Hadronic vacuum polarization from lattice QCD: ABGPT

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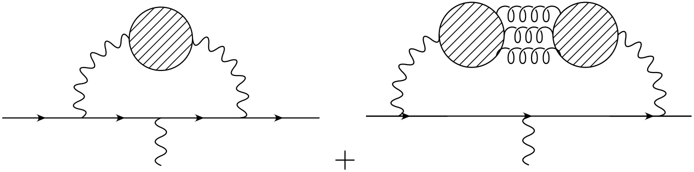
Hadronic vacuum polarization (HVP) contribution to g-2



The blobs (quark loops), which represent all possible intermediate hadronic states (ρ, $\pi\pi$, ...) are not calculable in perturbation theory, but can be calculated from

- dispersion relation + experimental cross-section for $e^+e^- \rightarrow$ hadrons
- first principles using lattice QCD

Lattice QCD method [Blum, 2003, Lautrup et al., 1971]

Using lattice QCD and continuum, ∞-volume pQED

$$a_\mu(\mathrm{HVP}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2\, f(q^2)\, \hat{\Pi}(q^2)$$

$\hat{\Pi}(q^2)$ is subtracted HVP, computed directly on Euclidean space-time lattice,

$$\begin{aligned} \hat{\Pi}(q^2) &= \Pi(q^2) - \Pi(0) \\ \Pi^{\mu\nu}(q) &= \int e^{iqx} \langle j^\mu(x) j^\nu(0) \rangle \qquad j^\mu(x) = \sum_i Q_i \bar{\psi}(x)\gamma^\mu\psi(x) \\ &= \Pi(q^2)(q^\mu q^\nu - q^2 \delta^{\mu\nu}) \end{aligned}$$

fit to a smooth function of Q^2, $a \to 0$, $V \to \infty$

direct-double-subtraction method

[Bernecker and Meyer, 2011, Lehner and Izubuchi, 2015]

$$\begin{aligned} \Pi(q^2) - \Pi(0) &= \sum_t \left(\frac{\cos qt - 1}{q^2} + \frac{1}{2} t^2 \right) C(t) \\ C(t) &= \frac{1}{3} \sum_{x,i} \Re \langle j_i(x,t) j_i(0) \rangle \end{aligned}$$

- $C(t)$: A_1 irrep of cubic group
- first moment subtracts $\Pi(0)$
- additional "-1" subtracts Π_{ii}, finite volume effect

Alternative: work in position space RBC/UKQCD

[Blum et al., 2015b, Blum et al., 2015a]

- reorder integral and sum
- contribution of $C(t)$ for each t is more apparent

$$
\begin{aligned}
\Pi(q^2) - \Pi(0) &= \sum_t \left(\frac{\cos qt - 1}{q^2} + \frac{1}{2} t^2 \right) C(t) \\
C(t) &= \frac{1}{3} \sum_{x,i} \langle j_i(x,t) j_i(0) \rangle \\
a_\mu^{\mathrm{HVP}} &= \sum_t \tilde{w}(t) C(t) \\
\tilde{w}(t) &= 2 \int_0^\infty \frac{d\omega}{\omega} f(\omega^2) \left[\frac{\cos \omega t - 1}{(2 \sin \omega t/2)^2} + \frac{t^2}{2} \right]
\end{aligned}
$$

$w(t)$ includes the continuum QED part of the diagram

Staggered fermions

- Four degenerate flavors when $a \to 0$ (doublers)

The Dirac operator (we omit Naik term),

$$M = 2m + \sum_\mu \eta_\mu(x) \left(U_\mu(x)\delta_{x+\mu,y} - U_\mu^\dagger(x-\mu)\delta_{x-\mu,y} \right)$$

where $\eta_\mu(x) = (-1)^{\sum_i^{\mu-1} x_i}$ are the staggered phases. We use the exactly conserved (point-split) vector current,

$$J^\mu(x) = -\frac{1}{2}\eta_\mu(x) \left(\bar{\chi}(x+\hat{\mu})U_\mu^\dagger(x)\chi(x) + \bar{\chi}(x)U_\mu(x)\chi(x+\hat{\mu}) \right)$$

- satisfies exact Ward Identity at $a \neq 0$, no Z_V
- point splitting induces contact terms, must subtract
- contact terms can be ignored if DDS is used since $C(0)$ does not contribute

All mode and low mode averaging RBC/UKQCD

[Izubuchi et al., 2013, Giusti et al., 2004, DeGrand and Schaefer, 2004]

$$
\begin{aligned}
\langle O \rangle &= \langle O^{\mathrm{exact}} \rangle - \langle O^{\mathrm{approx}} \rangle + \frac{1}{N} \sum_{i}^{N} \langle O_i^{\mathrm{approx}} \rangle \\
&- \frac{1}{N} \sum_{i}^{N} \langle O_i^{\mathrm{LM}} \rangle + \frac{1}{V} \sum_{i}^{V} \langle O_i^{\mathrm{LM}} \rangle
\end{aligned}
$$

- O^{approx} : correlation function computed from approximate quark propagators
- exact spectral decomposition of low modes of the dirac operator + "sloppy" (rel. stop. cond.) CG for the high modes
- O^{LM} : correlation function computed purely from exact low modes
- $V \gg N \gg 1$

Eigen-system of staggered fermions

$$
\begin{aligned}
M\psi_\lambda = \begin{pmatrix} 2m & M_{oe} \\ M_{eo} & 2m \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} &= (2m + i\lambda_n) \begin{pmatrix} n_o \\ n_e \end{pmatrix} \\
M_{oe} n_e &= i\lambda_n n_o \\
M_{eo} n_o &= i\lambda_n n_e
\end{aligned}
$$

and similarly for the preconditioned operator

$$
M^\dagger M\psi_\lambda = \begin{pmatrix} 2m & -M_{oe} \\ -M_{eo} & 2m \end{pmatrix} \begin{pmatrix} 2m & M_{oe} \\ M_{eo} & 2m \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix}
$$

$$
\begin{pmatrix} 4m^2 - M_{oe}M_{eo} & 0 \\ 0 & 4m^2 - M_{eo}M_{oe} \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} = (4m^2 + \lambda_n^2) \begin{pmatrix} n_o \\ n_e \end{pmatrix}
$$

Same eigenvectors as M with squared magnitude eigenvalues, and we can construct the even part from the odd

The eigenvalues come in pairs,

$$\begin{pmatrix} 2m & M_{oe} \\ M_{eo} & 2m \end{pmatrix} \begin{pmatrix} -n_o \\ n_e \end{pmatrix} = (2m - i\lambda_n) \begin{pmatrix} -n_o \\ n_e \end{pmatrix},$$

- $(-1)^x \psi(x) = (-n_o, n_e)$ is also an eigenvector with eigenvalue $-\lambda$.
- Use implicitly restarted Lanczos algorithm to generate $O(1000)$ low modes of $4m^2 - M_{oe} M_{eo}$
- Thus we can construct pairs of eigenvectors with $\pm i\lambda$ for each λ^2, $|n_o\rangle$

Spectral decomposition for staggered fermions

Define

$$
\begin{aligned}
|V\rangle &= \begin{pmatrix} M_{oe}|n_e\rangle \\ |n_e\rangle \end{pmatrix} \\
M^{-1} &= \sum_n \frac{1}{4m^2+\lambda_n^2} \begin{pmatrix} -\frac{2m}{\lambda_n^2}|V_o\rangle\langle V_o| & -|V_o\rangle\langle V_e| \\ -|V_e\rangle\langle V_o| & 2m|V_e\rangle\langle V_e| \end{pmatrix}
\end{aligned}
$$

- spectral decomp used in AMA procedure
- For LMA write correlator as the product of two local "meson fields" [Foley et al., 2005] instead of propagators

Meson fields and low mode average

The two-point function is

$$
\begin{aligned}
4J_\mu(t_x)J_\nu(t_y) \quad = \quad & \sum_{m,n} \\
& \sum_{\vec{x}} \frac{\langle m|x+\mu\rangle\, U_\mu^\dagger(x)\langle x|n\rangle}{\lambda_m} \sum_{\vec{y}} \frac{\langle n|y\rangle\, U_\nu(y)\langle y+\nu|m\rangle}{\lambda_n} \\
+ \quad & \sum_{\vec{x}} \frac{\langle m|x\rangle\, U_\mu(x)\langle x+\mu|n\rangle}{\lambda_m} \sum_{\vec{y}} \frac{\langle n|y\rangle\, U_\nu(y)\langle y+\nu|m\rangle}{\lambda_n} \\
+ \quad & \sum_{\vec{x}} \frac{\langle m|x+\mu\rangle\, U_\mu^\dagger(x)\langle x|n\rangle}{\lambda_m} \sum_{\vec{y}} \frac{\langle n|y+\nu\rangle\, U_\nu^\dagger(y)\langle y|m\rangle}{\lambda_n} \\
+ \quad & \sum_{\vec{x}} \frac{\langle m|x\rangle\, U_\mu(x)\langle x+\mu|n\rangle}{\lambda_m} \sum_{\vec{y}} \frac{\langle n|y+\nu\rangle\, U_\nu^\dagger(y)\langle y|m\rangle}{\lambda_n}
\end{aligned}
$$

where λ_m is shorthand for $2m \pm i\lambda_m$. So we need to construct the four point-split matrices

$$
(\Lambda_\mu(t))_{n,m} \quad = \quad \sum_{\vec{x}} \langle n|x\rangle\, U_\mu(x)\langle x+\mu|m\rangle(-1)^{(m+n)x+m}
$$

where we will order eigenvectors $m = 0, 1, 2, 3, \ldots, 2N_{\mathrm{low}}$ as $\lambda_0, -\lambda_0, \lambda_1, -\lambda_1, \ldots, -\lambda_{2N_{\mathrm{low}}}$.

Simulation details

Highly improved staggered quark (HISQ) fermion action

[Follana et al., 2007]

Symanzik gluon action

gauge field ensembles generated by MILC collaboration

[Bazavov et al., 2016]

- pion mass $m_\pi \approx 135$ MeV ($m_\pi L \lesssim 4$)
- lattice spacings $a = 0.06$, 0.086, and 012 fm
- lattice size $L/a = 48$, 64, 96
- lattice volume $\approx (5.5)^3\ fm^3$
- 2+1+1 flavors

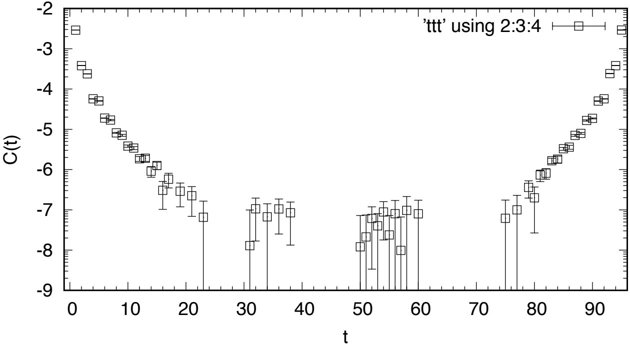
Warning: All of the following results are preliminary and are mostly for illustration

Euclidean time correlation function

Staggered fermions have oscillations (parity is broken, $a \neq 0$),

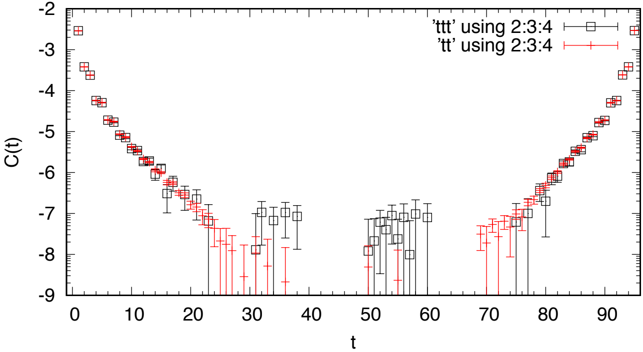
$$C(t) \;=\; \sum_m (-1)^{mt} |\langle 0|O|m\rangle|^2 \frac{\exp - E_m t}{2E_m}$$

Using only AMA (21 configurations, 16 meas/config):



Euclidean time correlation function

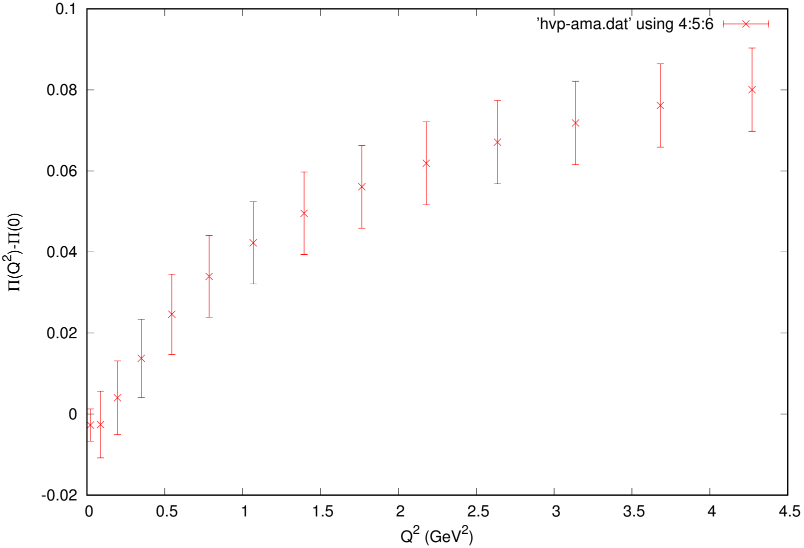
AMA+full volume LMA (2×1000 low modes, 21 configurations)



statistical errors are dramatically reduced, but long distance still poorly determined

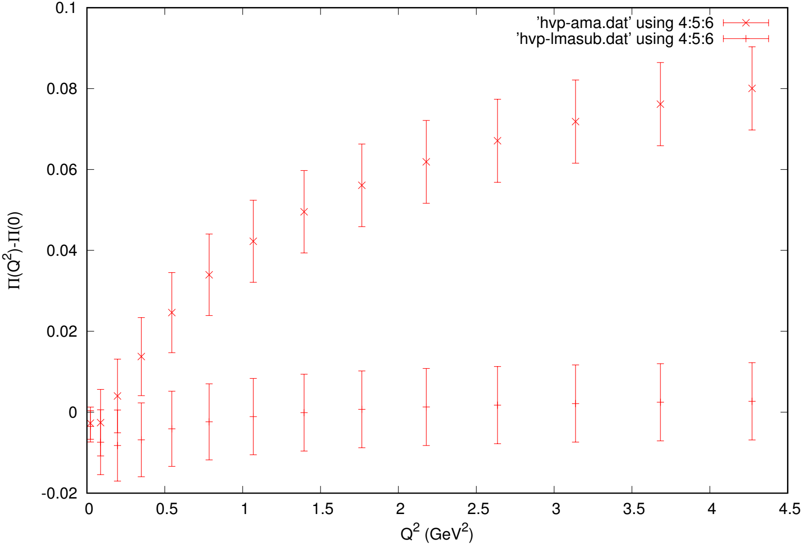
Hadronic vacuum polarization

Using only AMA (21 configurations, 16 meas/config):



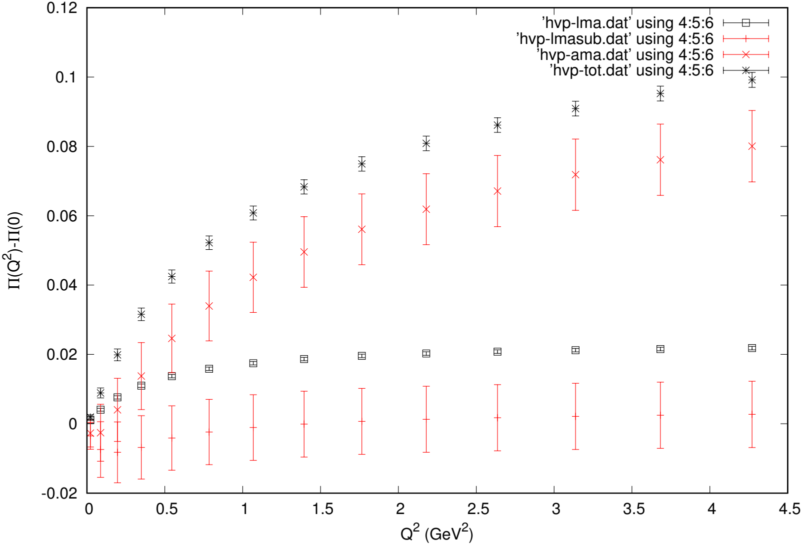
Hadronic vacuum polarization

Errors come almost entirely from low-modes! (16 pt src props)



Hadronic vacuum polarization

Full LMA dramatically reduces stat errors (factor of 5-10)

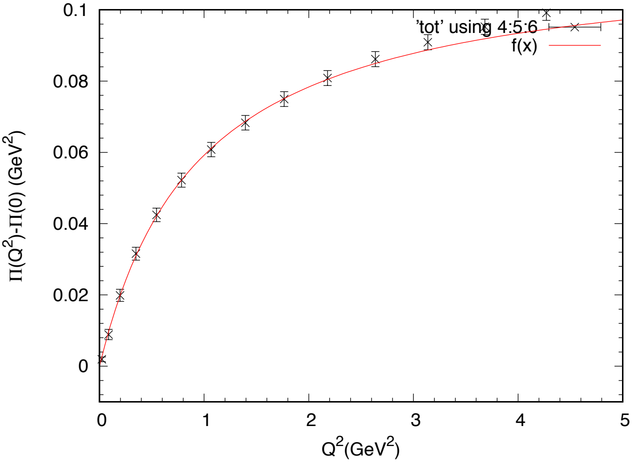


Model independent fit to HVP: Padé approximants

[Aubin et al., 2012]

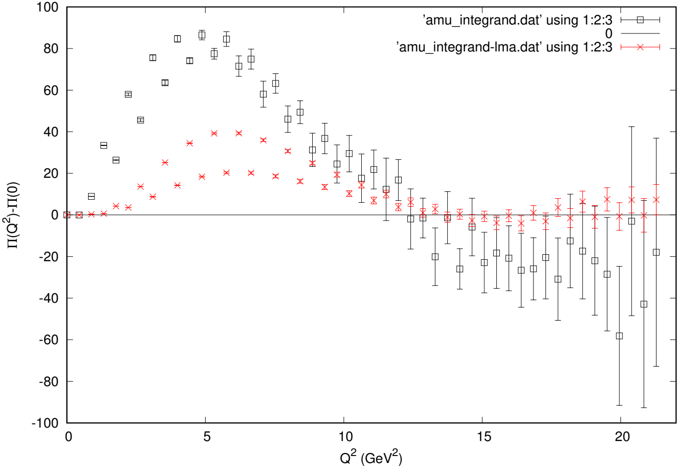
simple uncorrelated fit to [0,1] Padé,

$$\Pi(Q^2) - \Pi(0) = Q^2 \frac{a_1}{b_1 + Q^2}$$



Model independent sum: positon space method (RBC/UKQCD)

[Blum et al., 2015b, Blum et al., 2015a]

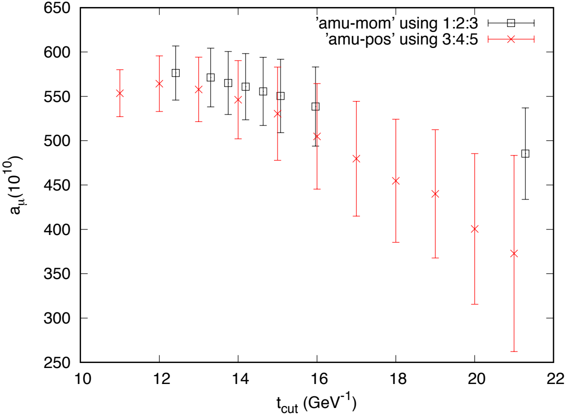


Good, but still need more low modes (up to $\sim m_s$, in progress)

Contribution to the muon anomaly (two flavors)

Selectively cut correlation function at long distance

- only contributes noise
- can correct with e^+e^- data [Bernecker and Meyer, 2011], RBC/UKQCD
- works same in either momentum or position space



Summary

- Lattice QCD calculations with physical masses, large boxes + improved measurement algorithms are powerful
- Increasing the number of LM, 1000$\rightarrow$2000-3000
- AMA+LMA dramatically reduces cost, push to sub percent errors
- AMA done for 0.06 fm (96^3), need LMA. Repeat for 0.12 fm, take $a \rightarrow 0$ limit
- study FV effects at fixed a

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