

Mellin-Barnes Approach to HVP and $g_\mu - 2$

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4th June 2017

First Workshop of the Muon $g - 2$ Theory Initiative
FERMILAB June 2017

*Based on Phys. Lett. B736 522 (2014), arXiv:1702.06783 [hep-ph]
and work with David Greynat (to be published).*

- There is a **Persistent Discrepancy** at the 3σ to 4σ level between Experimental and Theoretical evaluation of $a_\mu = \frac{1}{2}(g_\mu - 2)$.
- $a_\mu^{\text{HVP}} = (6.926 \pm 0.033) \times 10^{-8}$ is the contribution with the **largest error**.
- Possibility of an **alternative evaluation** of a_μ^{HVP} based on QCD first principles, with the help of Lattice QCD (LQCD).

Standard Representation of the HVP Contribution to the Muon Anomaly

Weighted Integral of the **Hadronic Spectral Function**

$$\begin{aligned} a_\mu^{\text{HVP}} &= \frac{\alpha}{\pi} \int_{4m_\pi^2}^\infty \frac{dt}{t} \int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)} \frac{1}{\pi} \text{Im}\Pi(t) \\ &= \frac{\alpha}{\pi} \int_0^1 dx (1-x) \underbrace{\int_{4m_\pi^2}^\infty \frac{dt}{t} \frac{\frac{x^2}{1-x} m_\mu^2}{t + \frac{x^2}{1-x} m_\mu^2}}_{\text{underbrace}} \frac{1}{\pi} \text{Im}\Pi(t) \end{aligned}$$

$$\sigma(t)_{[e^+ e^- \rightarrow (\gamma) \rightarrow \text{Hadrons}]} = \frac{4\pi^2 \alpha}{t} \frac{1}{\pi} \text{Im}\Pi(t)$$

Dispersion Relation

$$-\Pi(Q^2) = \int_0^\infty \frac{dt}{t} \underbrace{\frac{Q^2}{t+Q^2}}_{\frac{1}{\pi} \text{Im}\Pi(t)}, \quad Q^2 = -q^2 \geq 0.$$

Euclidean Representation (*B.E. Lautrup-E. de Rafael '69, EdeR '94*)

$$\begin{aligned} a_\mu^{\text{HVP}} &= \frac{\alpha}{\pi} \int_0^1 dx (1-x) \int_0^\infty \frac{dt}{t} \underbrace{\frac{\frac{x^2}{1-x} m_\mu^2}{t + \frac{x^2}{1-x} m_\mu^2}}_{\frac{1}{\pi} \text{Im}\Pi(t)}, \\ &= -\frac{\alpha}{\pi} \int_0^1 dx (1-x) \underbrace{\Pi\left(\frac{x^2}{1-x} m_\mu^2\right)}_{\text{LQCD}}, \quad Q^2 \equiv \frac{x^2}{1-x} m_\mu^2. \end{aligned}$$

Trading the Feynman x -integration by a Q^2 -integration (*T. Blum '03*)

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi} \int_0^\infty \frac{dQ^2}{Q^2} \sqrt{\frac{Q^2}{4m_\mu^2 + Q^2}} \left(\frac{\sqrt{4m_\mu^2 + Q^2} - \sqrt{Q^2}}{\sqrt{4m_\mu^2 + Q^2} + \sqrt{Q^2}} \right)^2 [-\Pi(Q^2)]$$

Moments of the Hadronic Spectral Function

Spectral Function Moments \Leftrightarrow Derivatives of $\Pi(Q^2)$ at $Q^2 = 0$

$$\int_{4m_\pi^2 \equiv t_0}^{\infty} \frac{dt}{t} \left(\frac{t_0}{t} \right)^{1+n} \frac{1}{\pi} \text{Im} \Pi(t) = \frac{(-1)^{n+1}}{(n+1)!} (t_0)^{n+1} \frac{\partial^{n+1}}{(\partial Q^2)^{n+1}} \Pi(Q^2)|_{Q^2=0}, n = 0, 1, 2 \dots$$

The Leading Moment provides a rigorous upper bound to a_μ^{HVP}

$$\begin{aligned} a_\mu^{\text{HVP}} &= \frac{\alpha}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)} \frac{1}{\pi} \text{Im} \Pi(t) \\ &\leq \frac{\alpha}{\pi} \frac{1}{3} \frac{m_\mu^2}{t_0} \underbrace{\int_{4m_\pi^2}^{\infty} \frac{dt}{t} \frac{t_0}{t} \frac{1}{\pi} \text{Im} \Pi(t)}_{\mathcal{M}(0)} = \left(\frac{\alpha}{\pi} \right) \frac{1}{3} \frac{m_\mu^2}{t_0} \underbrace{\left(-t_0 \frac{\partial}{\partial Q^2} \Pi(Q^2) \right)}_{\text{LQCD}}|_{Q^2=0} \end{aligned}$$

J.S. Bell-EdeR '69 (Coupling of the effective low-energy operator $\partial^\lambda F^{\mu\nu} \partial_\lambda F_{\mu\nu}$)

This provides a First Test for LQCD

Comment on the lowest-order HVP Contribution to the Electron $g_e - 2$

M. Davier '10

$$a_e^{(\text{HVP-lo})} = (1.875 \pm 0.017) \times 10^{-12}$$

$$a_e^{(\text{HVP-lo})} \leq \frac{\alpha}{\pi} \frac{1}{3} \frac{m_e^2}{4m_\pi^2} \underbrace{\int_{4m_\pi^2}^\infty \frac{dt}{t} \frac{4m_\pi^2}{t} \frac{1}{\pi} \text{Im}\Pi(t)}_{\mathcal{M}(0)} = \frac{\alpha}{\pi} \frac{1}{3} \frac{m_e^2}{4m_\pi^2} \underbrace{\left(-4m_\pi^2 \frac{\partial}{\partial Q^2} \Pi(Q^2) \right)_{Q^2=0}}_{\text{LQCD}}$$

This Upper Bound gives practically the result for $a_e^{(\text{HVP-lo})}$

Using Input values for $\mathcal{M}(0)$:

$$a_e^{(\text{HVP-lo})} (\text{Toy Model}) \leq 1.832 \times 10^{-12}$$

$$a_e^{(\text{HVP-lo})} (\text{BHLS}) \leq (1.835 \pm 0.013) \times 10^{-12}$$

$$a_e^{(\text{HVP-lo})} (\text{BMWc}) \leq (1.853 \pm 0.054) \times 10^{-12}$$

Experiment LARGER than UPPER-BOUNDS (for central values)

One may invoke *errors* to argue *consistency* but shows
Importance of making Precise Determinations of $\mathcal{M}(0)$

Moments \Rightarrow Mellin Transform of $\frac{1}{\pi} \text{Im}\Pi(t)$

The **Mellin Transform** of $\frac{1}{\pi} \text{Im}\Pi(t)$ is a function of (**s-complex**):

$$\underbrace{\mathcal{M}(s)}_{\mathcal{M}(s)} = \int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{t}{t_0} \right)^{s-1} \frac{1}{\pi} \text{Im}\Pi(t), \quad -\infty \leq \text{Re}(s) < 1, \quad t_0 \equiv 4m_\pi^2 \pm$$

The **Spectral Function Moments** correspond to $\mathcal{M}(0), \mathcal{M}(-1), \mathcal{M}(-2), \dots$

Representation of $\Pi(Q^2)$ as a function of $\mathcal{M}(s)$

$$\Pi(Q^2) = \int_{t_0}^{\infty} \frac{dt}{t} \underbrace{\frac{-Q^2}{t+Q^2}}_{\frac{1}{1+\frac{Q^2}{t}}} \frac{1}{\pi} \text{Im}\Pi(t), \quad Q^2 = -q^2 \geq 0.$$

Inserting

$$\frac{1}{1 + \frac{Q^2}{t}} = \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{Q^2}{t} \right)^{-s} \Gamma(s)\Gamma(1-s)$$

there follows a **Mellin-Barnes Representation** of HVP in the Euclidean:

$$\Pi(Q^2) = -\frac{Q^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{Q^2}{t_0} \right)^{-s} \Gamma(s)\Gamma(1-s) \underbrace{\mathcal{M}(s)}, \quad c_s \equiv \text{Re}(s) \in]0, 1[,$$

The **Mellin-Barnes Representation** of $\Pi(Q^2)$ is very useful for **asymptotic expansions** of $\Pi(Q^2)$ at Q^2 -small and Q^2 -large.

See e.g. Friot-Greynat-deR '08, Aguilar-Greynat-deR '12, Friot-Greynat '12, ...

Integral Representation of a_μ^{HVP} in terms of the Mellin Transform $\mathcal{M}(s)$

EdeR'14

From the Euclidean Representation to the Mellin-Barnes Representation

$$a_\mu^{\text{HVP}} = -\frac{\alpha}{\pi} \int_0^1 dx (1-x) \Pi \left(\frac{x^2}{1-x} m_\mu^2 \right), \quad Q^2 \equiv \frac{x^2}{1-x} m_\mu^2$$

Recall

$$\Pi \left(\frac{x^2}{1-x} m_\mu^2 \right) = -\frac{\frac{x^2}{1-x} m_\mu^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{\frac{x^2}{1-x} m_\mu^2}{t_0} \right)^{-s} \Gamma(s) \Gamma(1-s) \underbrace{\mathcal{M}(s)},$$

Insert in a_μ^{HVP} and do the x -integration:

$$a_\mu^{\text{HVP}} = \left(\frac{\alpha}{\pi} \right) \frac{m_\mu^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{m_\mu^2}{t_0} \right)^{-s} \mathcal{F}(s) \underbrace{\mathcal{M}(s)}, \quad c_s \equiv \text{Re}(s) \in]0, 1[$$

$$\mathcal{F}(s) = -\Gamma(3-2s) \Gamma(-3+s) \Gamma(1+s)$$

$$\mathcal{M}(s) = \underbrace{\int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{t}{t_0} \right)^{s-1} \frac{1}{\pi} \text{Im} \Pi(t)}_{\text{Mellin Transform of the Spectral Function}}$$

Mellin Transform of the Spectral Function

This Mellin-Barnes Representation provides a framework for an evaluation of a_μ^{HVP} from first principles.

The (so called) Ramanujan's Master Theorem

- Recall the Expansion of $\Pi(Q^2)$ for Q^2 -small:

$$\left\{ -\frac{t_0}{Q^2} \Pi(Q^2) \right\}_{Q^2 \rightarrow 0} = \left\{ \mathcal{M}(0) - \frac{Q^2}{t_0} \mathcal{M}(-1) + \left(\frac{Q^2}{t_0} \right)^2 \mathcal{M}(-2) - \left(\frac{Q^2}{t_0} \right)^3 \mathcal{M}(-3) + \dots \right\},$$

- Ramanujan's Theorem -*proof by Hardy uses the Mellin-Barnes representation-*

$$\int_0^\infty d \left(\frac{Q^2}{t_0} \right) \left(\frac{Q^2}{t_0} \right)^{s-1} \left\{ -\frac{t_0}{Q^2} \Pi(Q^2) \right\}_{Q^2 \rightarrow 0} = \Gamma(s) \Gamma(1-s) \mathcal{M}(s),$$

guarantees the convergence of **Discrete Moments** $\mathcal{M}(-n)$ to the **Full Mellin Transform** $\mathcal{M}(s)$.

Simple Illustration of Ramanujan's Theorem in QED

- Lowest Order Vacuum Polarization in QED for a fermion of mass m :

$$-\frac{4m^2}{Q^2} \Pi^{\text{QED}}(Q^2) \underset{Q^2 \rightarrow 0}{\sim} \sum_{n=0,1,2,\dots} (-1)^n \underbrace{\left(\frac{Q^2}{4m^2} \right)^n}_{\underbrace{\frac{\alpha}{\pi} \frac{1}{n+1} \frac{\sqrt{\pi}}{4} \frac{\Gamma(3+n)}{\Gamma(\frac{7}{2}+n)}}}_{\text{underbrace}}$$

- Then - *WITHOUT DOING the CALCULATION* - Ramanujan's Theorem implies:

$$\mathcal{M}^{\text{QED}}(s) \equiv \int_{4m^2}^\infty \frac{dt}{t} \left(\frac{t}{4m^2} \right)^{s-1} \frac{1}{\pi} \text{Im} \Pi^{\text{QED}}(t) = \underbrace{\frac{\alpha}{\pi} \frac{1}{1-s} \frac{\sqrt{\pi}}{4} \frac{\Gamma(3-s)}{\Gamma(\frac{7}{2}-s)}}_{n \rightarrow -s \text{ (analytic continuation)}}$$

OBSERVATION: This QED Mellin Function is of the Marichev's Class

Fractions involving Products of Gamma-Functions of the Type

$$\mathcal{M}(s) = C \prod_{i,j,k,l} \frac{\Gamma(a_i - s)\Gamma(c_j + s)}{\Gamma(b_k - s)\Gamma(d_l + s)}$$

The Mellin transforms of all (G)-Hypergeometric Functions are of this type

This “Suggests” the Following Approximation Procedure

which we illustrate below with the previous QED example

- Asymptotic Approximation: Matching at $s = 1$

$$\mathcal{M}^{\text{QED}}(s) \Rightarrow \mathcal{M}_0(s) = \frac{\alpha}{\pi} \frac{1}{3} \frac{1}{1-s} = \frac{\alpha}{\pi} \frac{1}{3} \frac{\Gamma(1-s)}{\Gamma(2-s)}$$

- First Marichev Approximation: Matching at $s = 1$ and $s = 0$ -i.e. the slope of Π -

$$\mathcal{M}^{\text{QED}}(s) \Rightarrow \mathcal{M}_1(s) = \frac{\alpha}{\pi} \frac{1}{3} \Gamma(1-s) \frac{\Gamma(b-1)}{\Gamma(b-s)}, \quad \Rightarrow b = \frac{9}{4}$$

- 2nd Approximation: Matching at $s = 1$ and $s = 0, -1$ -i.e. slope and curvature of Π -

$$\mathcal{M}^{\text{QED}}(s) \Rightarrow \mathcal{M}_2(s) = \frac{\alpha}{\pi} \frac{1}{3} \frac{1}{1-s} \frac{\Gamma(c-1)}{\Gamma(c-s)} \frac{\Gamma(d-s)}{\Gamma(d-1)}, \quad \Rightarrow c = \frac{7}{2}, d = 3$$

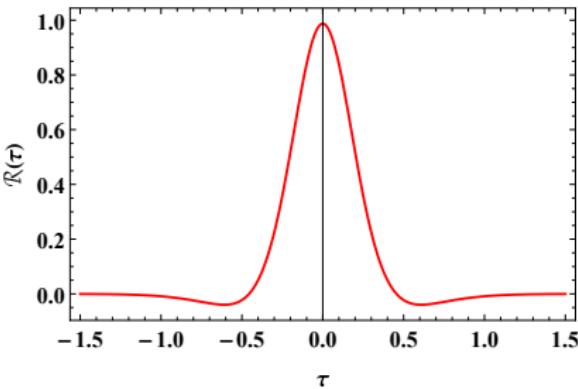
The 2nd Marichev Approximation is already the Exact Result for $\mathcal{M}^{\text{QED}}(s)$!!!

Application to $a^{\text{QED}}(\text{VP})$ (same mass for external and VP fermions)

$$a^{\text{QED}}(\text{VP}) = \left(\frac{\alpha}{\pi} \right) \frac{1}{4} \frac{1}{2\pi i} \int_{c_s - i\infty}^{c_s + i\infty} ds \left(\frac{1}{4} \right)^{-s} \mathcal{F}(s) \mathcal{M}^{\text{QED}}(s)$$

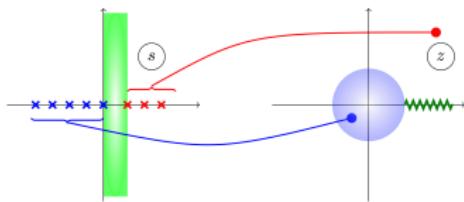
With s within the *fundamental strip*, e.g. $s = \frac{1}{2} - i\tau$:

$$\begin{aligned} a^{\text{QED}}(\text{VP}) &\doteq \left(\frac{\alpha}{\pi} \right)^2 \frac{1}{4} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \underbrace{\left[\frac{1}{1 + \tau^2} - \frac{10}{4 + \tau^2} + \frac{40}{25 + \tau^2} \right]}_{\Re(\tau)} \frac{\pi^2}{[\cosh(\pi\tau)]^2} \\ &= \left(\frac{\alpha}{\pi} \right)^2 \times \underbrace{0.01568742185910}_{\text{Reproduces exact result to this accuracy}} \Leftrightarrow \left(\frac{\alpha}{\pi} \right)^2 \underbrace{\left(\frac{119}{36} - \frac{\pi^2}{3} \right)}_{\Re(\tau)} \end{aligned}$$



Convergence of Mellin-Barnes Integrals (*work with David Greynat*)

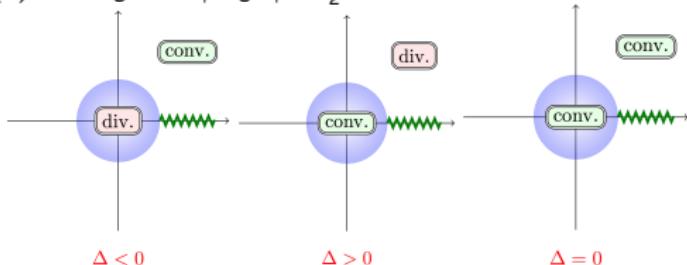
R.B. Paris–D. Kaminski, CUP'01, M. Passare–A.K. Tsikh–O.N. Zhdanov, Vieweg Verlag'94.



Mapping of the s -Mellin plane to the $z = \frac{\alpha^2}{4m^2}$ -plane of $\Pi(Q^2)$

$$I(z) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} z^{-s} \frac{\prod_{j=1}^m \Gamma(A_j s + B_j)}{\prod_{k=1}^n \Gamma(C_k s + D_k)} \quad \text{and} \quad \begin{cases} \Delta \doteq \sum_{j=1}^m A_j - \sum_{k=1}^n C_k \\ \alpha \doteq \sum_{j=1}^m |A_j| - \sum_{k=1}^n |C_k| \end{cases}$$

$I(z)$ converges for $|\arg z| < \frac{\pi}{2}\alpha$



If $\Delta = 0$ and $\alpha > 0$ the two series are analytic continuation of each other

The conditions $\Delta = 0$ and $\alpha > 0$ plus the fact that $\mathcal{M}(s)$ has No Poles for $s < 1$ imply the following:

General Structure of the Successive Marichev Approximations

The Successive Approximants with $\mathcal{M}(0), \mathcal{M}(-1) \dots, \mathcal{M}(-N+1)$ input

$$\Pi_N(Q^2) = -\frac{Q^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{Q^2}{t_0} \right)^{-s} \Gamma(s)\Gamma(1-s) \mathcal{M}_N(s)$$

$$\mathcal{M}_N(s) = \text{Cte} \frac{\Gamma(1-s)}{\Gamma(2-s)} \prod_k \frac{\Gamma(a_k-s)}{\Gamma(a_k-1)} \frac{\Gamma(b_k-1)}{\Gamma(b_k-s)} \quad N \text{ even}$$

$$\mathcal{M}_N(s) = \text{Cte} \Gamma(1-s) \prod_k \frac{\Gamma(a_k-s)}{\Gamma(a_k-1)} \frac{\Gamma(b_k-1)}{\Gamma(b_k-s)} \frac{\Gamma(c_N-1)}{\Gamma(c_N-s)} \quad N \text{ odd}$$

With Cte fixed by the residue of $\mathcal{M}(s)$ at $s = 1$

and $\text{Re } a_k > 1, \text{Re } b_k > 1, \dots, \text{Re } c_N > 1$ by matching to $\mathcal{M}(0), \mathcal{M}(-1), \dots, \mathcal{M}(-N+1)$

The Approximants $\Pi_N(Q^2)$ Converge to $\Pi(Q^2)$

The same conditions $\Delta = 0$ and $\alpha > 0$ apply to a_μ^{HVP}

At each successive N -Approximation

$$a_\mu^{\text{HVP}}(N) = \left(\frac{\alpha}{\pi}\right) \frac{m_\mu^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{m_\mu^2}{t_0}\right)^{-s} \mathcal{F}(s) \mathcal{M}_N(s), \quad c_s \equiv \text{Re}(s) \in]0, 1[$$

$$\mathcal{F}(s) = -\Gamma(3 - 2s) \Gamma(-3 + s) \Gamma(1 + s) \quad \text{has} \quad \Delta = 0 \quad \text{and} \quad \alpha > 0$$

Therefore each $a_\mu^{\text{HVP}}(N)$ is a convergent Mellin-Barnes Integral.

CONCLUSION

- With $\Delta = 0$ and $\alpha > 0$ there exists Mellin Marichev Approximations $\mathcal{M}_N(s)$ which coincide with the exact $\mathcal{M}(s)$ at $s = 0, -1, \dots, -N + 1$
- $\mathcal{M}_N(s) \Rightarrow \Pi_N(Q^2)$ -with its analytic continuation- and hence $a_\mu^{\text{HVP}}(N)$
- Ramanujan's theorem guarantees the convergence $\mathcal{M}_N(s) \rightarrow \mathcal{M}(s)$ after a finite number of steps and hence to $\Pi(Q^2)$ and a_μ^{HVP}

Test with the Proper 4th Order VP in QED

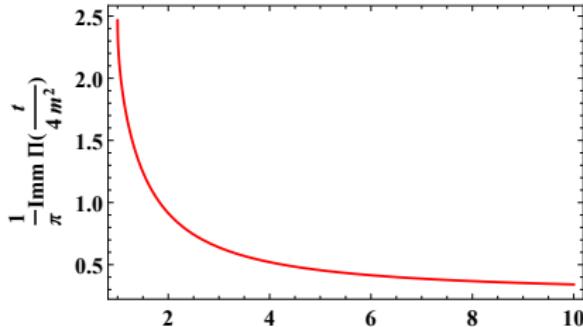
$$\begin{aligned} \frac{1}{\pi} \text{Im}\Pi_{4\text{th}}^{\text{QED}}(t) = & \left(\frac{\alpha}{\pi}\right)^2 \left\{ \delta \left(\frac{5}{8} - \frac{3}{8}\delta^2 - \left(\frac{1}{2} - \frac{1}{6}\delta^2 \right) \log \left[64 \frac{\delta^4}{(1-\delta^2)^3} \right] \right) \right. \\ & + \left(\frac{11}{16} + \frac{11}{24}\delta^2 - \frac{7}{48}\delta^4 + \left(\frac{1}{2} + \frac{1}{3}\delta^2 - \frac{1}{6}\delta^4 \right) \log \left[\frac{(1+\delta)^3}{8\delta^2} \right] \right) \log \left[\frac{1+\delta}{1-\delta} \right] \\ & \left. + 2 \left(\frac{1}{2} + \frac{1}{3}\delta^2 - \frac{1}{6}\delta^4 \right) \left(2 \text{Li}_2 \left[\frac{1-\delta}{1+\delta} \right] + \text{Li}_2 \left[-\frac{1-\delta}{1+\delta} \right] \right) \right\} \theta(t-4m^2) \end{aligned}$$

where $\delta = \sqrt{1 - \frac{4m^2}{t}}$

Källen-Sabry '54, Lautrup-deR '68

$$\frac{1}{\pi} \text{Im}\Pi_{4\text{th}}^{\text{QED}}(t) \underset{t \rightarrow 4m^2}{\sim} \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{\pi^2}{4} - 2\sqrt{\frac{t}{4m^2} - 1} + \mathcal{O} \left[\left(\frac{t}{4m^2} - 1 \right) \right] \right\}$$

$$\frac{1}{\pi} \text{Im}\Pi_{4\text{th}}^{\text{QED}}(t) \underset{t \rightarrow \infty}{\sim} \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{1}{4} + \frac{3}{4} \frac{4m^2}{t} + \mathcal{O} \left[\left(\frac{4m^2}{t} \right)^2 \log \left(\frac{t}{4m^2} \right) \right] \right\}$$



Successive $\mathcal{M}_N(s)$ Mellin Approximants to $\mathcal{M}(s)_{\text{4th}}^{\text{QED}}$ and a_μ^{VP}

Mignaco-Remiddi '69

$$a_\mu^{\text{VP}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \frac{673}{108} - \frac{41}{81}\pi^2 - \frac{4}{9}\pi^2 \log(2) - \frac{4}{9}\pi^2 \log^2(2) + \frac{4}{9}\log^4(2) - \frac{7}{270}\pi^4 + \frac{13}{18}\zeta(3) + \frac{32}{3}\text{PolyLog}\left[4, \frac{1}{2}\right] \right\} = \left(\frac{\alpha}{\pi}\right)^3 0.0528707$$

$$a_\mu^{\text{VP}}(\textcolor{red}{N}) = \left(\frac{\alpha}{\pi}\right) \frac{1}{4} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau e^{-i\tau \log(4)} 2 \mathcal{F}\left(\frac{1}{2} - i\tau\right) \mathcal{M}_{\textcolor{red}{N}}\left(\frac{1}{2} - i\tau\right)$$

- $\textcolor{red}{N} = 1$ With $C = \frac{1}{4}$ and $\mathcal{M}(0)$ matching

$$\mathcal{M}_1(s) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{4} \Gamma(1-s) \frac{\Gamma(b-1)}{\Gamma(b-s)} \Leftrightarrow b = 1.24695122$$

$$a_\mu^{\text{VP}}(1) = \left(\frac{\alpha}{\pi}\right)^3 0.0500007 \quad 5\% - \text{accuracy}$$

- $\textcolor{red}{N} = 2$ With $C = \frac{1}{4}$ and $\mathcal{M}(0), \mathcal{M}(-1)$ matching

$$\mathcal{M}_2(s) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{4} \frac{1}{1-s} \frac{\Gamma(c-1)}{\Gamma(c-s)} \frac{\Gamma(d-s)}{\Gamma(d-1)} \Leftrightarrow c = 1.11485, d = 1.46508$$

$$a_\mu^{\text{VP}}(2) = \left(\frac{\alpha}{\pi}\right)^3 0.0531447 \quad 0.52\% - \text{accuracy}$$

Higher Order $\mathcal{M}_N(s)$ Approximants to $\mathcal{M}(s)_{\text{4th}}^{\text{QED}}$ and a_μ^{VP}

- $N = 3$

With $C = \frac{1}{4}$ and $\mathcal{M}(0), \mathcal{M}(-1), \mathcal{M}(-2)$ matching

$$\mathcal{M}_3(s) = \frac{1}{4} \Gamma(1-s) \frac{\Gamma(c-1)}{\Gamma(c-s)} \frac{\Gamma(d-s)}{\Gamma(d-1)} \frac{\Gamma(e-1)}{\Gamma(e-s)}$$

The parameters c, d and e are fixed by the Moment-Matching Equations
Polynomial Equations:

$$\frac{1}{4} \frac{1}{c-1} (d-1) \frac{1}{e-1} = \mathcal{M}_{\text{4th}}^{\text{QED}}(0)$$

$$\frac{1}{c} d \frac{1}{e} = \frac{\mathcal{M}_{\text{4th}}^{\text{QED}}(-1)}{\mathcal{M}_{\text{4th}}^{\text{QED}}(0)}$$

$$2 \frac{1}{c+1} (d+1) \frac{1}{e+1} = \frac{\mathcal{M}_{\text{4th}}^{\text{QED}}(-2)}{\mathcal{M}_{\text{4th}}^{\text{QED}}(-1)}$$

with solutions:

$$c = 1.16361, \quad d = 2.52855, \quad \text{and} \quad e = 3.30712$$

$$a_\mu^{\text{VP}}(3) = \left(\frac{\alpha}{\pi}\right)^3 0.0528678 \quad 0.004\% - \text{accuracy}$$

When should one STOP?

- $N = 4$ With $C = \frac{1}{4}$ and $\mathcal{M}(0), \mathcal{M}(-1), \mathcal{M}(-2), \mathcal{M}(-3)$ matching

$$a_\mu^{\text{VP}}(4) = \left(\frac{\alpha}{\pi}\right)^3 0.0528711 \quad 0.00075\% - \text{accuracy}$$

- $N = 5$ With $C = \frac{1}{4}$ and $\mathcal{M}(0), \mathcal{M}(-1), \mathcal{M}(-2), \mathcal{M}(-3), \mathcal{M}(-4)$ matching

$$a_\mu^{\text{VP}}(5) = \left(\frac{\alpha}{\pi}\right)^3 0.0528706 \quad 0.00018\% - \text{accuracy}$$

This is the **BEST** one can do in this case !

Limited by the numerical approximation of the input Moments.

If e.g. one tries the next $N = 6$ approximation there are only
 “unphysical solutions” to the Matching Equations

They are *complex numbers* with *real parts* within the *fundamental strip*
 in contradiction with the basic assumptions.

Reconstruction of the $\Pi_N(Q^2)$ in terms of Meijer's Functions

Recall the General Structure of Marichev Approximations

$$\mathcal{M}_N(s) = \text{Cte} \frac{\Gamma(1-s)}{\Gamma(2-s)} \prod_k \frac{\Gamma(a_k-s)}{\Gamma(a_k-1)} \frac{\Gamma(b_k-1)}{\Gamma(b_k-s)} \quad N \text{ even}$$

$$\mathcal{M}_N(s) = \text{Cte} \Gamma(1-s) \prod_k \frac{\Gamma(a_k-s)}{\Gamma(a_k-1)} \frac{\Gamma(b_k-1)}{\Gamma(b_k-s)} \frac{\Gamma(c_N-1)}{\Gamma(c_N-s)} \quad N \text{ odd}$$

The Corresponding Self-Energy Functions $\Pi_N(Q^2)$, $N = 1, 2, 3, \dots$

$$\Pi_N(Q^2) = -\frac{Q^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{Q^2}{t_0}\right)^{-s} \Gamma(s)\Gamma(1-s) \mathcal{M}_N(s)$$

are Meijer Functions (*recognized by Mathematica*)

Example: The $N = 2$ approximant in 4th order QED (slope and curvature)

$$\mathcal{M}_{4\text{th}}^{\text{QED}}(s) \Rightarrow \mathcal{M}_{N=2}^{\text{QED}}(s) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{4} \frac{\Gamma(1-s)}{\Gamma(2-s)} \frac{\Gamma(c-1)}{\Gamma(c-s)} \frac{\Gamma(d-s)}{\Gamma(d-1)}, \quad c = 1.11485, \quad d = 1.46508$$

$$\Pi_{N=2}^{\text{QED}}(Q^2) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{4} \frac{\Gamma(c-1)}{\Gamma(d-1)} G_{3,3}^{1,3} \left(z \middle| \begin{matrix} 0, 0, 1-d \\ 0; -1, 1-c \end{matrix}\right), \quad z = \frac{Q^2}{4m^2}$$

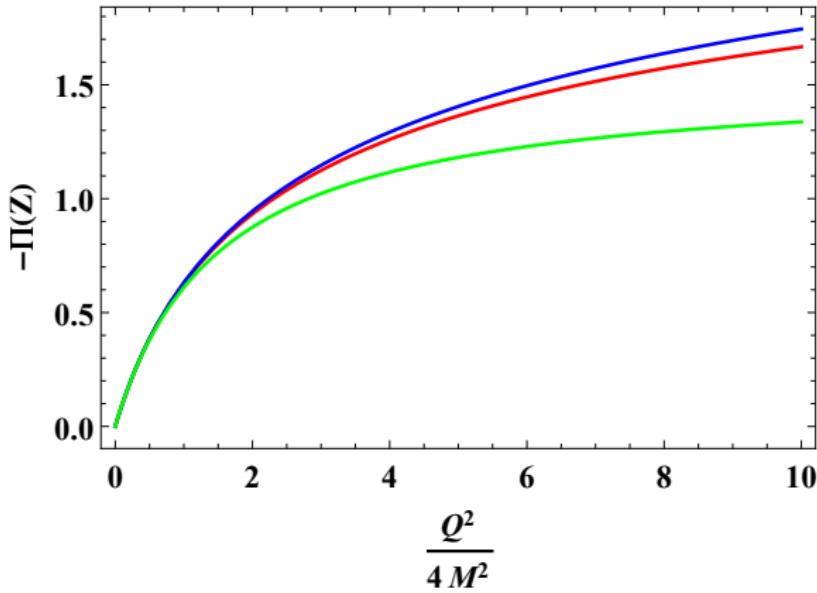
Shapes of the Self-Energy Function $\Pi(Q^2)$ in $(\frac{\alpha}{\pi})^2$ units

Approximations with only slope and curvature of $\Pi(Q^2)$ at $Q^2 = 0$

Shape of the exact $\Pi_{4\text{th}}^{\text{QED}}$ ($Z \equiv \frac{Q^2}{4m^2}$) in Red

Shape of the Marichev $N = 2$ approximant $\Pi_{N=2}^{\text{QED}}(Z)$ in Blue

Shape of the Padé approximant (slope and curvature) in Green



$$\begin{aligned}\mathcal{M}(0) &= (0.707094 \pm 0.005200) \times 10^{-3} \\ \mathcal{M}(-1) &= (0.114518 \pm 0.000901) \times 10^{-3} \\ \mathcal{M}(-2) &= (0.029589 \pm 0.000391) \times 10^{-3} \\ \mathcal{M}(-3) &= (0.011515 \pm 0.000221) \times 10^{-3}\end{aligned}$$

$$a_{\mu}^{\text{HVP}}(\text{BHLS}) = (683.50 \pm 4.75) \times 10^{-10}$$

Mellin-Marichev APPROXIMATIONS with $C = \frac{\alpha}{\pi} \frac{1}{3} N_c \frac{10}{9}$ (central values only)

- $N = 1$ With C and $\mathcal{M}(0)$ matching
 $a_{\mu}^{\text{HVP}}(N = 1) = 668.01 \times 10^{-10}$ 2.3% accuracy
- $N = 2$ With C and $\mathcal{M}(0), \mathcal{M}(-1)$ matching
 $a_{\mu}^{\text{HVP}}(N = 2) = 683.81 \times 10^{-10}$ 0.05% accuracy
- $N = 3$ With C and $\mathcal{M}(0), \mathcal{M}(-1), \mathcal{M}(-2)$ matching
 $a_{\mu}^{\text{HVP}}(N = 3) = 684.81 \times 10^{-10}$ 0.2% accuracy
- $N = 4$ With C and $\mathcal{M}(0), \mathcal{M}(-1), \mathcal{M}(-2), \mathcal{M}(-3)$ matching
produces already *unphysical solutions*

Application to LQCD determination of $\mathcal{M}(0)$ and $\mathcal{M}(-1)$ (*BMWc'17*)

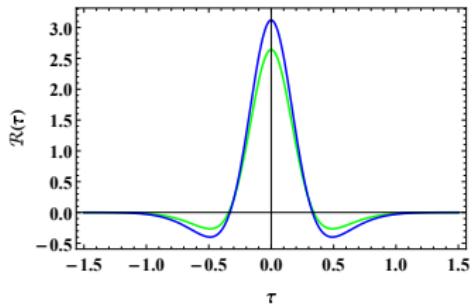
With charm subtracted:

$$\mathcal{M}(0)_{\text{BMWc}} = (0.704 \pm 0.021) \times 10^{-3} \quad \text{and} \quad \mathcal{M}(-1)_{\text{BMWc}} = (0.101 \pm 0.007) \times 10^{-3}$$

Integral Representation of a_μ^{HVP} in terms of Marichev Mellin Transforms

$$a_\mu^{\text{HVP}}(\text{Second}) = \left(\frac{\alpha}{\pi}\right) \frac{m_\mu^2}{t_0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \left(\frac{m_\mu^2}{t_0}\right)^{-\left(\frac{1}{2}-i\tau\right)} \mathcal{F}\left(\frac{1}{2}-i\tau\right) \mathcal{M}_{(0,-1)}^{(1,2)}\left(\frac{1}{2}-i\tau\right)$$

$$\mathcal{R}(\tau) = \frac{1}{C} \left(\frac{m_\mu^2}{t_0}\right)^{-\left(\frac{1}{2}-i\tau\right)} \mathcal{F}\left(\frac{1}{2}-i\tau\right) \left[\mathcal{M}_{(0,-1)}^{(1,2)}\left(\frac{1}{2}-i\tau\right)\right] \left[\mathcal{M}_{(0)}^{(1,2)}\left(\frac{1}{2}-i\tau\right)\right]$$



The predicted values using these Marichev Interpolations are:

$$a_\mu^{\text{HVP}}(\text{First}) = (6.23 \pm 0.18) \times 10^{-8} \quad a_\mu^{\text{HVP}}(\text{Second}) = (6.81 \pm 0.30) \times 10^{-8}$$

CONCLUSIONS

- A precise determination of the lowest $\mathcal{M}(0)$ -moment, i.e. *a precise determination of the slope of the HVP function at the origin from LQCD*, provides an *excellent test* to compare with the determinations using experimental data, and it is practically the calculation of a_e^{HVP} (l.o.).
- The Method of Successive Mellin-Approximations of the Marichev-Type is well defined and the QED examples discussed show very rapid convergence.
It is particularly *well adapted to QCD* because it incorporates the leading pQCD short-distance behaviour and does not require matching of scales. It only requires the *LQCD determination of a few moments* i.e. of a few derivatives of $\Pi(Q^2)$ at $Q^2 = 0$.
- The preliminary tests with the BHLS-results and the LQCD (BMWc'16) indicate that, with a precise determination of the $\mathcal{M}(0)$ and $\mathcal{M}(-1)$ moments, i.e. *the slope and curvature of $\Pi(Q^2)$ at $Q^2 = 0$* , one may be able to reach an accuracy for a_μ^{HVP} comparable to the experimental determination.