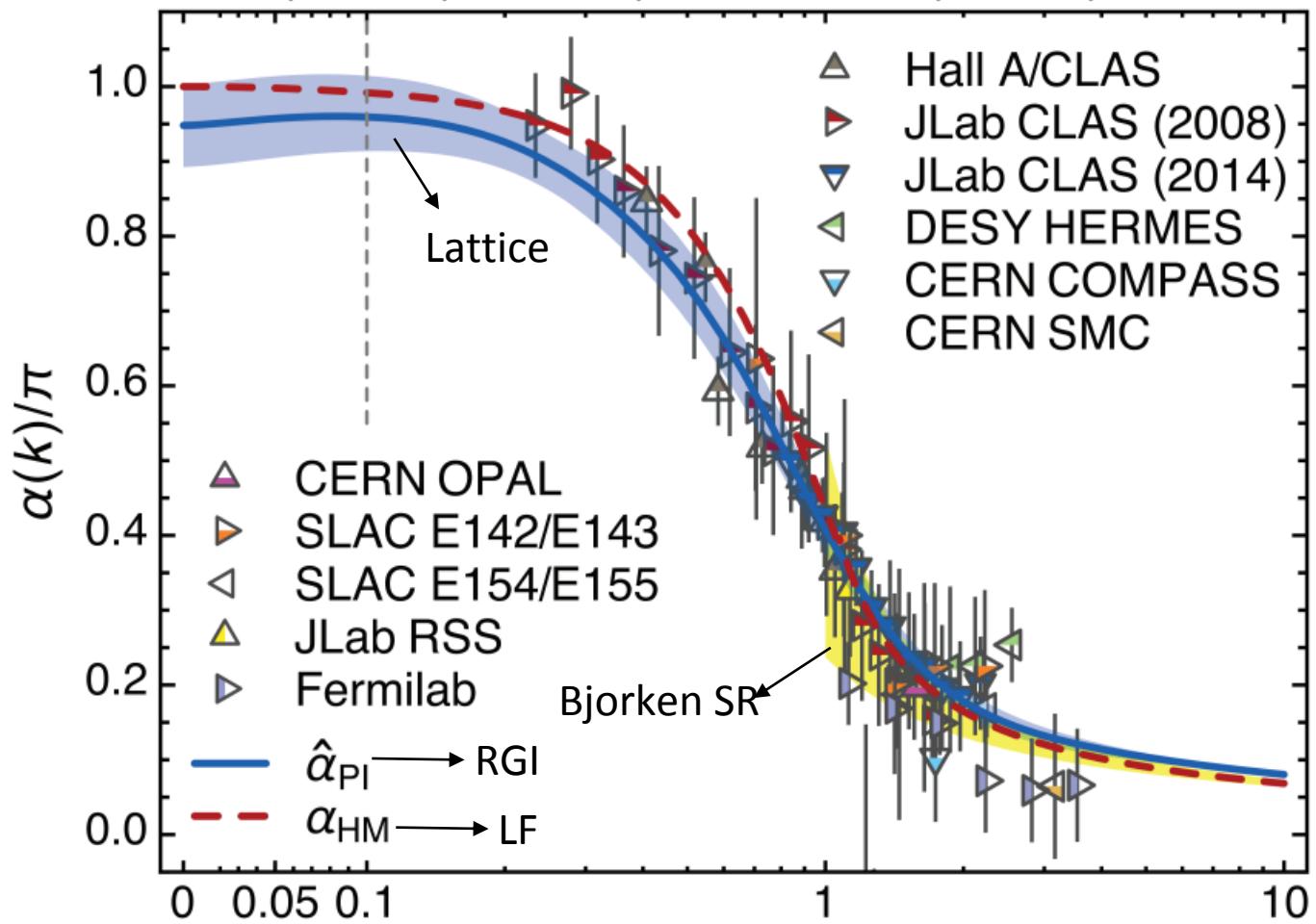
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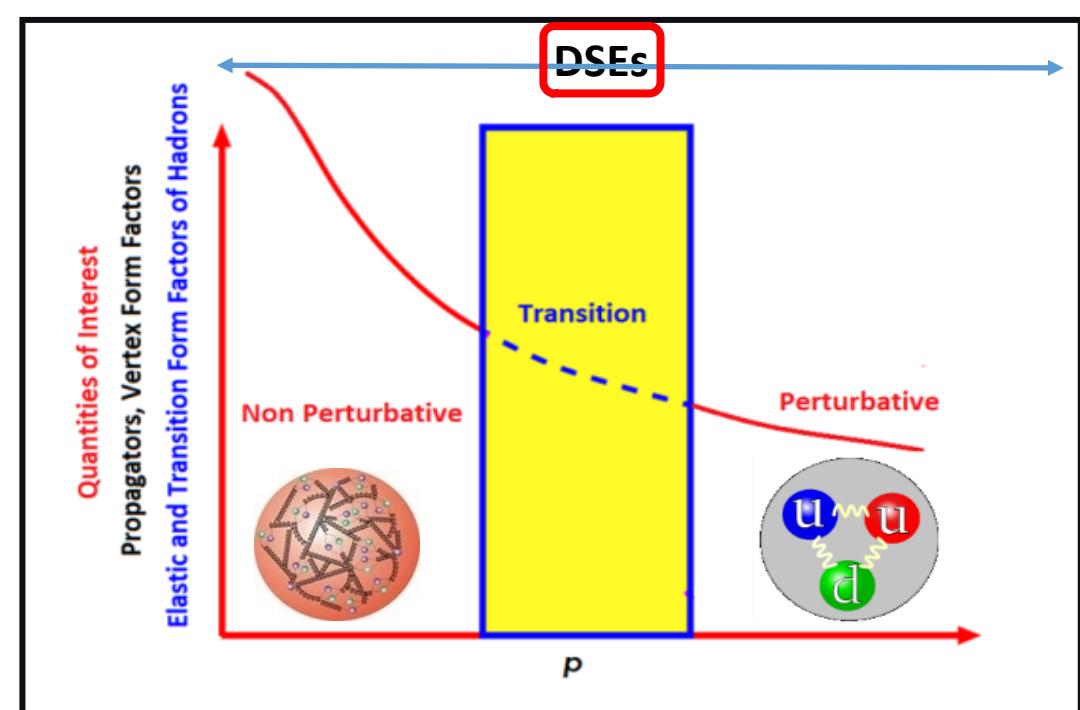
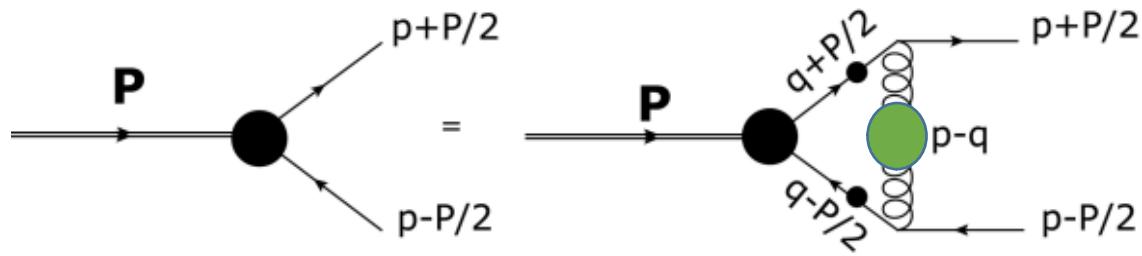
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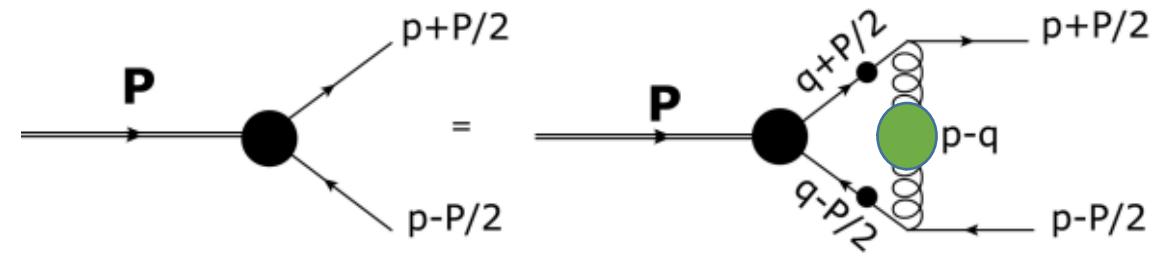
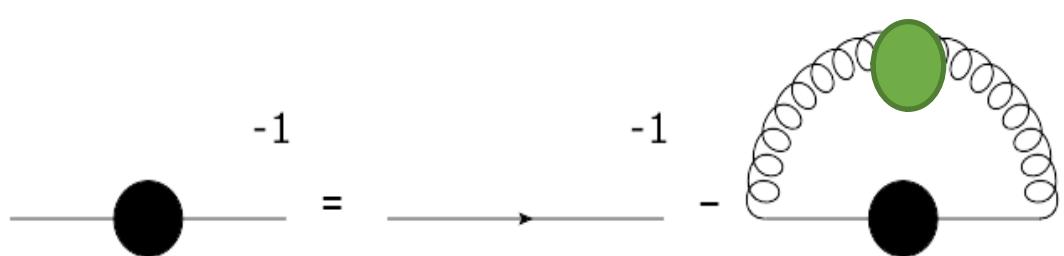
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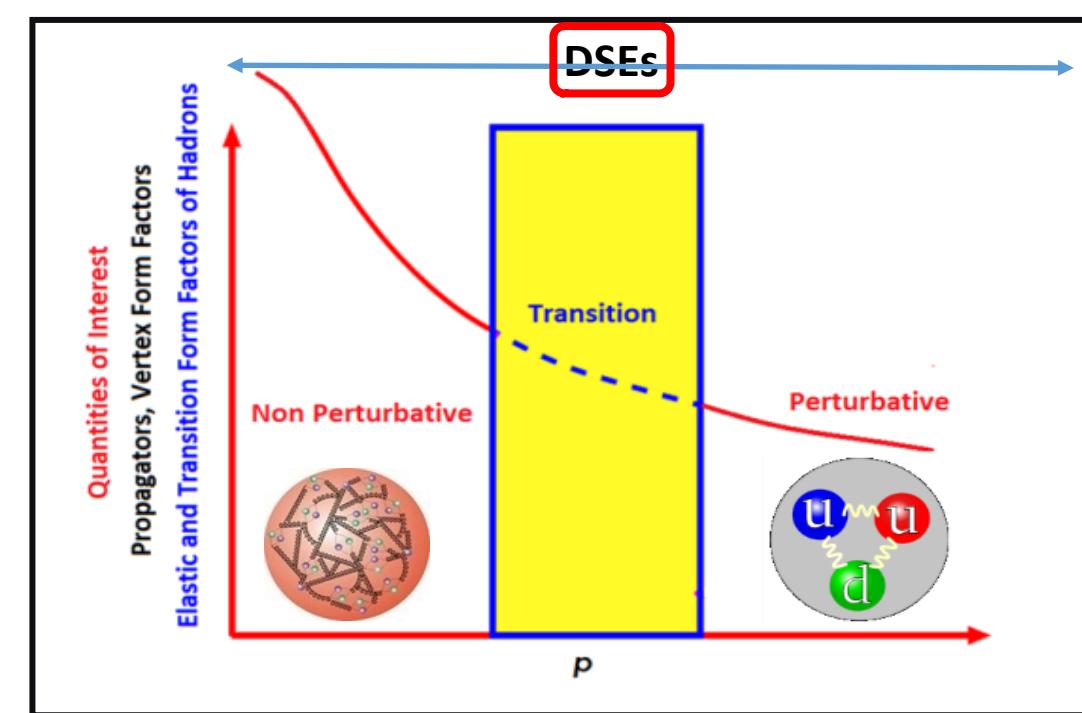
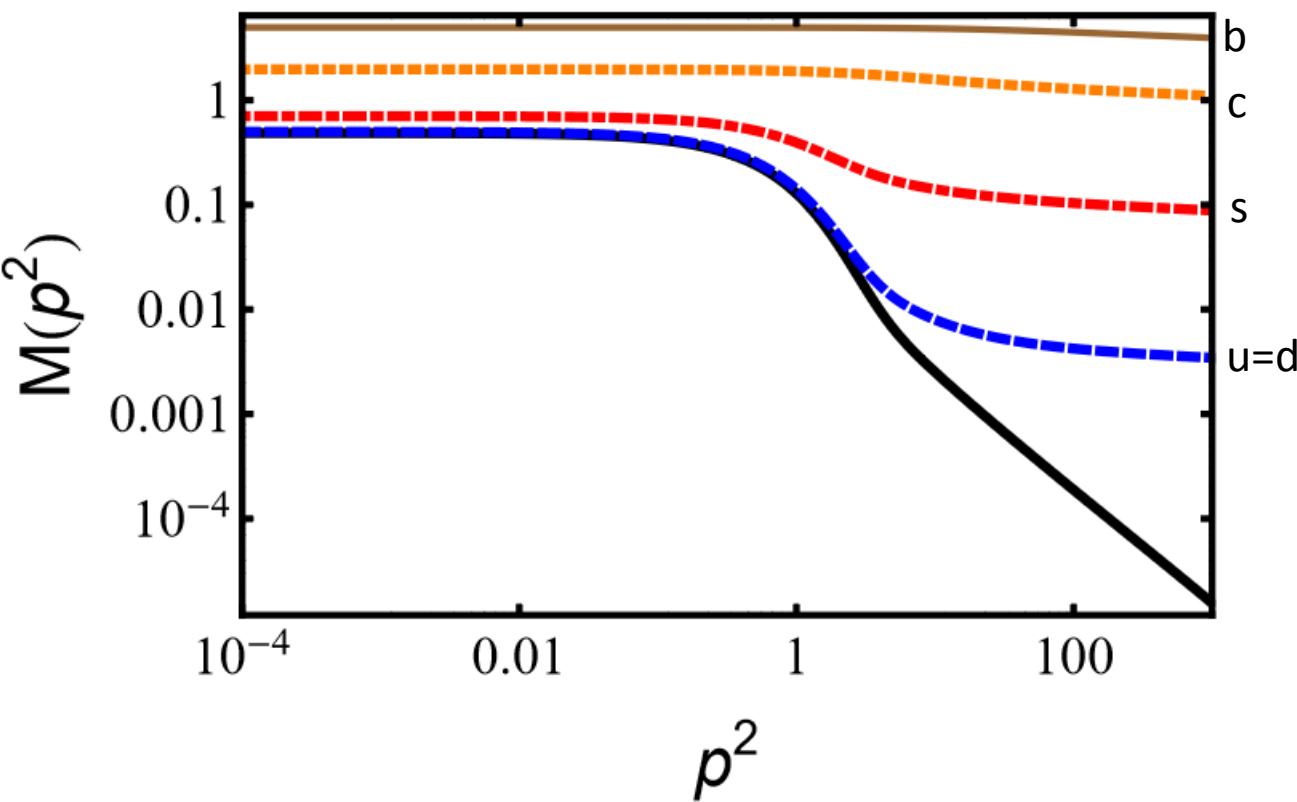
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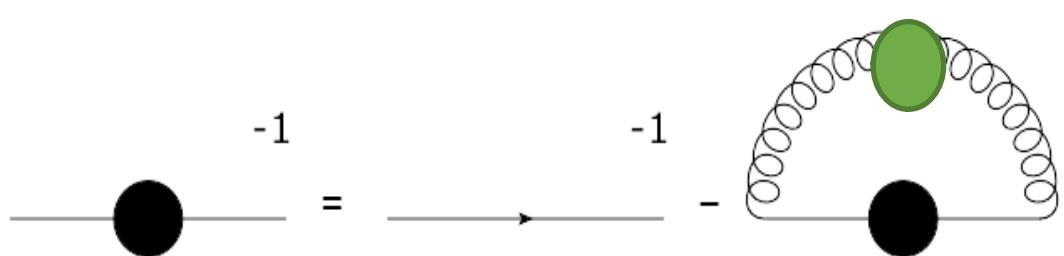


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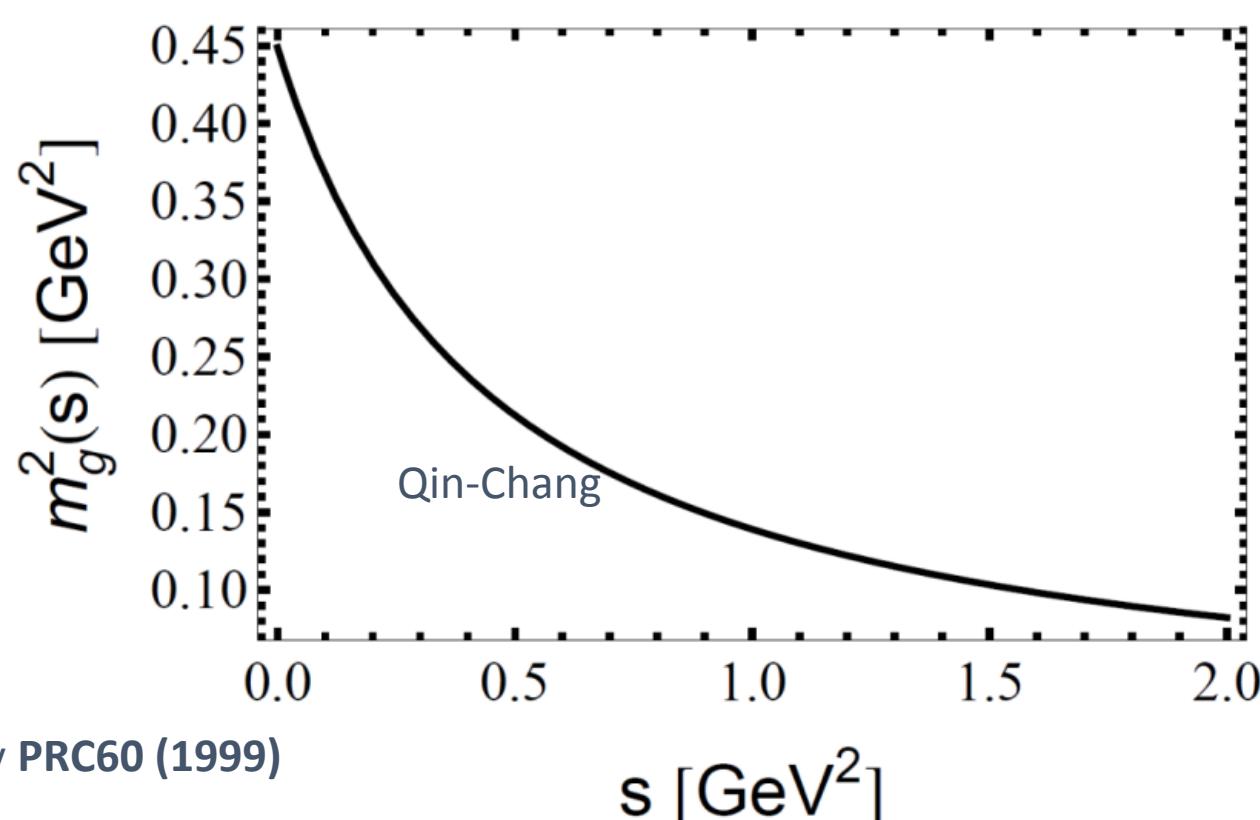
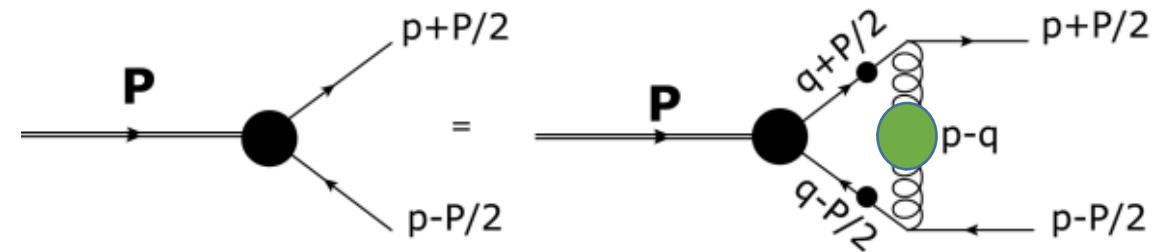
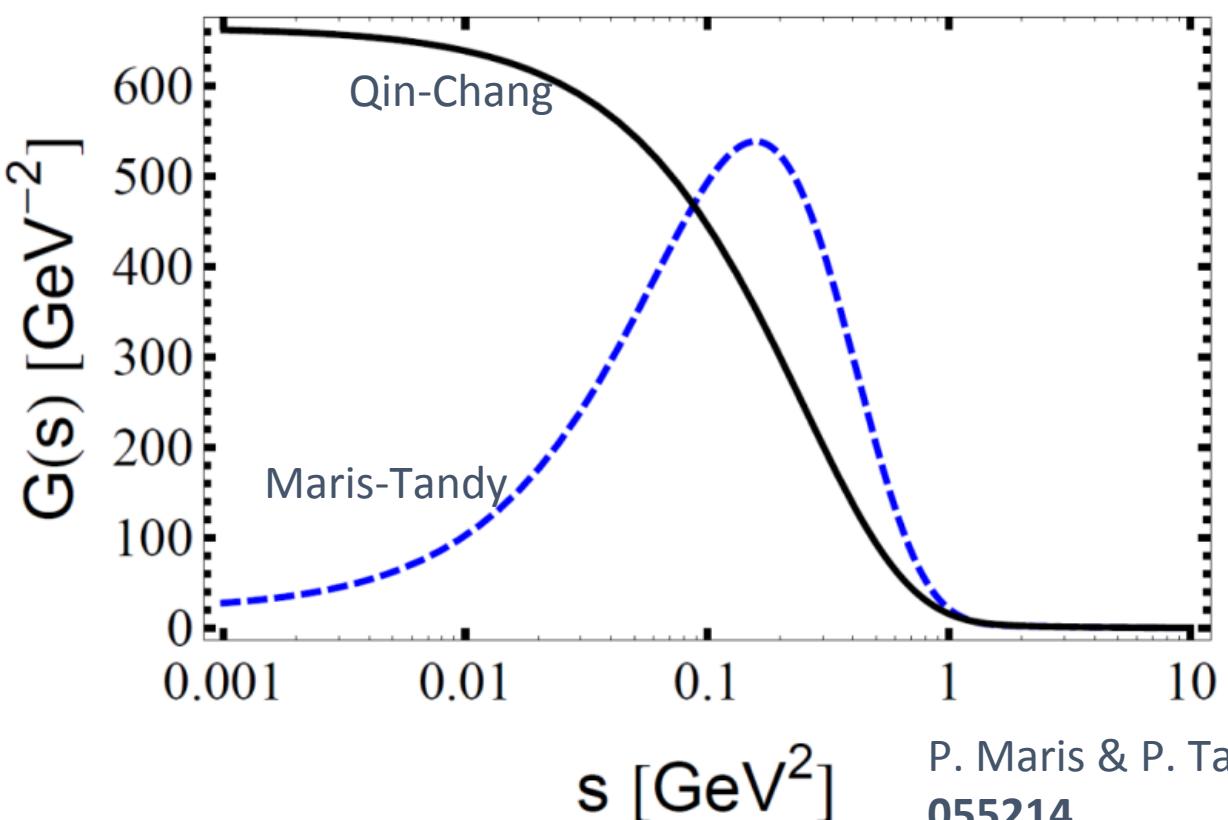
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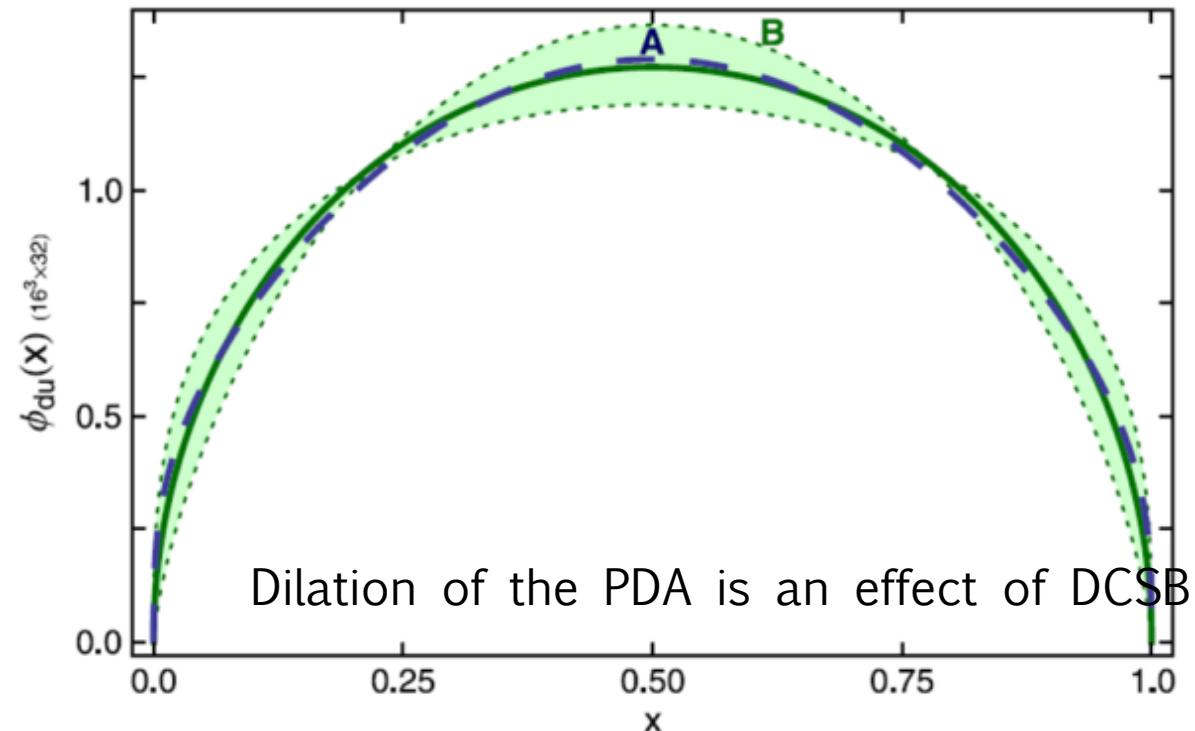
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$N_f = 2 + 1$ domain-wall fermions and nonperturbative renormalisation of lattice operators linear extrapolation to physical pion mass, $\overline{\text{MS}}$ -scheme at $\zeta = 2$ GeV, two lattice volumes.

Pion PDA

Precise agreement of DSE with lattice-QCD result (R. Arthur et al., Phys.Rev. D83 (2011) 074505).



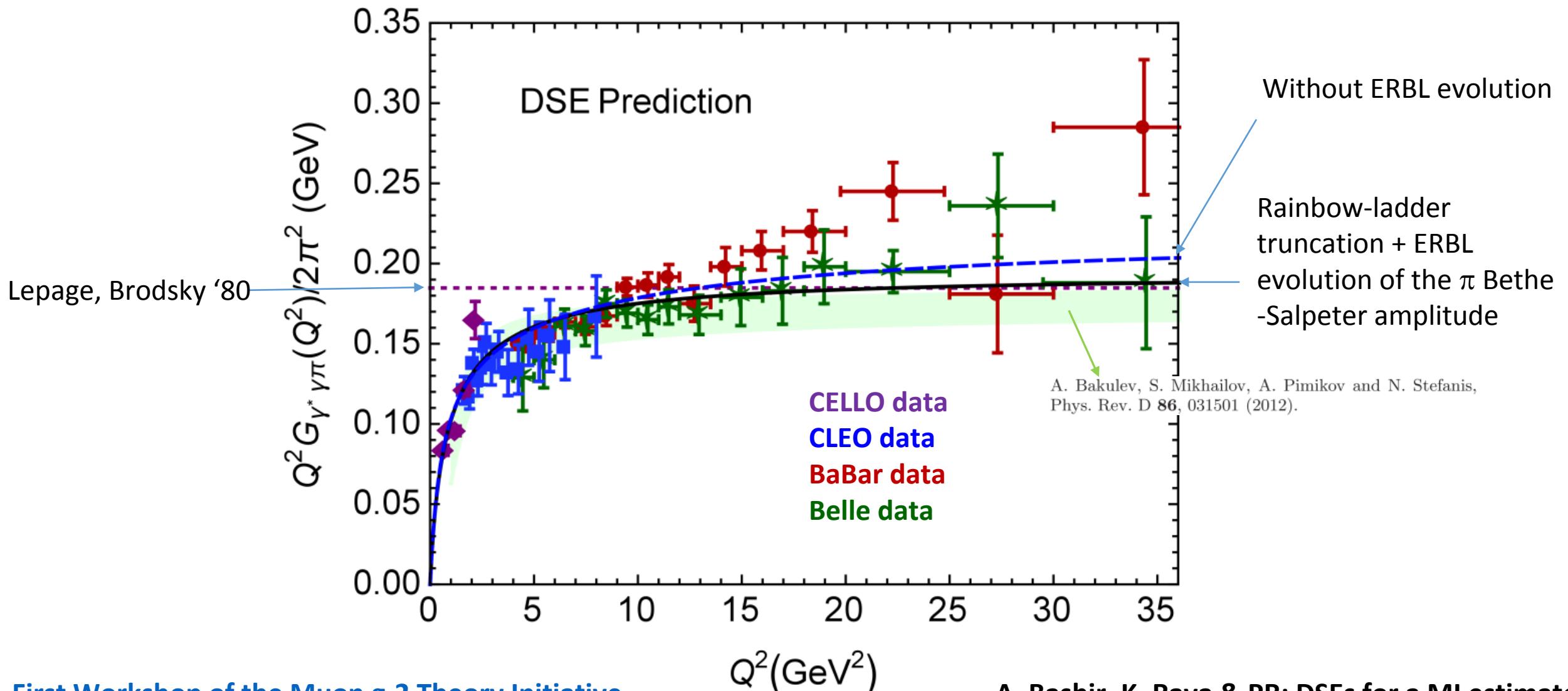
- A: DSE prediction
Phys.Rev.Lett. 110 (2013)
no.13, 132001.
 $\langle (2x - 1)^2 \rangle_{DSE} = 0.25$
- B: Inferred PDA from lattice
Phys.Lett. B731 (2014) 13-18.
 $\langle (2x - 1)^2 \rangle_{lQCD} = 0.25(1)(2)$

Phys.Lett. B731 (2014) 13-18. “*Distribution amplitudes of light-quark mesons from lattice QCD*”

Jorge Segovia, Lei Chang, Ian C. Cloët, Craig D. Roberts, Sebastian M. Schmidt, Hong-shi Zong

π^0 TFF from Dyson-Schwinger equations

K. Raya, L. Chang, A. Bashir, J. J. Cobos-Martínez, L. X. Gutiérrez-Guerrero, C. D. Roberts, P. C. Tandy Phys. Rev. D93 (2016) no.7, 074017



Evaluation of $a_\mu^{\pi^0\text{-pole}}$ with DSE input: Conclusions

We have evaluated $a_\mu^{\pi^0\text{-pole}}$ varying parameters in the ranges discussed previously

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See backup

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What's next? Obtain **fully off-shell π^0 TFF** (Si-Xue Qin, Chen Chen, Cedric Mezrag, Craig D. Roberts; arXiv:1702.06100[nucl-th])

Using the LMD+V parametrization we will compute $a_\mu^{\pi^0\text{-exchange}}$

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After BaBar & Belle data
Some personal reference values...

Pole contributions

$$a_\mu^{\pi^0, HLB} = (5.75 \pm 0.06) \cdot 10^{-10}$$

Exchange contributions

$$a_\mu^{\pi^0, HLB} = (6.66 \pm 0.21) \cdot 10^{-10}$$

According to Roig, Guevara & López-Castro,
Phys.Rev. D89 (2014) no.7, 073016

↑
Roig & Sanz-Cillero PLB733 (2014) 158

↓
Similar results in Kampf & Novotny
Phys. Rev. D84 (2011), 014036

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$$a_\mu^{\eta', HLbL} = (1.08 \pm 0.09) \cdot 10^{-10}$$

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$$a_\mu^{\eta, HLbL} = (2.04 \pm 0.44) \cdot 10^{-10}$$

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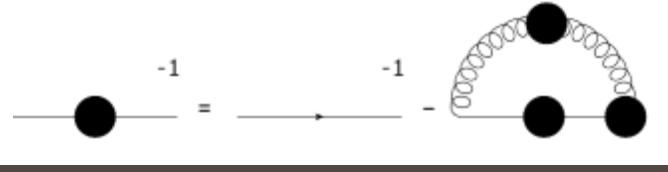
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$$a_\mu^{BSM} \lesssim 288 \times 10^{-11}$$

BACKUP SLIDES

Quark Propagator



- The renormalised DSE for the quark propagator (gap equation) is:

$$S^{-1}(p, \zeta) = [\mathcal{Z}_{2F} S_0^{-1}(p)] + \mathcal{Z}_{1F} \int_q^\Lambda g^2 D_{\mu\nu}(p-q, \zeta) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma_\nu^a(p, q; \zeta)$$

- A general solution is written as:

$$S(p, \zeta) = Z(p^2; \zeta^2) (i\gamma \cdot p + M(p^2))^{-1} = (i\gamma \cdot p A(p^2; \zeta^2) + B(p^2; \zeta^2))^{-1}$$

- The simplest, yet **symmetry preserving** truncation, is the **rainbow truncation**. With $k = p - q$ and $G(k^2)$ being an effective coupling, we have:

$$S^{-1}(p, \zeta) = [\mathcal{Z}_{2F} S_0^{-1}(p)] + \int_q^\Lambda G((p-q)^2) D_{\mu\nu}^0(p-q, \zeta) \frac{\lambda^a}{2} \gamma_\mu S(q) \frac{\lambda^a}{2} \gamma_\nu,$$

$$\mathcal{Z}_{1F} g^2 D_{\mu\nu}(k) \Gamma_\nu^a(q, p) \rightarrow k^2 G(k^2) D_{\mu\nu}^2(k) \frac{\lambda^a}{2} \gamma_\nu ,$$



Bethe-Salpeter equation



- The two-particle bound state equation is the Bethe-Salpeter equation (BSE). It is written as:

$$\Gamma_M(p; P) = \int_q^\Lambda K(q, p; P) S(q^+) \Gamma_M(q; P) S(q^-), \quad q^\pm = q \pm P/2$$

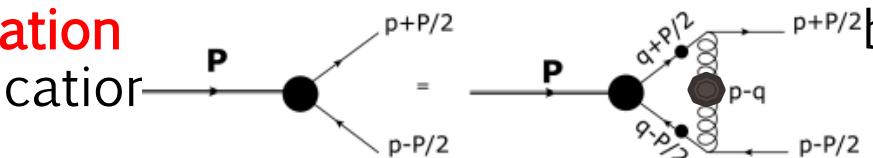
- The Interaction kernel, $K(q, p; P)$, is related to the truncation of the gap equation via the axial vector Ward-Takahashi identity (*Phys.Lett. B733 (2014) 202-208*, Qin et al.):

$$[\Sigma(p^+) \gamma_5 + \gamma_5 \Sigma(p^-)] = \int_q^\Lambda K(q, p; P) [\gamma_5 S(q^-) + S(q^+) \gamma_5]$$

- It implies:

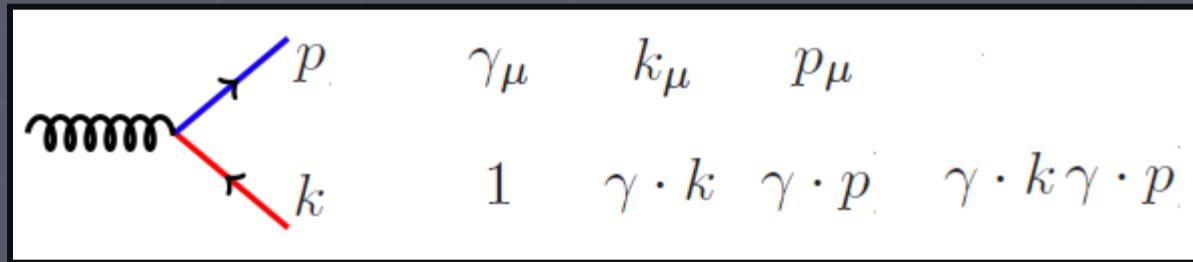
$$K(p, q; P) = -G((p - q)^2) (p - q)^2 D_{\mu\nu}^0(p - q) \frac{\lambda^a}{2} \gamma_\mu \times \frac{\lambda^a}{2} \gamma_\nu$$

- This corresponds to the **ladder truncation** truncation, it is called Rainbow-ladder truncation



The Quark-Gluon Vertex

- In addition to the the gluon propagator, quark-gluon vertex is another object which enters the quark SDE.



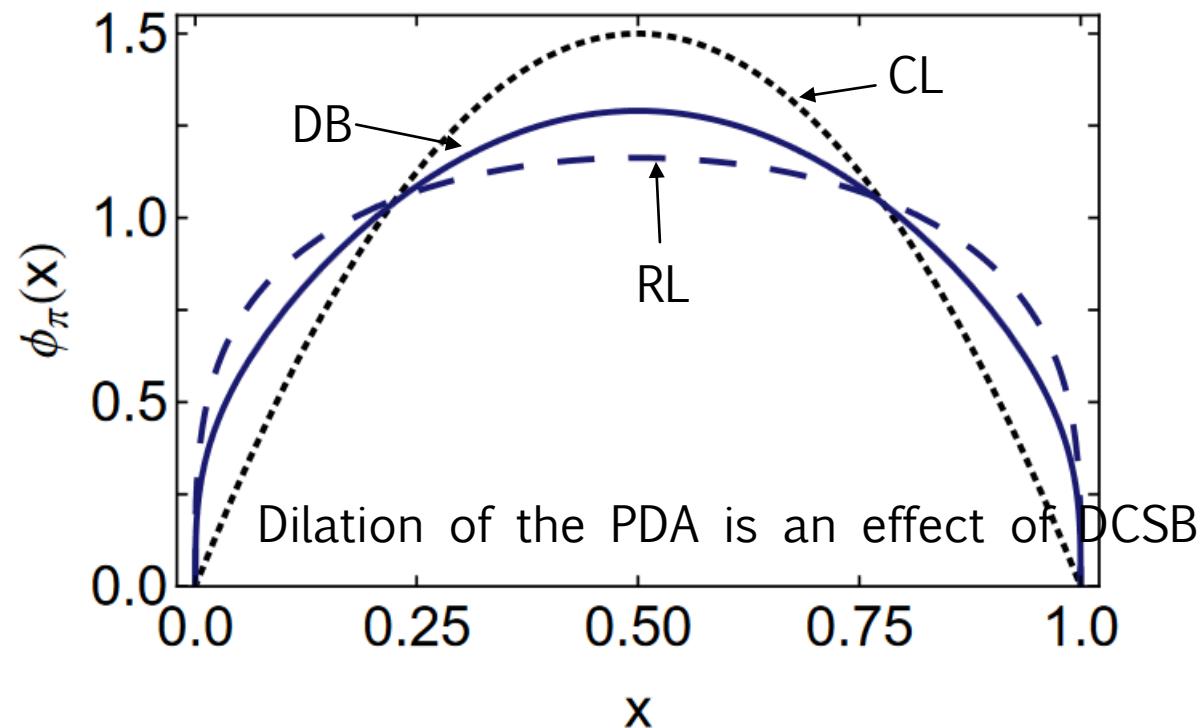
- Quark gluon vertex consists of 12 linearly independent Dirac structures.
- 6 of these 12 structures are generated dynamically in the chiral limit.

$$\gamma^\mu \not{k}, \gamma^\mu \not{p}, k^\mu, p^\mu, k^\mu \not{k} \not{p}, p^\mu \not{k} \not{p}$$

- Thus DCSB manifests itself not only in the quark propagator but also the quark-gluon vertex.

Pion PDA

Pion PDA at hadronic scale ($\zeta=2$ GeV), pion PDA is a broad concave function of x .

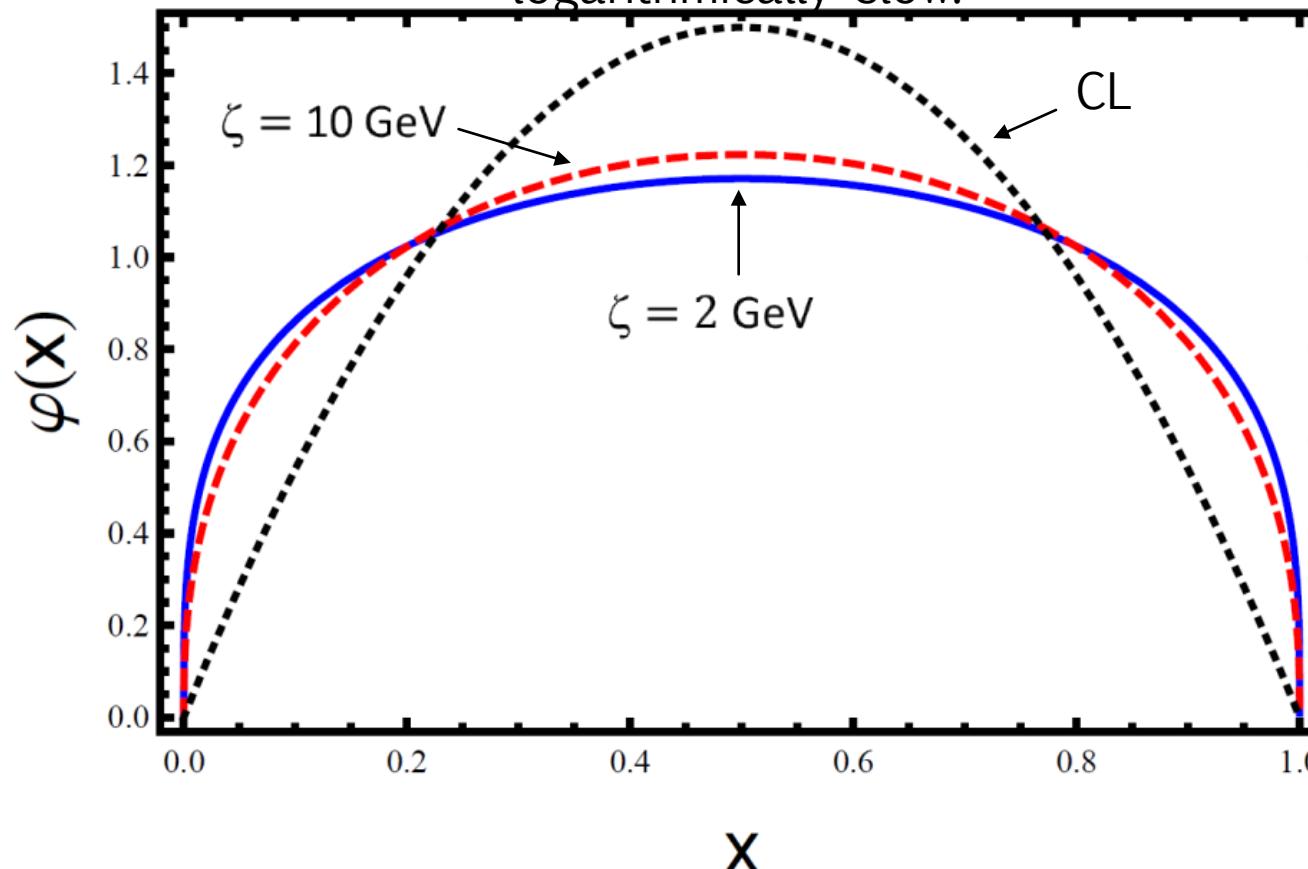


Phys.Rev.Lett. 110 (2013) no.13, 132001

“Imaging dynamical chiral symmetry breaking: pion wave function on the light front”
Lei Chang, Ian C. Cloët, J. Javier Cobos-Martinez, Craig D. Roberts, Sebastian. M. Schmidt, Peter. C. Tandy

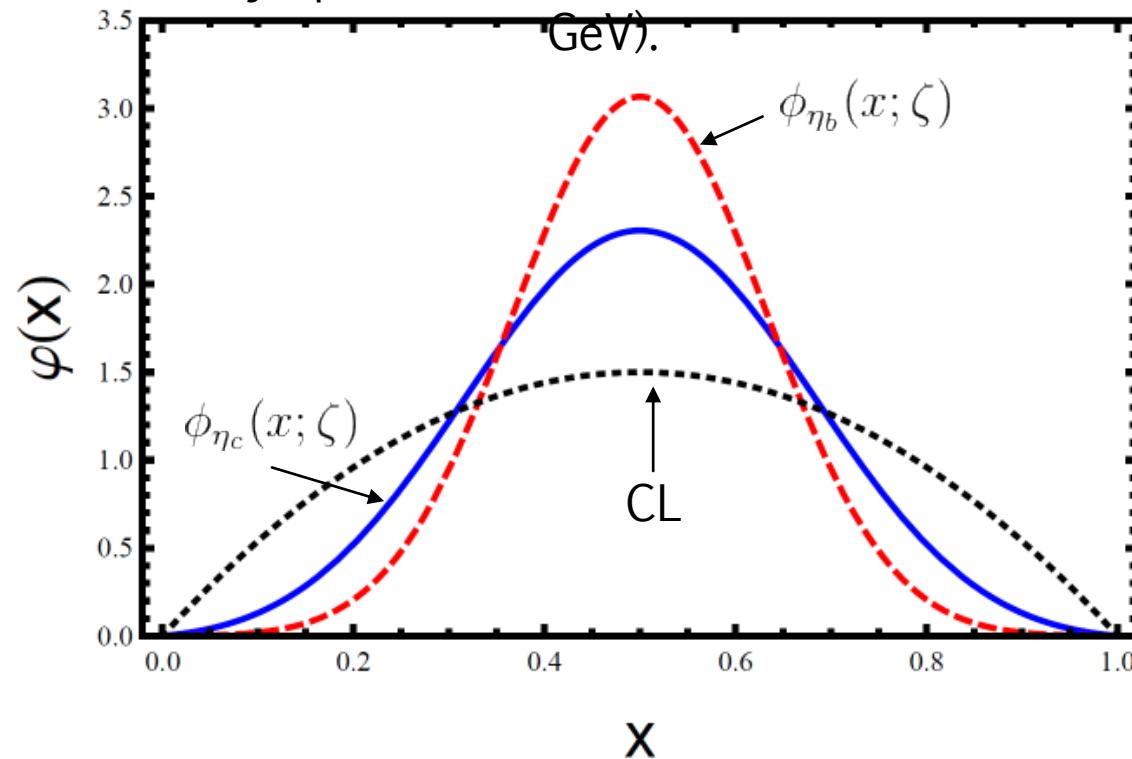
Pion PDA

Evolution of PDA from a hadron scale towards its asymptotic form is logarithmically slow.



Heavy quarkonia PDAs

Unlike pion PDA, heavy quarkonia PDAs are narrow at real-life scales ($\zeta=2$

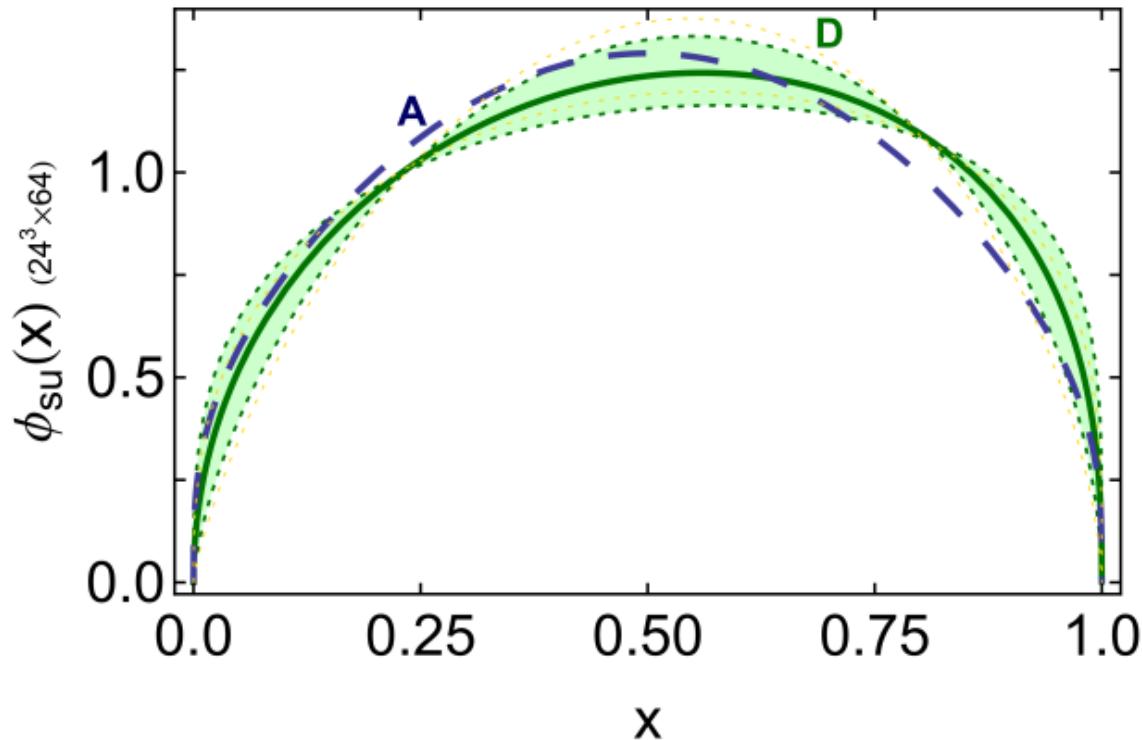
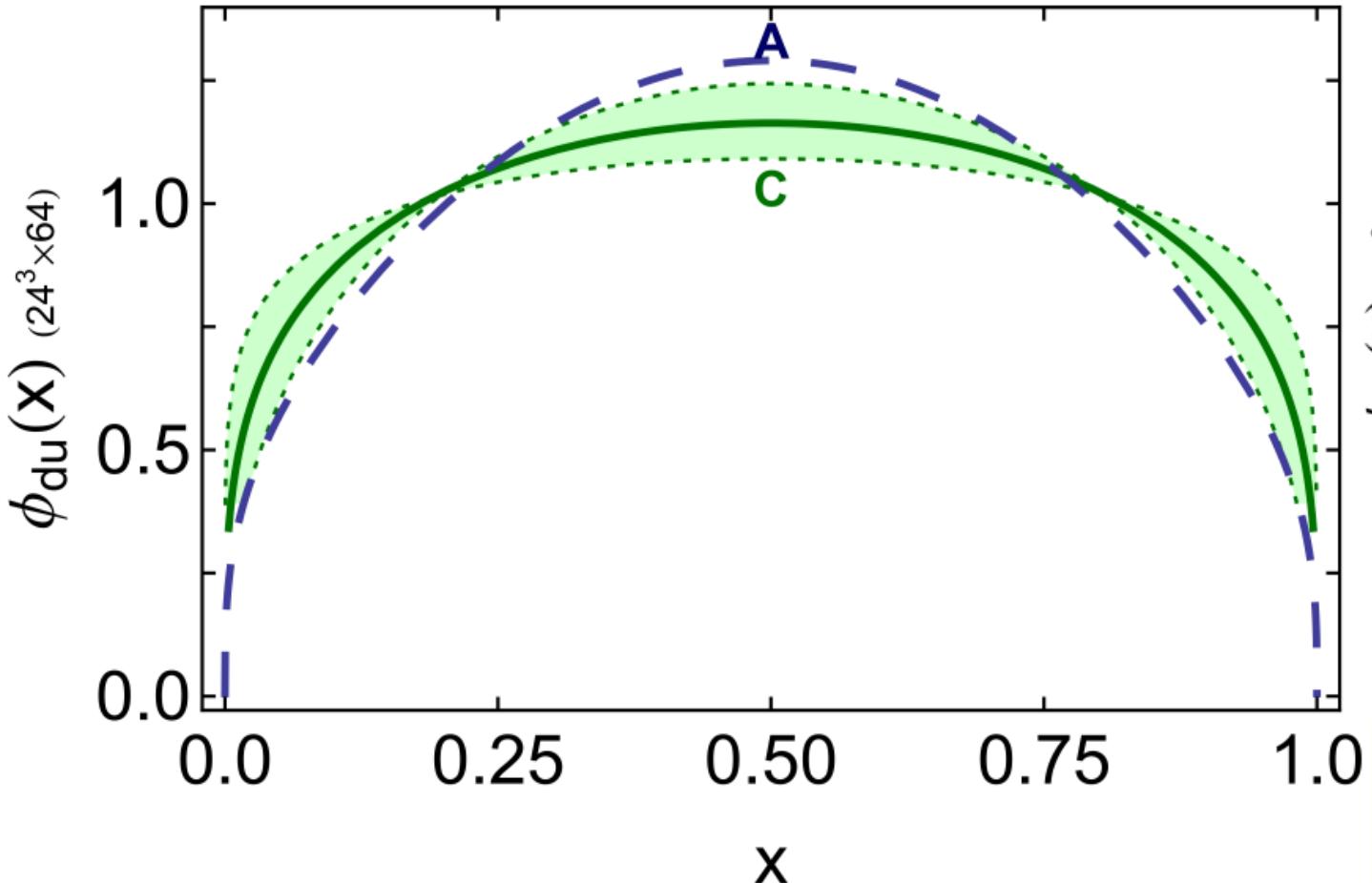


Phys.Lett. B753 (2016) 330-335

“Leading-twist parton distribution amplitudes of S-wave heavy-quarkonia”

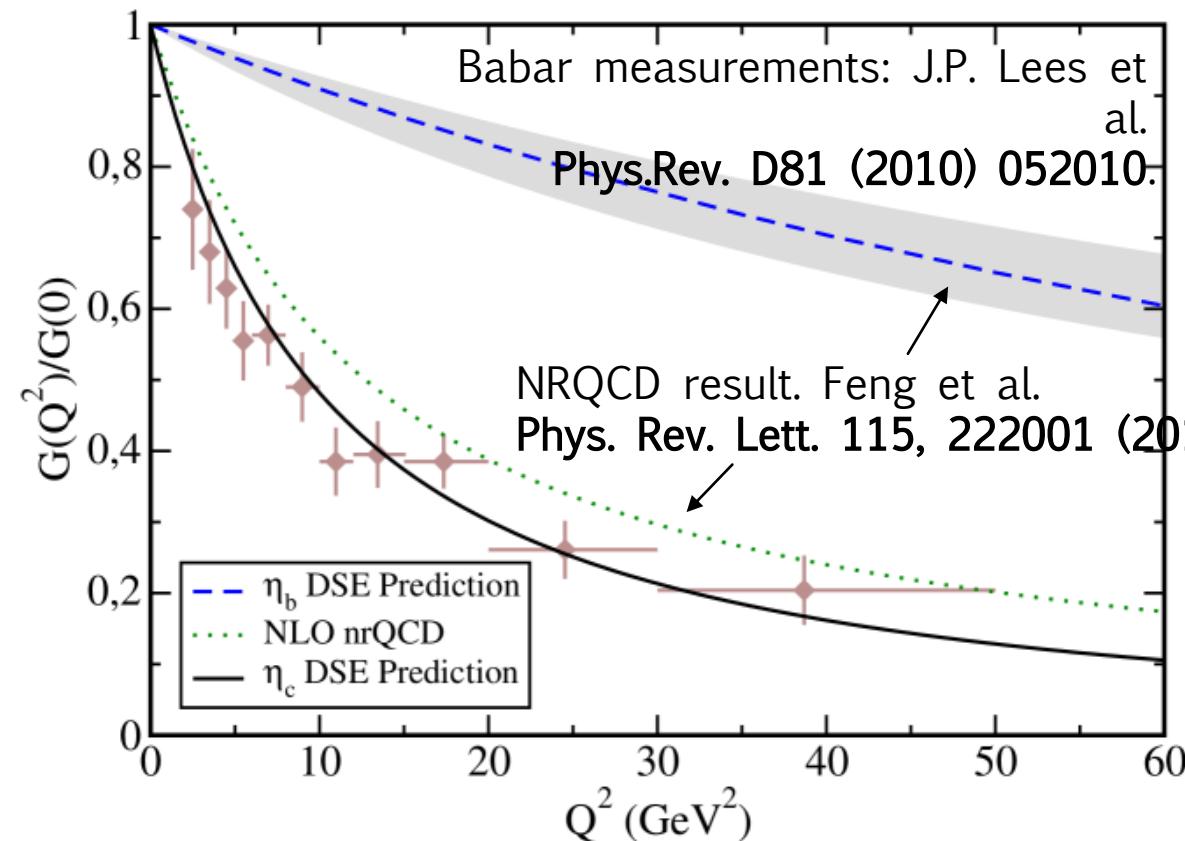
Minghui Ding, Fei Gao, Lei Chang, Yu-Xin Liu, Craig D. Roberts

Other PDA comparisons



Phys.Lett. B731 (2014) 13-18. “*Distribution amplitudes of light-quark mesons from lattice QCD*”

η_c, η_b transition form factors



- $\eta_{c,b}$ TFF DSE prediction:

$$\Gamma[\eta_c \rightarrow \gamma\gamma] = 6.1 \text{ keV}, r = 0.16 \text{ fm.}$$

$$\Gamma[\eta_b \rightarrow \gamma\gamma] = 0.52 \text{ keV}, r = 0.04 \text{ fm.}$$

Our result matches the available data and the empirical value of the η_c interaction radius ($r=0.17 \text{ fm}$).

NNLO η_c result of NRQCD is vastly different from the data. Our agreement with data tells that NRQCD is not a reliable effective field theory for exclusive processes involving charmonia, however, it is for bottomonia.

Phys.Rev. D95 (2017) no.7, 074014.

“Partonic structure of neutral pseudoscalars via two photon transition form factors”

K. R., M. Ding, A. Bashir, L. Chang, C.D. Roberts

Perturbation theory integral representations (PTIRs)

- The quark propagator may be expressed as:

$$S(p; \zeta) = -i\gamma \cdot p \sigma_v(p^2; \zeta) + \sigma_s(p^2; \zeta)$$

- The **numerical solutions** are parametrized in terms of N pairs of complex conjugate poles:

$$\sigma_v(q) = \sum_{k=1}^N \left(\frac{z_k}{q^2 + m_k^2} + \frac{z_k^*}{q^2 + m_k^{*2}} \right), \quad \sigma_s(q) = \sum_{k=1}^N \left(\frac{z_k m_k}{q^2 + m_k^2} + \frac{z_k^* m_k^*}{q^2 + m_k^{*2}} \right).$$

$\Im m_j \neq 0 \forall j$

- Constrained to the UV conditions of the free quark propagator form. For our computations, we found that N=2 is adequate.

- Phys.Rev. D67 (2003) 054019. “Confinement phenomenology in the Bethe-Salpeter equation”

M. S. Bhagwat, M. A. Pichowsky, and P. C. Tandy

Perturbation theory integral representations (PTIRs)

- On the other hand, BSAs may be written as in the Nakanishi representation. We split the amplitude in IR and UV:

$$A(q, P) = \int_{-1}^1 dz \int_0^\infty d\Lambda \left[\frac{\rho^i(z, \Lambda)}{(q^2 + zq \cdot P + \Lambda^2)^{m+n}} + \frac{\rho^u(z, \Lambda)}{(q^2 + zq \cdot P + \Lambda^2)^n} \right] .$$

- In principle, one should plug into the BSE the above expression for the BSA and solve for $\rho(z, \Lambda)$, as described in Nakanishi's work, **Phys. Rev. 130 1230-1235 (1963)**.
- However, what we do, is to solve directly for the BSA, and match the Nakanishi-like representation to the numerical solution.

Perturbation theory integral representations (PTIRs)

- On the other hand, BSA may be written as in the Nakanishi-like representation (**Phys. Rev. 130 1230-1235 (1963)**). Our particular choice was first described in **Phys. Rev. Lett. 110 (2013) no.13, 132001** (Chang et al.), and refined in **Phys. Rev. D93 (2016) no.7, 074017** (KR et al.):

$$E^u(k; P) = c_E^u \int_{-1}^1 dz \rho_{\nu_E^u}(z) \hat{\Delta}_{\Lambda_E^u}^{1+\alpha}(k_z^2) \quad F^u(k, P) = c_F^u \int_{-1}^1 dz \rho_{\nu_F^u}(z) k^2 \Lambda_F^u \Delta_{\Lambda_F^u}^{2+\alpha}(k_z^2)$$

$$G^u(k, P) = c_G^u \int_{-1}^1 dz \rho_{\nu_G^u}(z) \Lambda_G^u \Delta_{\Lambda_G^u}^{2+\alpha}(k_z^2) \quad \mathcal{F}^i(k; P) = c_{\mathcal{F}}^i \int_{-1}^1 dz \rho_{\nu_{\mathcal{F}}^i}(z) \left[a_{\mathcal{F}} \hat{\Delta}_{\Lambda_{\mathcal{F}}^i}^4(k_z^2) + a_{\mathcal{F}}^- \hat{\Delta}_{\Lambda_{\mathcal{F}}^i}^5(k_z^2) \right]$$

- Where Λ , ν , a , c , are parameters **fitted** to the numerical data. The following definitions apply:

$$\hat{\Delta}_{\Lambda}(s) = \Lambda \Delta_{\Lambda}(s), \quad \Delta_{\Lambda}(s) = (s + \Lambda^2)^{-1}, \quad k_z^2 = k^2 + z k \cdot P. \quad \rho_{\nu}(z) \sim (1 - z^2)^{\nu}$$

- $H(k, P)$ is negligible for pion and η_c ; $G(k, P)$ and $H(k, P)$ are negligible for η_b .

$$\Gamma_{\pi}(\hat{k}; P) = i\gamma_5 \left[E_{\pi}(\hat{k}; P) + \gamma \cdot P F_{\pi}(\hat{k}; P) \right] \quad (\mathcal{F} = E, F, G)$$

$$+ \gamma \cdot \hat{k} \hat{k} \cdot P G_{\pi}(\hat{k}; P) + \sigma_{\mu\nu} \hat{k}_{\mu} P_{\nu} H_{\pi}(\hat{k}; P)$$

$$\mathcal{F}(k; P) = \mathcal{F}^i(k; P) + \mathcal{F}^u(k; P)$$

	z_1	m_1	z_s	m_2			
Perturbatio	(0.44, 0.014)	(0.54, 0.23)	(0.19, 0)	(-1.21, -0.65)			
	c^i	c^u	ν^i	ν^u	a	Λ^i	Λ^u
E	$1 - c_E^u$	0.03	-0.74	1.08	2.75	1.32	1.0
F	0.51	$c_E^u/10$	0.96	0.0	$2.78/\Lambda_F^i$	1.09	1.0
G	0.18	$2c_F^u$	ν_F^i	0.0	$5.73/[\Lambda_G^i]^3$	0.94	1.0

- On the other hand, in **Phys. Rev. 130 1230-1235 (1963)**, refined in **Phys. Rev. D93 (2016) no.7, 074017** (KR et al.), and refined in **Phys. Rev. D93 (2016) no.7, 074017** (KR et al.):

$$E^u(k; P) = c_E^u \int_{-1}^1 dz \rho_{\nu_E^u}(z) \hat{\Delta}_{\Lambda_E^u}^{1+\alpha}(k_z^2) \quad F^u(k, P) = c_F^u \int_{-1}^1 dz \rho_{\nu_F^u}(z) k^2 \Lambda_F^u \Delta_{\Lambda_F^u}^{2+\alpha}(k_z^2)$$

$$G^u(k, P) = c_G^u \int_{-1}^1 dz \rho_{\nu_G^u}(z) \Lambda_G^u \Delta_{\Lambda_G^u}^{2+\alpha}(k_z^2) \quad \mathcal{F}^i(k; P) = c_{\mathcal{F}}^i \int_{-1}^1 dz \rho_{\nu_{\mathcal{F}}^i}(z) \left[a_{\mathcal{F}} \hat{\Delta}_{\Lambda_{\mathcal{F}}^i}^4(k_z^2) + a_{\mathcal{F}}^- \hat{\Delta}_{\Lambda_{\mathcal{F}}^i}^5(k_z^2) \right]$$

- Where Λ , ν , a , c , are parameters **fitted** to the numerical data. The following definitions apply:

$$\hat{\Delta}_{\Lambda}(s) = \Lambda \Delta_{\Lambda}(s), \quad \Delta_{\Lambda}(s) = (s + \Lambda^2)^{-1}, \quad k_z^2 = k^2 + z k \cdot P. \quad \rho_{\nu}(z) \sim (1 - z^2)^{\nu}$$

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$$+ \gamma \cdot \hat{k} \hat{k} \cdot P G_{\pi}(\hat{k}; P) + \sigma_{\mu\nu} \hat{k}_{\mu} P_{\nu} H_{\pi}(\hat{k}; P) \quad \mathcal{F}(k; P) = \mathcal{F}^i(k; P) + \mathcal{F}^u(k; P)$$

Parton Distribution Amplitudes

- Valence-quark distribution amplitude (PDA) is the probability density of having a quark-antiquark bound state, with momentum fraction x and $1-x$, respectively.
- The PDA is a projection of the system's Bethe-Salpeter wave-function onto the light-front. It is therefore process independent and hence plays a crucial role in explaining and understanding a wide range of a given meson's properties and interactions.
- Given a pseudoscalar meson with total momentum P , a resolution scale ζ and a light-cone four-vector n ($n^2 = 0, n \cdot P = -m_\pi$), the PDA reads as:

$$f_M \phi_M(x; \zeta) = Z_2(\Lambda; \zeta) \int_q^\Lambda \delta(n \cdot q^+ - xn \cdot P) \gamma_5 \gamma \cdot n \chi_M(q; P), \quad \chi_M(q; P) = S(q^+) \Gamma_M(q; P) S(q^-)$$

- The moments of the distribution are given by:

$$\langle x^m \rangle = \int_0^1 dx x^m \phi_M(x; \zeta), \quad f_M (n \cdot P)^{m+1} \langle x^m \rangle = \text{tr}_{CD} Z_2 \int_q^\Lambda (n \cdot q^+)^m \gamma_5 \gamma \cdot n \chi_M(q; P)$$

Parton Distribution Amplitudes

- According to **Phys. Rev. D22, 2157 (1980)** by G. Peter Lepage, Stanley J. Brodsky, in the neighborhood of the conformal limit, it is written in terms of 3/2-Gegenbauer polynomials.
- PDA should evolve with the resolution scale $\zeta^2=Q^2$ through the ERBL evolution equations (see **Phys. Lett. B87, 359(1979)** and **Phys. Lett. B94, 245 (1980)**).
- Evolution enables the dressed-quark and antiquark degrees of freedom, to split into less well-dressed partons via the addition of gluons and sea quarks in the manner prescribed by QCD dynamics.
- The asymptotic form (conformal limit) of the PDA is the well known result:

$$\phi^{cl}(x) = 6x(1 - x)$$

Quark-photon vertex

- We employ the ansatz explained in Phys.Rev.Lett. 111 (2013) no.14, 141802, Phys.Rev. D93 (2016) no.7, 074017 and Phys.Rev. D95 (2017) no.7, 074014
- With the following definitions (m = meson mass):

$$\Delta_F = [F(k_f^2) - F(k_i^2)]/[k_f^2 - k_i^2] \quad \mathcal{E} = \sqrt{Q^2/4 + m^2} - m$$

$$s = 1 + s_0 \text{Exp}[-\mathcal{E}/M_E] \quad M_E = \{p | p^2 = M^2(p^2), p^2 > 0\}$$

- The vertex ansatz is:

$$\chi_\mu(k_f, k_i) = \gamma_\mu \Delta_{k^2 \sigma_v} + [s \gamma \cdot k_f \gamma_\mu \gamma \cdot k_i + \bar{s} \gamma \cdot k_i \gamma_\mu \gamma \cdot k_f] \Delta_{\sigma_v} \\ + [s(\gamma \cdot k_f \gamma_\mu + \gamma_\mu \gamma \cdot k_i) + \bar{s}(\gamma \cdot k_i \gamma_\mu + \gamma_\mu \gamma \cdot k_f)] i \Delta_{\sigma_s}$$

↓
Axial anomaly

Quark-photon vertex

- The vertex is constructed through the gauge technique (**R. Delbourgo and P. C. West, J. Phys. A10, 1049 (1977)**), satisfies the longitudinal Ward-Green-Takahashi identity, is free of kinematic singularities, reduces to the bare vertex in the free-field limit, and has the same Poincaré transformation properties as the bare vertex.
- WTI ensures charge conservation, therefore $F_\pi(Q^2 = 0) = 1$ defines the charge. Owing to the WTI and the exponential damping of the transverse terms, the presence of s does not affect neither the qualitative nor quantitative behavior of EFF.
- Our vertex is written in terms of dressing functions which characterizes the quark propagator. Up to transverse pieces associated with s , $S(k_f)\Gamma_\mu(k_f, k_i)S(k_i)$ and $\chi_\mu(k_f, k_i)$ are equivalent. Nothing material would be gained in keeping them equal.
- We spared the need to solve the DSE for the quark-photon vertex, and instead, we can employ the PTIRs of $S(p)$. This expedites the computation of the form factors.

Quark-photon vertex

- The transverse terms have been included because, owing to the Abelian anomaly (**Phys. Rev. C82 (2010) 065202**), it is impossible to simultaneously conserve the vector and axial vector currents. We have thus included a momentum redistribution factor s_0 .

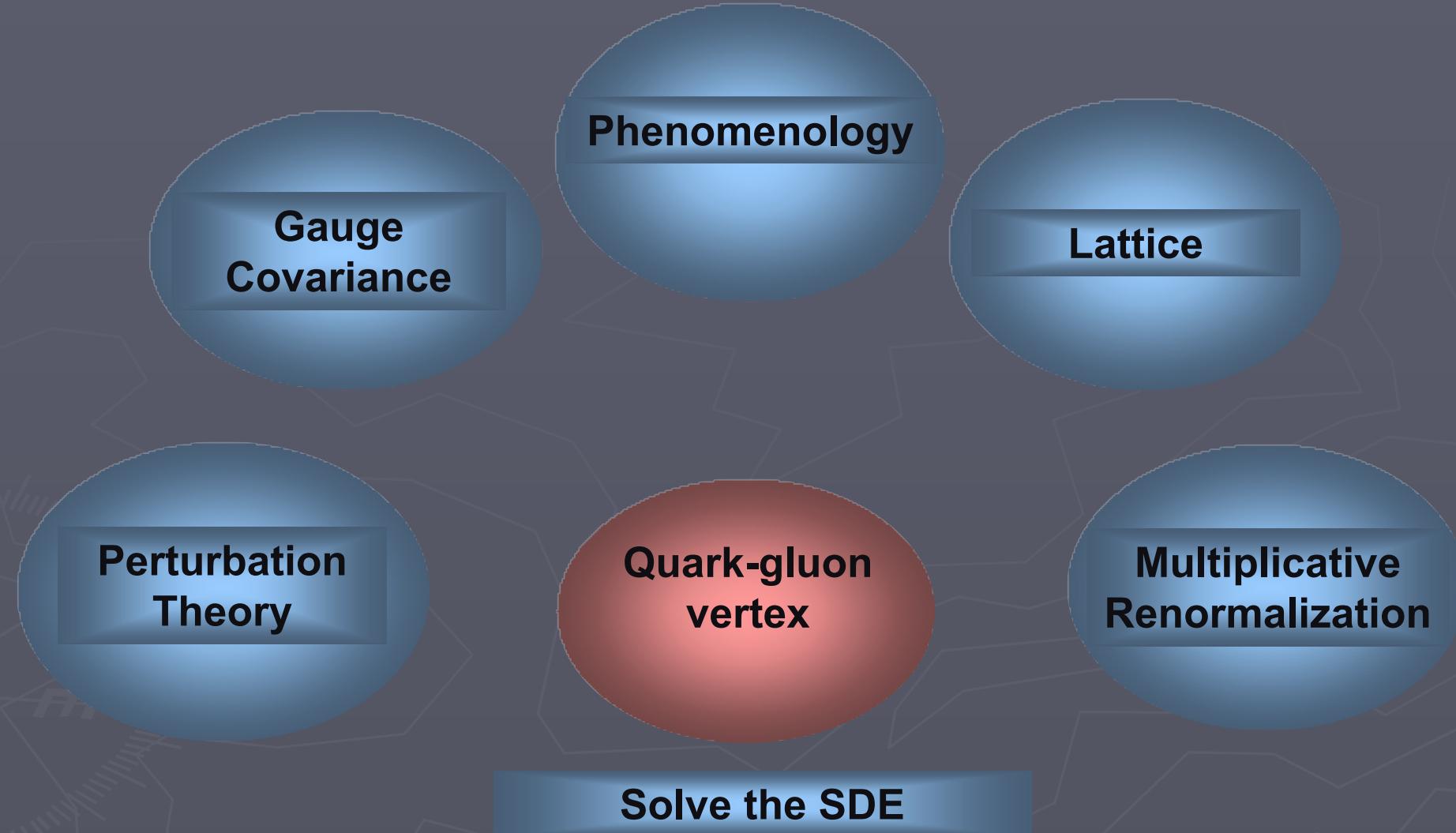
$$2f_\pi G(Q^2 = 0) = 1 \Rightarrow s_0 = 1.9.$$

- In the case of $\eta_{c,b}$, $|G(Q^2 = 0)|$ is fixed by the meson's decay width. This imposes a condition on the value of s_0 .

$$\begin{aligned} \Gamma[M \rightarrow \gamma\gamma] &= \frac{1}{4}\pi\alpha_{\text{em}}^2 m_M^2 |G_M(Q^2 = 0)|^2 \\ &= \frac{8\pi\alpha_{\text{em}}^2 e_{M^q}^4 f_M^2}{m_M^2} \left\{ \begin{array}{ll} \stackrel{\eta_c}{=} 6.1 \text{ keV} & s_0 = 0.89 \\ \stackrel{\eta_b}{=} 0.52 \text{ keV} & , s_0 = 0.23 \end{array} \right. \end{aligned}$$

- Fixing $|G_{\eta_c}(Q^2 = 0)|$ to PDG value, $\Gamma[\eta_c \rightarrow \gamma\gamma] = 5.1 \text{ keV}$, leads to an identical result. The values of decay constants are taken from **Phys. Lett. B753, 330-335 (2016)**, Ding et al.

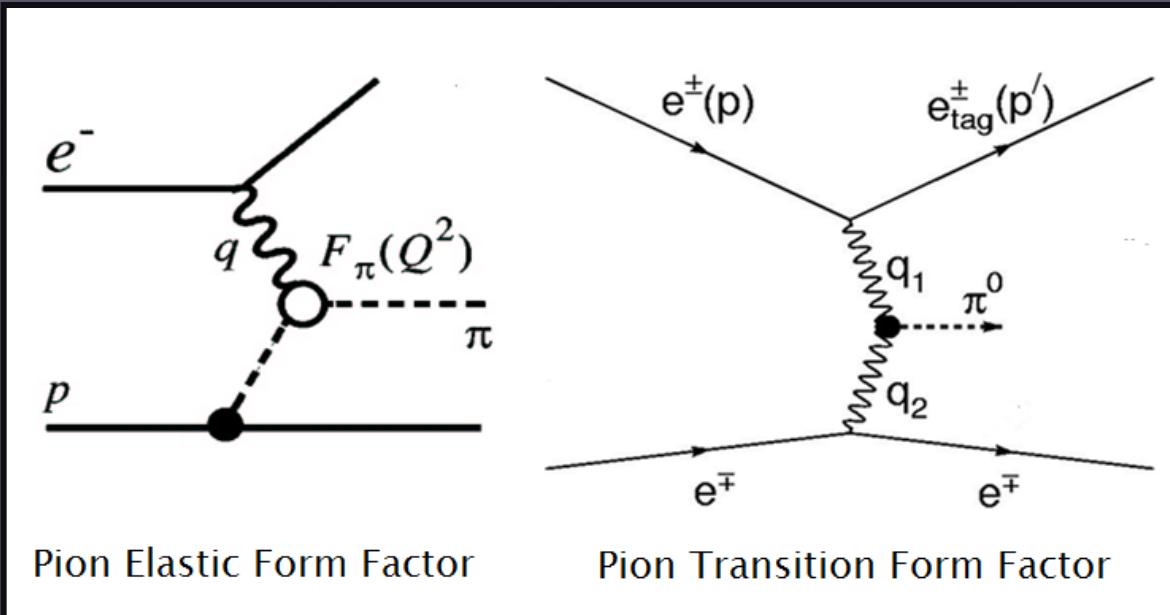
The Quark-Gluon Vertex



"Symmetry preserving truncations of the gap and Bethe-Salpeter equations",
D. Binosi, J. Papavassiliou, S-X Qin, C.D. Roberts, Phys. Rev. D93 096010 (2016).

The Quark-Photon Vertex

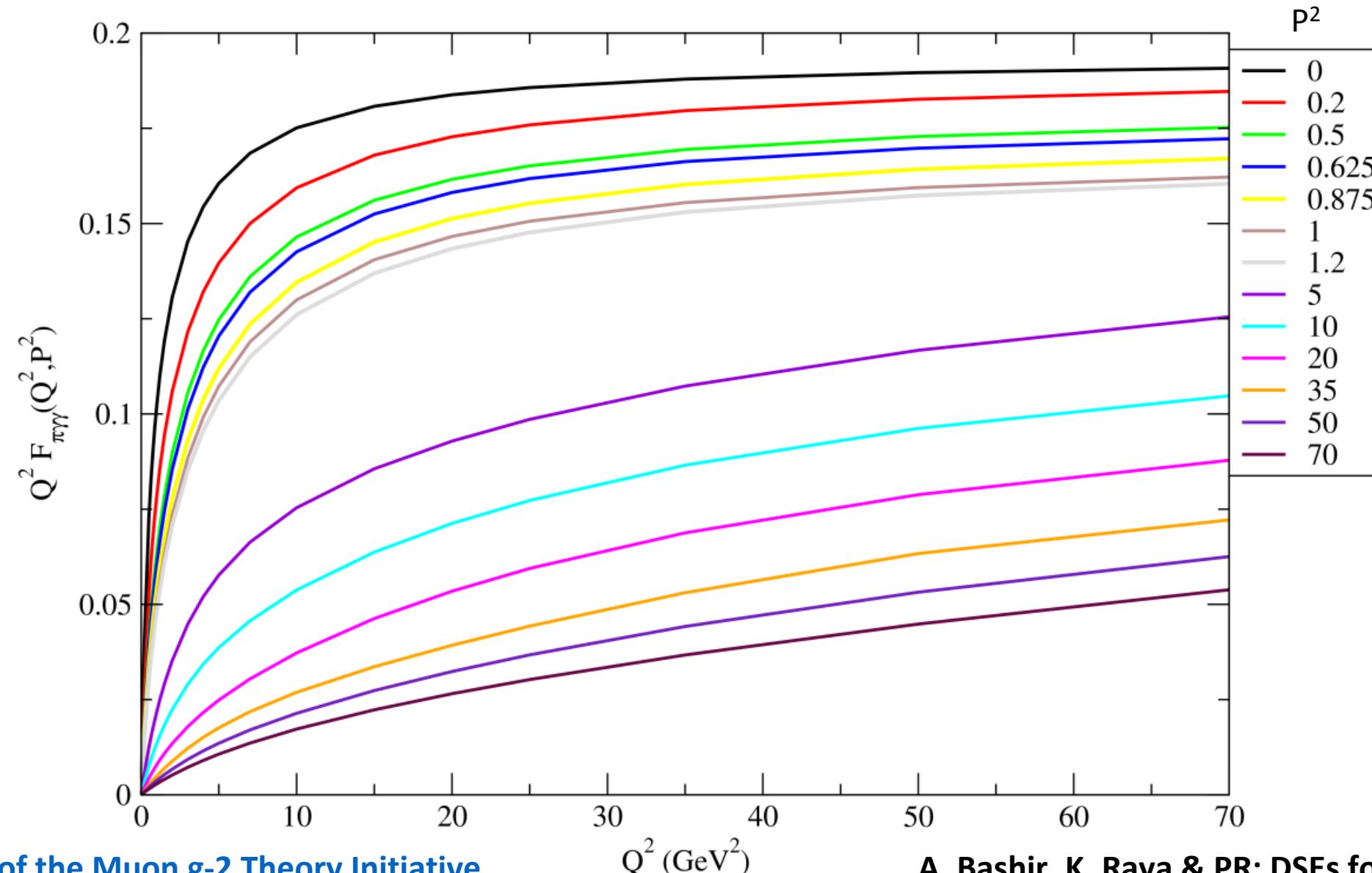
In studying the elastic or transition form factors of hadrons, it is the photon which probes its constituents, highlighting the importance of the quark-photon vertex.



Fortunately, both the quark-photon & the quark-gluon vertices require the same number of basis tensors for their description. So a unified approach is possible.

π^0 TFF from Dyson-Schwinger equations

Data generated by Khépani Raya from K. R. et. al. Phys. Rev. D93 (2016) no.7, 074017



How to parametrize DSE π^0 TFF?

$\Gamma(\pi^0 \rightarrow \gamma\gamma)$ is well described by **ABJ** result: chiral corrections are small $\rightarrow \lim_{Q^2 \rightarrow 0} \mathcal{F}_{\pi^0 \gamma^* \gamma}(m_\pi^2, -Q^2, 0) = \frac{-1}{4\pi^2 F_\pi}$

BL behaviour must be fulfilled: $\lim_{Q^2 \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma^* \gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$

The TFF must comply with the **OPE** constraints on $\langle VVP \rangle$: M. Knecht, A. Nyffeler, Eur. Phys. J. C **21** (2001) 659

$$\lim_{\lambda \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma^* \gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

$$\lim_{\lambda \rightarrow \infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2} \frac{\langle \bar{\psi} \psi \rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right) \longrightarrow \lim_{\lambda \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma^* \gamma^*}((\lambda q_1 + q_2)^2, (\lambda q_1)^2, q_2^2) = -\frac{2}{3} \frac{F_0}{\langle \bar{\psi} \psi \rangle_0} \Pi_{VT}(q_2^2) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$\lim_{\lambda \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_2^2, (\lambda q_1)^2, (q_2 - \lambda q_1)^2) = \frac{2F_0}{3} \frac{1}{\lambda^2} \frac{1}{q_1^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

↓ At external vertex

$$\Pi_{VT}(0) = -\frac{\langle \bar{\psi} \psi \rangle_0}{2} \chi \quad \lim_{\lambda \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma^* \gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = -\frac{2}{3} \frac{F_0}{\langle \bar{\psi} \psi \rangle_0} \Pi_{VT}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$\chi = -(3.05 \pm 0.20) \text{ GeV}^{-2}$: μ uncertainty

$$\chi_u = -(2.08 \pm 0.08) \text{ GeV}^{-2}, \quad \chi_d = -(2.02 \pm 0.09) \text{ GeV}^{-2}, \quad \text{Bali et. al. Phys.Rev. D86 (2012) 094512}$$

$\overline{\text{MS}}, \mu = 2 \text{ GeV} : \quad \chi_u = -(2.08 \pm 0.08) \text{ GeV}^{-2}, \quad \chi_d = -(2.02 \pm 0.09) \text{ GeV}^{-2}$
 $\chi = -(3.05 \pm 0.20) \text{ GeV}^{-2} @ \mu = 1 \text{ GeV}$

How to parametrize DSE π^0 TFF? (II)

In addition to the previous constraints (**ABJ, BL, KN, NSVVZ**), one also has the following ones:

$$\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, -Q^2, -Q^2)}{\mathcal{F}_{\pi^0\gamma\gamma}(0, 0, 0)} = \frac{8}{3}\pi^2 F_0^2 \left\{ \frac{1}{Q^2} - \frac{8}{9}\frac{\delta^2}{Q^4} + \dots \right\} \quad \delta^2 = (0.2 \pm 0.02) \text{ GeV}^2$$

Light-by-light scattering with one real photon has been studied by Melnikov & Vainshtein (Phys.Rev. D70 (2004) 113006).

In the limit $q_1^2 \approx q_2^2 \gg q_3^2$ it can be related to the (anomalous) amplitude

$$T_{\mu_3\rho}^{(a)} = i \int d^4z e^{iq_3 z} \langle 0 | T\{j_{5\rho}^{(a)}(z) j_{\mu_3}(0)\} | \gamma \rangle \text{ depending on two invariant functions that are fixed by ABJ}$$

$$w_L^{(a)}(q^2) = 2w_T^{(a)}(q^2) = -\frac{2}{q^2}$$

Finally, there is $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) \xrightarrow[Q^2 \rightarrow \infty]{} 2F_\pi/(3Q^2)$

V. A. Nesterenko and A. V. Radyushkin, Sov. J. Nucl. Phys. **38**, 284 (1983) [Yad. Fiz. **38**, 476 (1983)].
 V. A. Novikov *et al.*, Nucl. Phys. B **237**, 525 (1984).

How to parametrize DSE π^0 TFF? (III)

There are many proposed models in the market. One needs LMD+V to comply with all constraints discussed before **BUT BL'**:

$$\lim_{Q^2 \rightarrow \infty} \mathcal{F}_{\pi^0 \gamma^* \gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

general P²

$$\mathcal{F}_{\pi^0 \gamma^* \gamma^*}^{\text{LMD+V}}(p_\pi^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{\mathcal{P}(q_1^2, q_2^2, p_\pi^2)}{\mathcal{Q}(q_1^2, q_2^2)},$$

$$\begin{aligned} \mathcal{P}(q_1^2, q_2^2, p_\pi^2) = & q_1^2 q_2^2 (q_1^2 + q_2^2 + p_\pi^2) + h_1 (q_1^2 + q_2^2)^2 + h_2 q_1^2 q_2^2 + h_3 (q_1^2 + q_2^2) p_\pi^2 + h_4 p_\pi^4 \\ & + h_5 (q_1^2 + q_2^2) + h_6 p_\pi^2 + h_7, \end{aligned}$$

$$\mathcal{Q}(q_1^2, q_2^2) = (q_1^2 - M_{V_1}^2)(q_1^2 - M_{V_2}^2)(q_2^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2),$$

$$p_\pi^2 \equiv (q_1 + q_2)^2.$$

Jegerlehner & Nyffeler, Phys.Rept. 477 (2009) 1-110; Nyffeler, Phys.Rev. D79 (2009) 073012

Fully off-shell !!

How to parametrize DSE π^0 TFF? (III)

Jegerlehner & Nyffeler, Phys.Rept. 477 (2009) 1-110; Nyffeler, Phys.Rev. D79 (2009) 073012

Fully off-shell !!

$$\mathcal{P}(q_1^2, q_2^2, p_\pi^2) = q_1^2 q_2^2 (q_1^2 + q_2^2 + p_\pi^2) + h_1 (q_1^2 + q_2^2)^2 + h_2 q_1^2 q_2^2 + h_3 (q_1^2 + q_2^2) p_\pi^2 + h_4 p_\pi^4 + h_5 (q_1^2 + q_2^2) + h_6 p_\pi^2 + h_7$$

0

ABJ: $h_7 = -N_c M_{V_1}^4 M_{V_2}^4 / (4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$

BL:

$$h_5 = \underbrace{6 M_{V_1}^2 M_{V_2}^2}_{\sim 7.7 \text{ GeV}^4} + \delta_{BL}$$

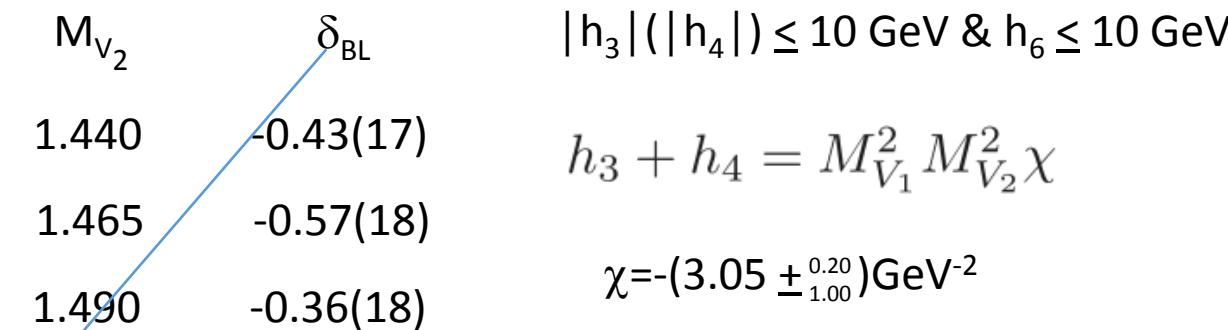
$$\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, -Q^2, -Q^2)}{\mathcal{F}_{\pi^0\gamma\gamma}(0, 0, 0)} = \frac{8}{3}\pi^2 F_0^2 \left\{ \frac{1}{Q^2} - \frac{8}{9} \frac{\delta^2}{Q^4} + \dots \right\} \rightarrow h_2 = -4(M_{V_1}^2 + M_{V_2}^2) + (16/9)\delta^2 \simeq -10.63 \text{ GeV}^2$$

$$\Pi_{VT}^{LMD+V}(p^2) = -\langle \bar{\psi} \psi \rangle_0 \frac{p^2 + c_{VT}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \quad c_{VT} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2} \rightarrow h_1 + h_3 + h_4 = 2c_{VT}$$

h_3 (h_4) & h_6 are still free parameters. We will fit δ_{BL} to DSE data in the region relevant for a_μ ($Q_i^2 \leq 10 \text{ GeV}^2$)
According to different estimates $|h_3| (|h_4|) \leq 10 \text{ GeV}$ & $h_6 \leq 10 \text{ GeV}$

Evaluation of $a_\mu^{\pi^0\text{-pole}}$ with DSE input

We have evaluated $a_\mu^{\pi^0\text{-pole}}$ varying parameters in the ranges discussed previously



$$|h_3|(|h_4|) \leq 10 \text{ GeV} \& h_6 \leq 10 \text{ GeV}$$

$$h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$$

$$\chi = -(3.05 \pm ^{0.20}_{1.00}) \text{ GeV}^{-2}$$

The only relevant variations are h_3 (h_4) & δ_{BL}

In agreement with the trend shown by data (theory) below 10 GeV²

$$a_\mu^{\pi^0\text{-pole}} = (6.26 \pm 0.08) 10^{-10}$$

Some caveats on **Fischer, Goecke & Williams** Eur.Phys.J. A47 (2011) 28; Phys.Rev. D83 (2011) 094006,
Erratum: Phys.Rev. D86 (2012) 099901 & Phys.Rev. D87 (2013) no.3, 034013

- Their off-shell prescription is based on an axial-vector WTI which holds only for the leading amplitude (Si-Xue Qin, Craig D. Roberts, S. M. Schmidt Phys.Lett. B733 (2014) 202-208)
- Use of PTIRs or extrapolations?
- Consistency with axial anomaly in the study of η/η' TFFs?
- Use of phenomenology to constrain dressing functions?
- Double-counting?
- ...?