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# Hadronic Vacuum Polarisation from Lattice QCD: Results from Mainz/CLS

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First Workshop of the Muon  $g - 2$  Theory Initiative

*Q Center / FNAL*

3 – 6 June 2017



# Lattice QCD approach to HVP

- \* Convolution integral over Euclidean momenta: *[Lautrup & de Rafael; Blum]*

$$a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2 (\Pi(Q^2) - \Pi(0))$$

- \* Weight function  $f(Q^2)$  strongly peaked near muon mass

- \* **Direct method:** determine  $\Pi(Q^2)$  from VP tensor

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ \cdot x} \langle J_\mu(x) J_\nu(0) \rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2)$$

$$J_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \dots$$



- Determine  $\Pi(0)$  and Padé representation of  $\Pi(Q^2)$  from fits

$$Q^2 \leq Q_{\text{cut}}^2 \approx 0.1 - 0.5 \text{ GeV}^2$$

# Lattice QCD approach to HVP

## \* Time-momentum representation:

[Bernecker & Meyer]

$$a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{f}(x_0) G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$

$$\tilde{f}(x_0) = 4\pi^2 \int_0^\infty dQ^2 f(Q^2) \left[ x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{1}{2} Q x_0\right) \right]$$

## \* Kernel $\tilde{f}(x_0)$ admits expansion in $x_0 m_\mu$ which is accurate to $O(10^{-6})$

[arXiv:1705.01775]

## \* Control long-distance behaviour of $G(x_0)$

$$G(x_0) = \begin{cases} G(x_0)_{\text{data}}, & x_0 \leq x_{0,\text{cut}} \\ G(x_0)_{\text{ext}}, & x_0 > x_{0,\text{cut}} \end{cases}$$

## \* $G(x_0)$ dominated by two-pion state for $x_0 \rightarrow \infty$

# Lattice QCD approach to HVP

- \* Time moments:

[Chakraborty et al.]

$$G_{2n} \equiv a \sum_{x_0} x_0^{2n} G(x_0) = (-1)^n \frac{\partial^{2n}}{\partial \omega^{2n}} \left\{ \omega^2 \Pi(\omega^2) \right\}_{\omega^2=0}$$

- \* Expansion of VPF at low  $Q^2$ :  $\Pi(Q^2) = \Pi_0 + \sum_{j=1}^{\infty} Q^{2j} \Pi_j$

- \* Coefficients:  $\Pi(0) \equiv \Pi_0 = -\frac{1}{2} G_2, \quad \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}$

- \* Resumming time moments yields time-momentum representation:

$$\begin{aligned} \hat{\Pi}(Q^2) &= \frac{1}{Q^2} \int_0^{\infty} dx_0 G(x_0) \left[ Q^2 x_0^2 - 4 \sin^2 \left( \frac{1}{2} Q x_0 \right) \right] \\ &= \frac{1}{Q^2} \int_{-\infty}^{\infty} dx_0 G(x_0) \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(Q x_0)^{2k+2}}{(2k+2)!} \end{aligned}$$

# Lattice QCD approach to HVP

- \* Construct low-energy representation of  $\Pi(Q^2)$  from time moments
- \* Control large- $x_0$  regime (c.f. TMR)

$$G_{2n} \equiv a \sum_{x_0} x_0^{2n} G(x_0)$$

- \* Time moments: input for Mellin-Barnes representation of  $a_\mu^{\text{hvp}}$

*[E de Rafael 2014, 2017, Benayoun et al. 2016]*

# The Mainz $(g - 2)_\mu$ project

## Collaborators:

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H. Meyer, A. Nyffeler, V. Pascalutsa, A. Risch, HW

M. Della Morte, A. Francis, J. Green, V. Gülpers,  
B. Jäger, G. Herdoíza



## Topics:

- \* Hadronic vacuum polarisation
- \* Hadronic light-by-light scattering
- \* Running of  $\alpha_{em}$  and  $\sin^2\theta_W$
- \* Determination of  $\alpha_s$  from vacuum polarisation function

# Current data sets

- \* CLS consortium — “Coordinated Lattice Simulations”
- \*  $N_f = 2$  flavours of  $O(a)$  improved Wilson fermions
- \* Three values of the lattice spacing:  $a = 0.076, 0.066, 0.049$  fm
- \* Pion masses and volumes:  $m_\pi^{\min} = 185$  MeV,  $m_\pi L > 4$

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## Future:

- \*  $N_f = 2+1$  flavours of  $O(a)$  improved Wilson fermions; tree-level Symanzik gauge action; open boundary conditions
- \* Five values of the lattice spacing; physical pion mass

# Current data sets

arXiv:1705.1775

The hadronic vacuum polarization contribution  
to the muon  $g - 2$  from lattice QCD

M. Della Morte<sup>a</sup>, A. Francis<sup>b</sup>, V. Gülpers<sup>c</sup>, G. Herdoíza<sup>d</sup>, G. von Hippel<sup>e</sup>, H. Horch<sup>e</sup>,  
B. Jäger<sup>f</sup>, H.B. Meyer<sup>e,g</sup>, A. Nyffeler<sup>e</sup>, H. Wittig<sup>e,g</sup>

Run	$L/a$	$\beta$	$\kappa$	$m_\pi L$	$a$ [fm]	$m_\pi$ [MeV]	$N_{\text{cfg}}$	$N_{\text{meas}}$
A3	32	5.20	0.13580	6.0	0.0755(9)(7)	495	251	1004
A4	32	5.20	0.13590	4.7	0.0755(9)(7)	381	400	1600
A5	32	5.20	0.13594	4.0	0.0755(9)(7)	331	251	1004
B6	48	5.20	0.13597	5.0	0.0755(9)(7)	281	306	1224
E5	32	5.30	0.13625	4.7	0.0658(7)(7)	437	1000	4000
F6	48	5.30	0.13635	5.0	0.0658(7)(7)	311	300	1200
F7	48	5.30	0.13638	4.2	0.0658(7)(7)	265	250	1000
G8	64	5.30	0.13642	4.0	0.0658(7)(7)	185	325	4588
N5	48	5.50	0.13660	5.2	0.0486(4)(5)	441	347	1388
N6	48	5.50	0.13667	4.0	0.0486(4)(5)	340	559	2236
O7	64	5.50	0.13671	4.2	0.0486(4)(5)	268	149	2384

\* Focus on methodology and systematics

# Standard Method

- \* Lattice observable:

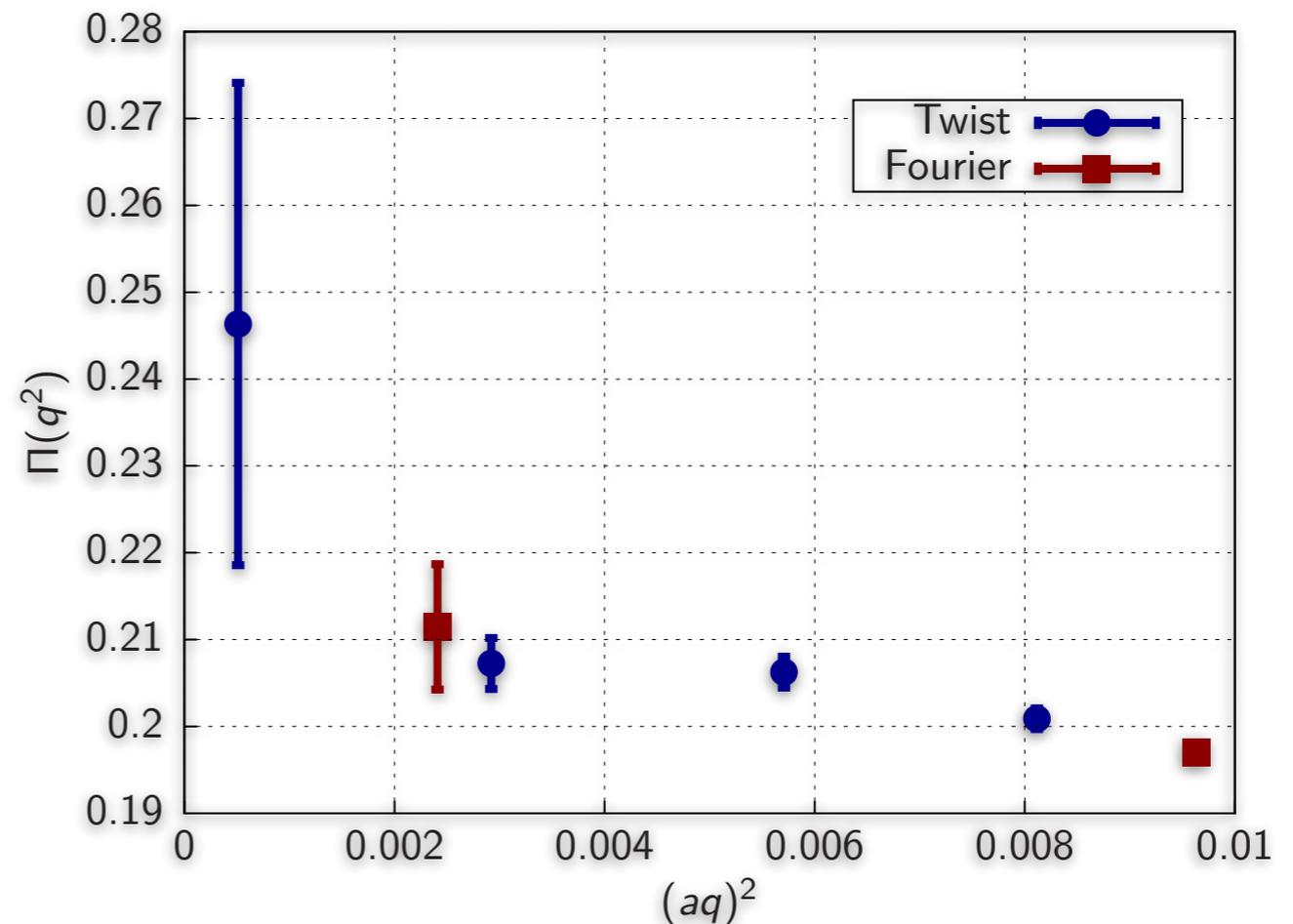
$$a^4 \sum_f q_f^2 Z_V \sum_x \left( e^{iQ(x+a\hat{\mu}/2)} - 1 \right) \langle V_{\mu,f}^{\text{con}}(x) V_{\nu,f}^{\text{loc}}(0) \rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(\hat{Q})$$

- \* Use twisted boundary conditions to reach smaller values of  $Q^2$ :

[Della Morte et al. 2011]

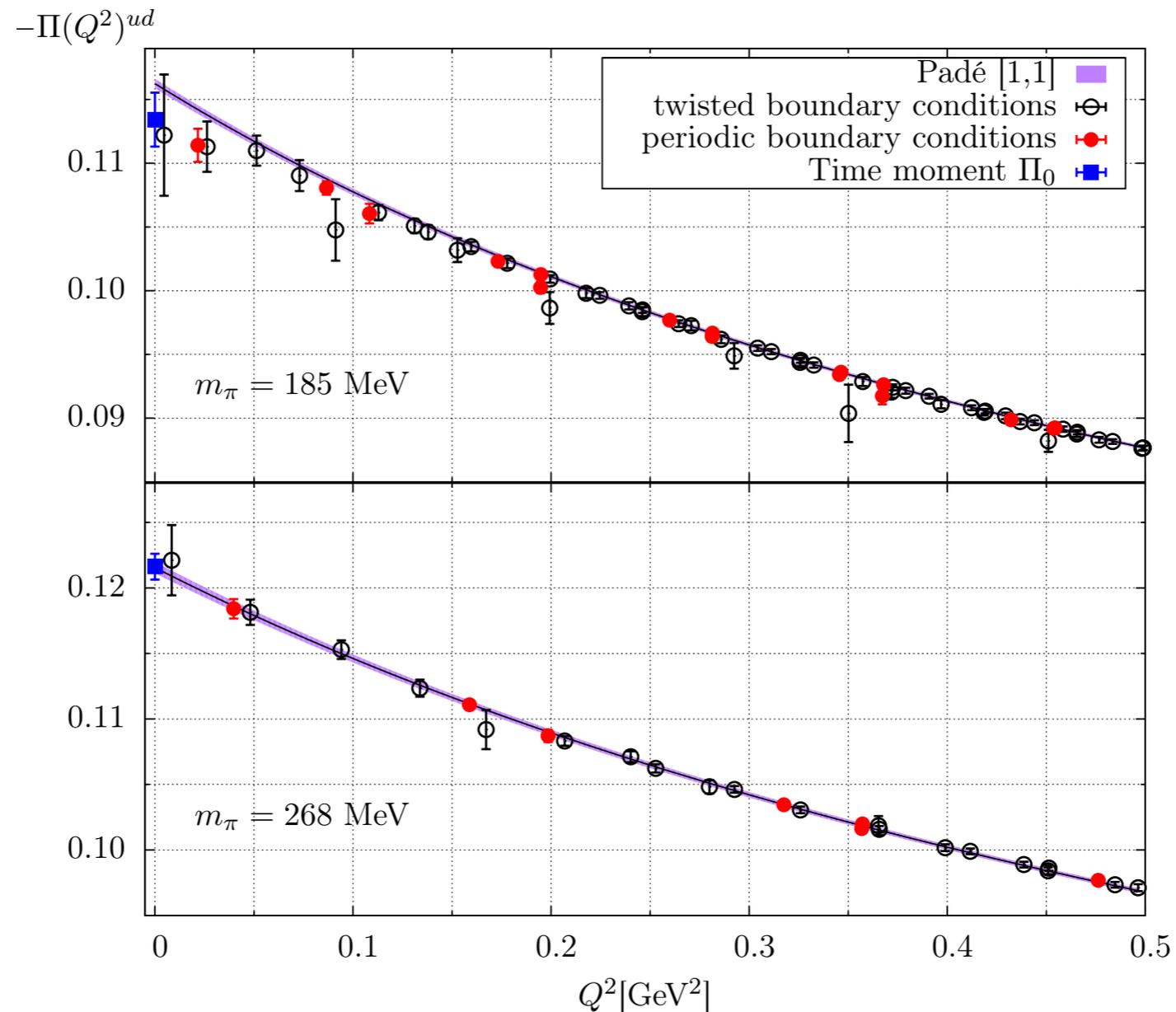
$$\psi(x + Le_\mu) = e^{i\theta_\mu} \psi(x)$$

$$\Rightarrow Q_\mu = \frac{2\pi}{L} + \frac{\theta_\mu}{L}$$



# Standard Method

- \* Fit  $\Pi(Q^2)$  to low-order Padé approximants for  $Q^2 \leq Q_{\text{cut}}^2 \approx 0.5 \text{ GeV}^2$



- \* Intercept  $\Pi(0)$  agrees well with  $\Pi_0$  determined from time moment

# Time-momentum representation

\* Lattice observable: 
$$G^f(x_0) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} q_f^2 Z_V \langle V_{k,f}^{\text{con}}(x_0, \vec{x}) V_{k,f}^{\text{loc}}(0) \rangle$$

$$(a_\mu^{\text{hvp}})^f = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{f}(x_0) G^f(x_0)$$

\* Control tail of integrand:

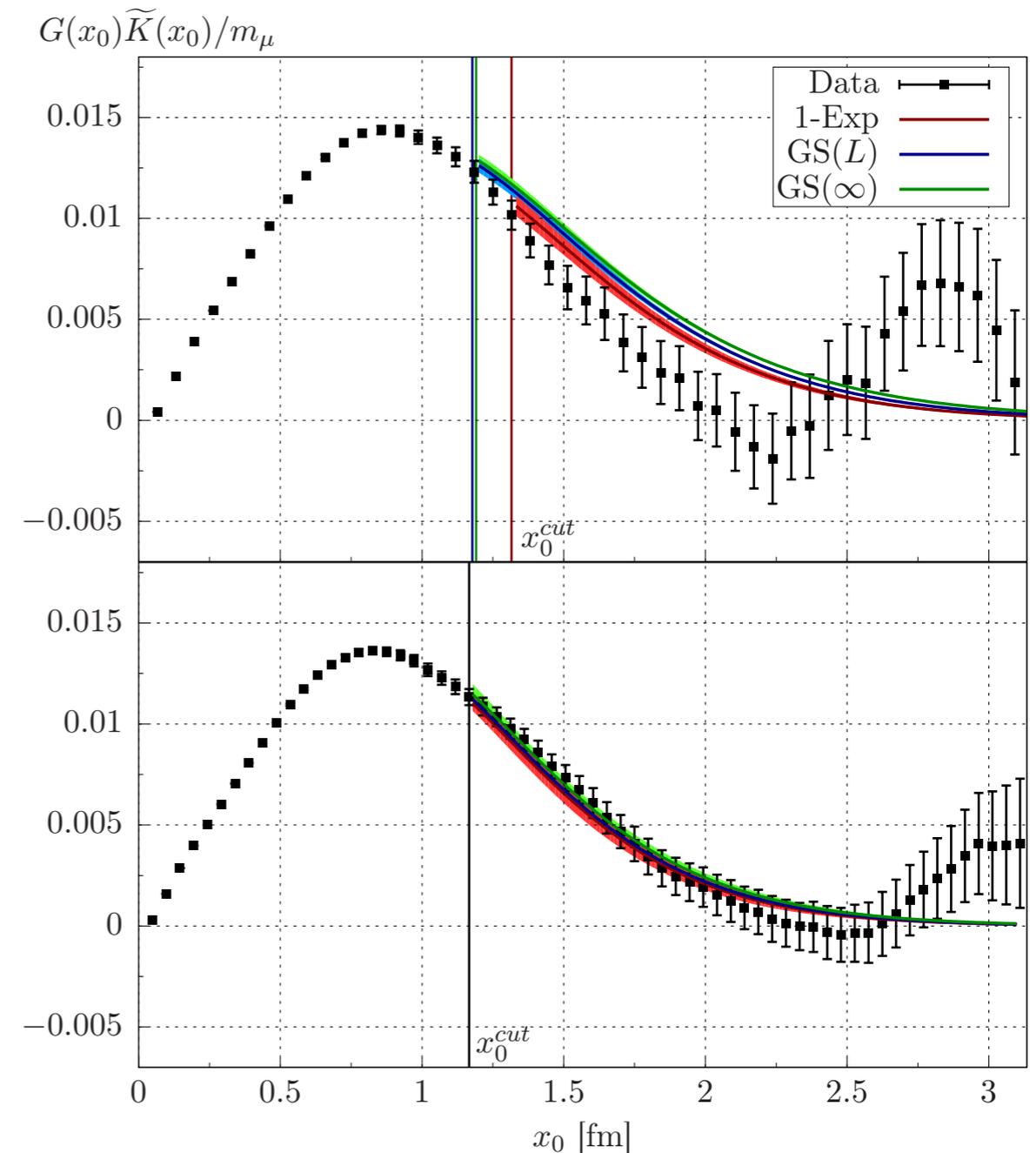
- Naive single exponential:

$$G(x_0)_{\text{ext}} = A e^{-m_\rho x_0}$$

- Single exponential plus 2-pion state:

$$G(x_0)_{\text{ext}} = A e^{-m_\rho x_0} + B e^{-E_{2\pi}(\vec{p})x_0}$$

- Gounaris-Sakurai parameterisation of timeline pion form factor



# TMR analysis of finite-volume effects

- \* Large- $x_0$  behaviour of  $G(x_0, L)$  and  $G(x_0, \infty)$ :
  - $G(x_0, \infty)$  contains a continuum of states with  $E \geq 2m_\pi$
  - $G(x_0, L)$ : discrete energy levels:  $E \geq 2 \sqrt{m_\pi^2 + (2\pi/L)^2}$

⇒ Long-distance regime related to finite-volume effects

- \* Isospin decomposition:

$$G(x_0) = G^{\rho\rho}(x_0) + G^{I=0}(x_0), \quad G^{\rho\rho}(x_0) = \frac{9}{10} G^{ud}(x_0)$$

- \* Long-distance behaviour of  $G^{\rho\rho}(x_0)$  constrained by time-like pion form factor  $F_\pi(\omega)$

# TMR analysis of finite-volume effects

- \* Iso-vector correlator in infinite volume:

$$G^{\rho\rho}(x_0) = \int_0^\infty d\omega \omega^2 \rho(\omega^2) e^{-\omega|x_0|} = \frac{1}{48\pi^2} \int_0^\infty d\omega \omega^2 (1 - 4m_\pi^2/\omega^2)^{3/2} |F_\pi(\omega)|^2 e^{-\omega x_0}$$

- \* Approximate  $F_\pi(\omega)$  by **Gounaris-Sakurai** parameterisation:  $(m_\rho, \Gamma_\rho)$

- \* **Finite volume:** 
$$G^{\rho\rho}(x_0, L) = \sum_n |A_n|^2 e^{-\omega_n x_0}, \quad \omega_n = 2 \sqrt{m_\pi^2 + k_n^2}$$

- Fix  $m_\rho$  from fits to smeared correlation function
- Determine  $\Gamma_\rho$  from correlator  $G(x_0, L)$  using  $m_\rho$  as input
- Determine energy levels  $\omega_n$  and amplitudes  $A_n$  via Lüscher formalism and GS

$$\delta_{11}(k) + \phi\left(\frac{kL}{2\pi}\right) = n\pi, \quad n = 1, 2, \dots$$

$$|F_\pi(\omega)|^2 = \left\{ (z\phi'(z))_{z=kL/2\pi} + k \frac{\partial \delta_1(k)}{\partial k} \right\} \frac{3\pi\omega^2}{2k^2} |A|^2$$

# TMR analysis of finite-volume effects

- \* **Infinite volume:**

Compute  $G^{\rho\rho}(x_0)$  beyond  $x_{0,\text{cut}}$  using GS parameterisation

$$G^{\rho\rho}(x_0) = \int_0^\infty d\omega \omega^2 \rho(\omega^2) e^{-\omega|x_0|} = \frac{1}{48\pi^2} \int_0^\infty d\omega \omega^2 (1 - 4m_\pi^2/\omega^2)^{3/2} |F_\pi(\omega)|^2 e^{-\omega x_0}$$

- \* Determine finite-volume shift from  $G(x_0, \infty) - G(x_0, L)$

- \* At  $m_\pi = 185$  MeV,  $L = 4.0$  fm,  $m_\pi L = 4.0$ :

$$a_\mu^{\text{hvp}}(\infty) - a_\mu^{\text{hvp}}(L) \approx 3.5\%$$

- \* Assign **20%** uncertainty to determination of volume shift

# Disconnected Contributions

- \* Electromagnetic current correlator with  $u, d, s$  quarks:

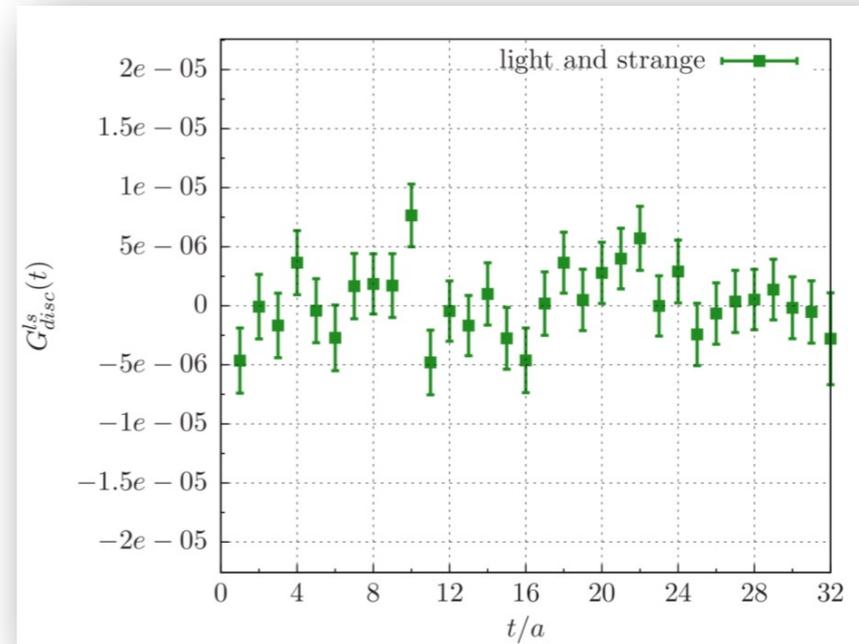
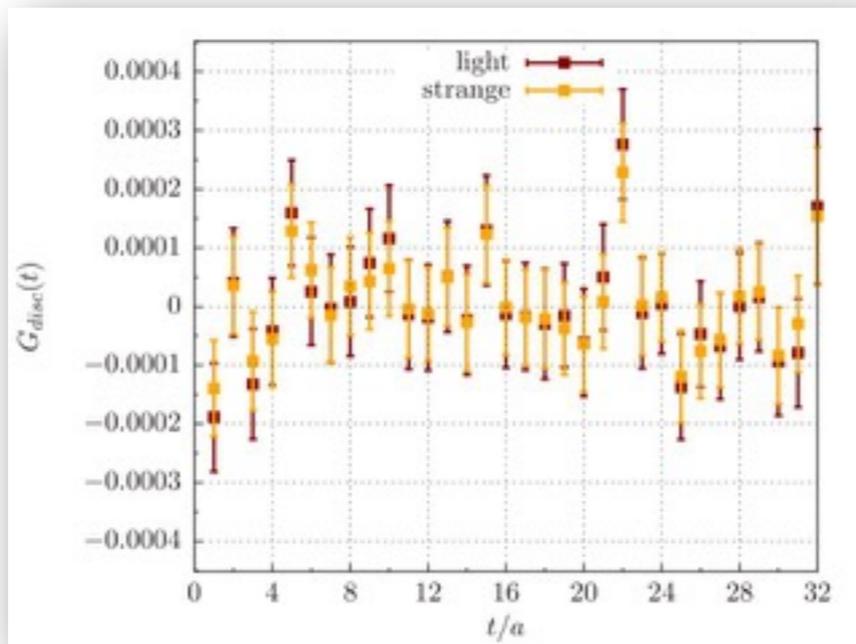
$$G(x_0) := -a \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle, \quad J_k = \frac{2}{3} \bar{u} \gamma_k u - \frac{1}{3} \bar{d} \gamma_k d - \frac{1}{3} \bar{s} \gamma_k s$$

- \* Identify connected and disconnected contributions:

$$G(x_0) = G^{ud}(x_0) + G^s(x_0) - G_{\text{disc}}(x_0)$$

$$G^{\text{disc}}(x_0) = -\frac{1}{9} \left\langle \left( \Delta^{ud}(x_0) - \Delta^s(x_0) \right) \left( \Delta^{ud}(0) - \Delta^s(0) \right) \right\rangle$$

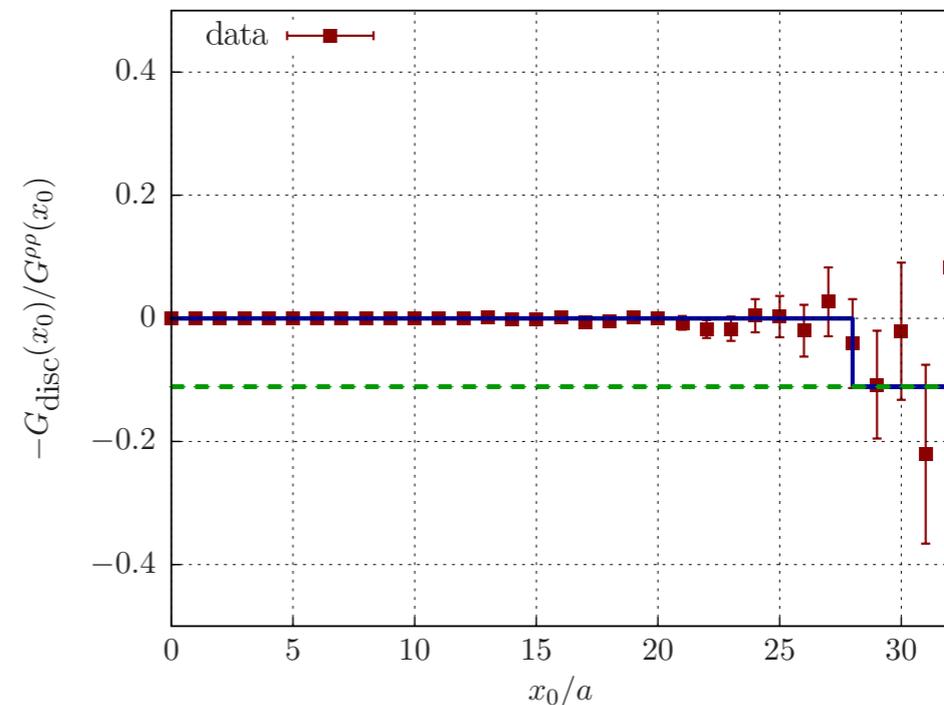
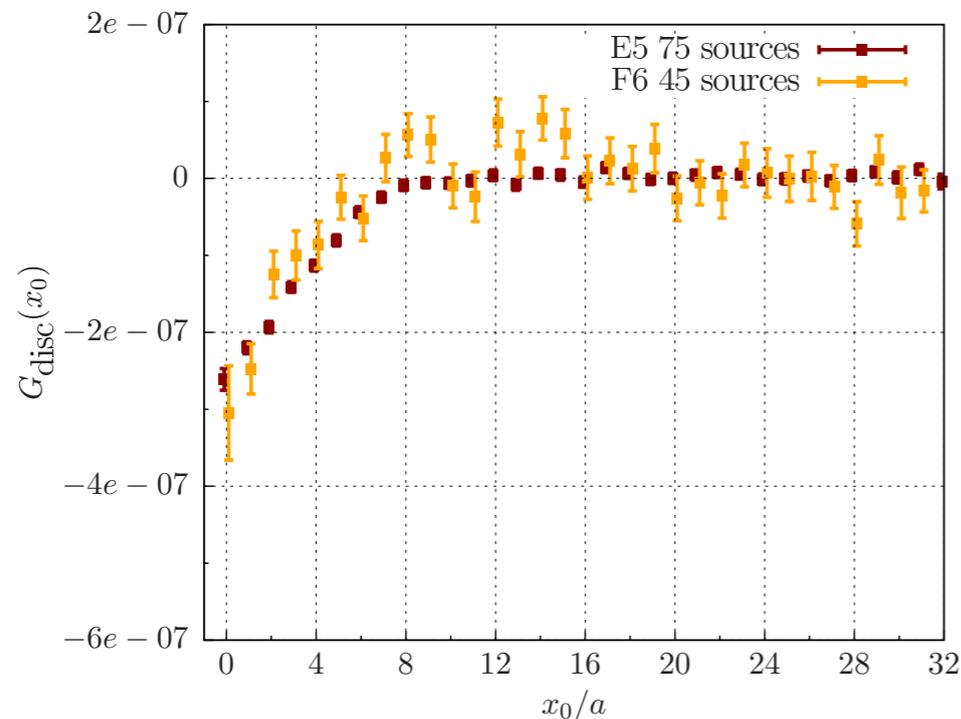
$$\Delta^f(x_0) \sim \gamma_k \text{ } \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \curvearrowright \text{---} \end{array}$$



[Gülpers et al., arXiv:1411.7592; V. Gülpers, PhD Thesis 2015]

# Disconnected Contributions

- \* Disconnected contributions evaluated on subset of ensembles



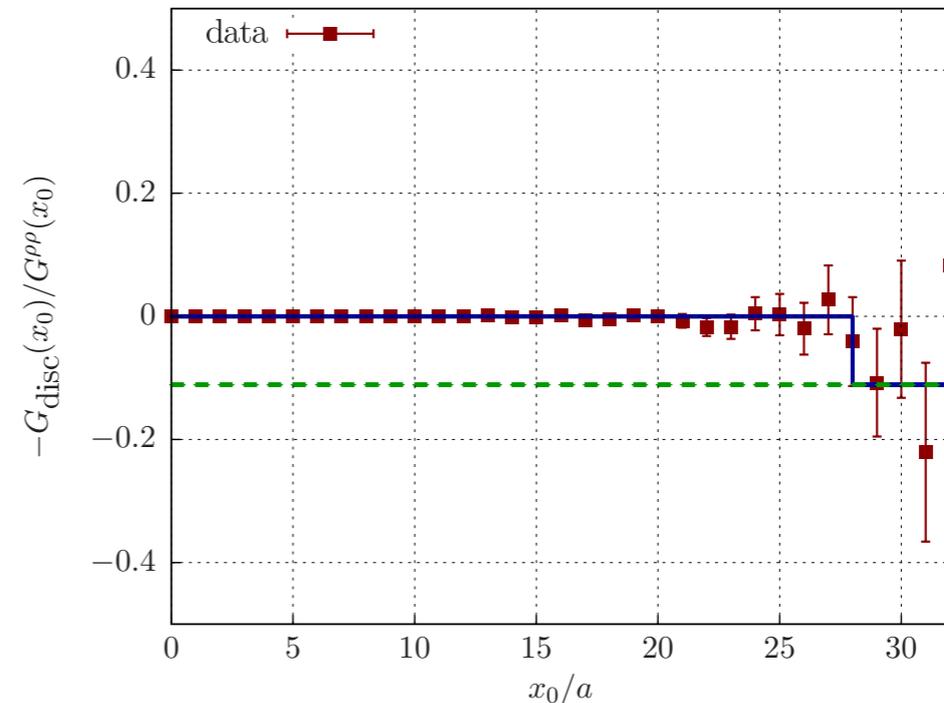
- \* Asymptotic behaviour: 
$$-\frac{G_{\text{disc}}(x_0)}{G^{\rho\rho}(x_0)} \xrightarrow{x_0 \rightarrow \infty} -\frac{1}{9}, \quad G^{\rho\rho}(x_0) = \frac{9}{10} G^{\text{ud}}(x_0)$$

- \* Data compatible with: 
$$-\frac{G_{\text{disc}}(x_0)}{G^{\rho\rho}(x_0)} = \begin{cases} 0, & x_0 \leq x_0^* \\ -1/9, & x_0 > x_0^* \end{cases}$$

# Disconnected Contributions

\* Disconnected contributions evaluated on subset of ensembles

Run	$N_{\text{cfg}}$	$N_r$	$T/a$	$x_0^*$	$\Delta a_\mu^{\text{hvp}}$
E5	1000	75	64	25	0.7%
				28	0.3%
F6	300	45	96	22	1.8%
				23	1.5%



⇒ Upper bound on disconnected contribution:

$$(a_\mu^{\text{hvp}})_{\text{disc}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{f}(x_0) (-G_{\text{disc}}(x_0)) = -\left(\frac{\alpha}{\pi}\right)^2 \int_{x_0^*}^\infty dx_0 \tilde{f}(x_0) \frac{1}{9} G^{\rho\rho}(x_0)$$

$$\Rightarrow \Delta a_\mu^{\text{hvp}} \equiv -\frac{(a_\mu^{\text{hvp}})_{\text{disc}}}{(a_\mu^{\text{hvp}})_{\text{con}}} \leq 2\%$$

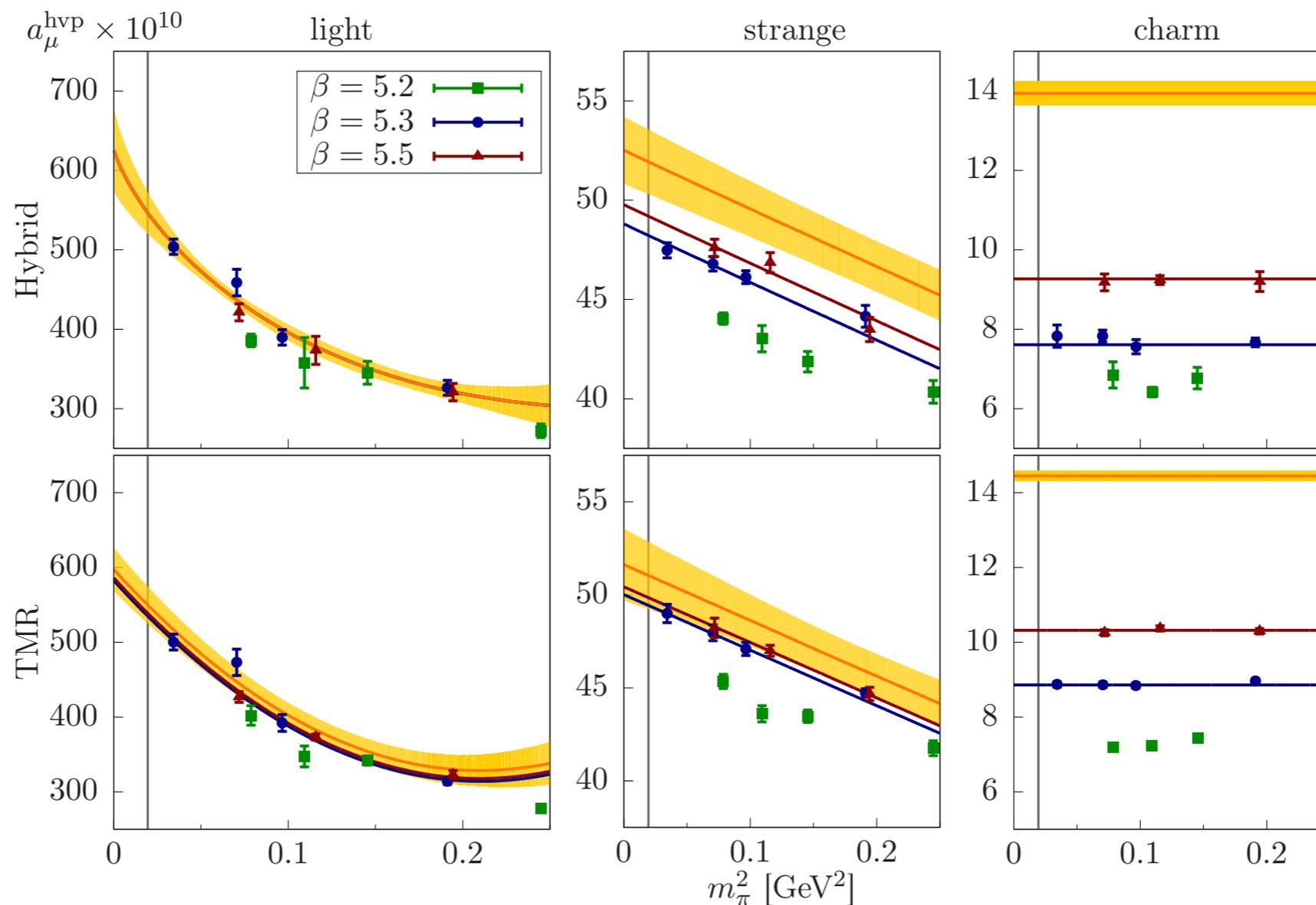
# Chiral and continuum extrapolations

- \* Employ variety of phenomenological *ansätze* to fit  $a_\mu^{\text{hvp}}$  e.g.

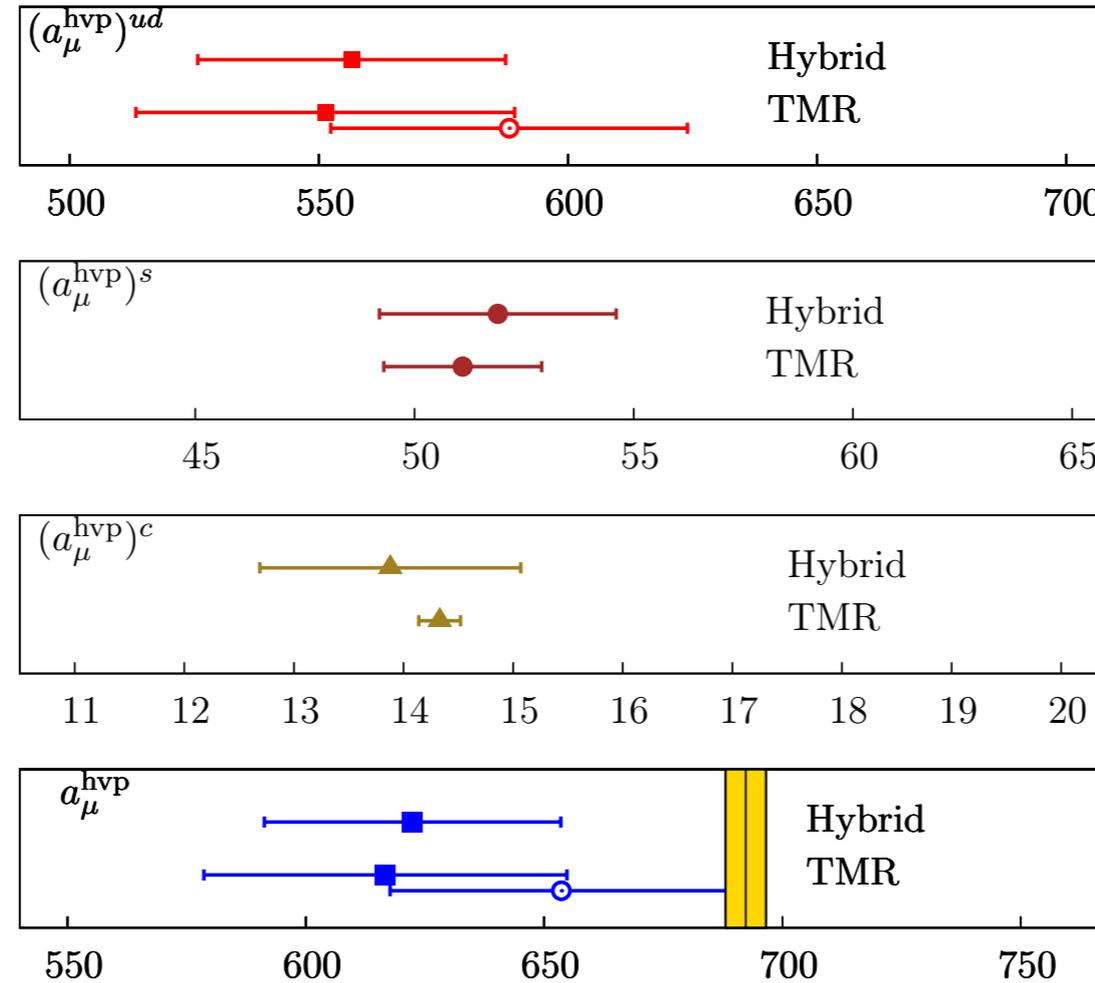
$$\alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a$$

$$\beta_1 + \beta_2 m_\pi^2 + \beta_3 m_\pi^4 + \beta_4 a$$

$$\gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a$$



# Results at the physical point



- \* Systematic errors estimated from distribution of fit variations
- \* Good consistency between “hybrid” method and TMR
- \* Finite-volume corrections sizeable

# Scale setting error

\* Evaluate  $\delta a_\mu^{\text{hvp}} = \left| a \frac{da_\mu^{\text{hvp}}}{da} \right| \frac{\delta a}{a}$

\* TMR:  $a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{f}(x_0) G(x_0)$        $x_0 \tilde{f}'(x_0) - \tilde{f}(x_0) = J(x_0)$

$$\Rightarrow a \frac{da_\mu^{\text{hvp}}}{da} = -a_\mu^{\text{hvp}} + \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 G(x_0) J(x_0)$$

\* Use expansion of  $\tilde{f}(x_0)$  to obtain  $J(x_0)$

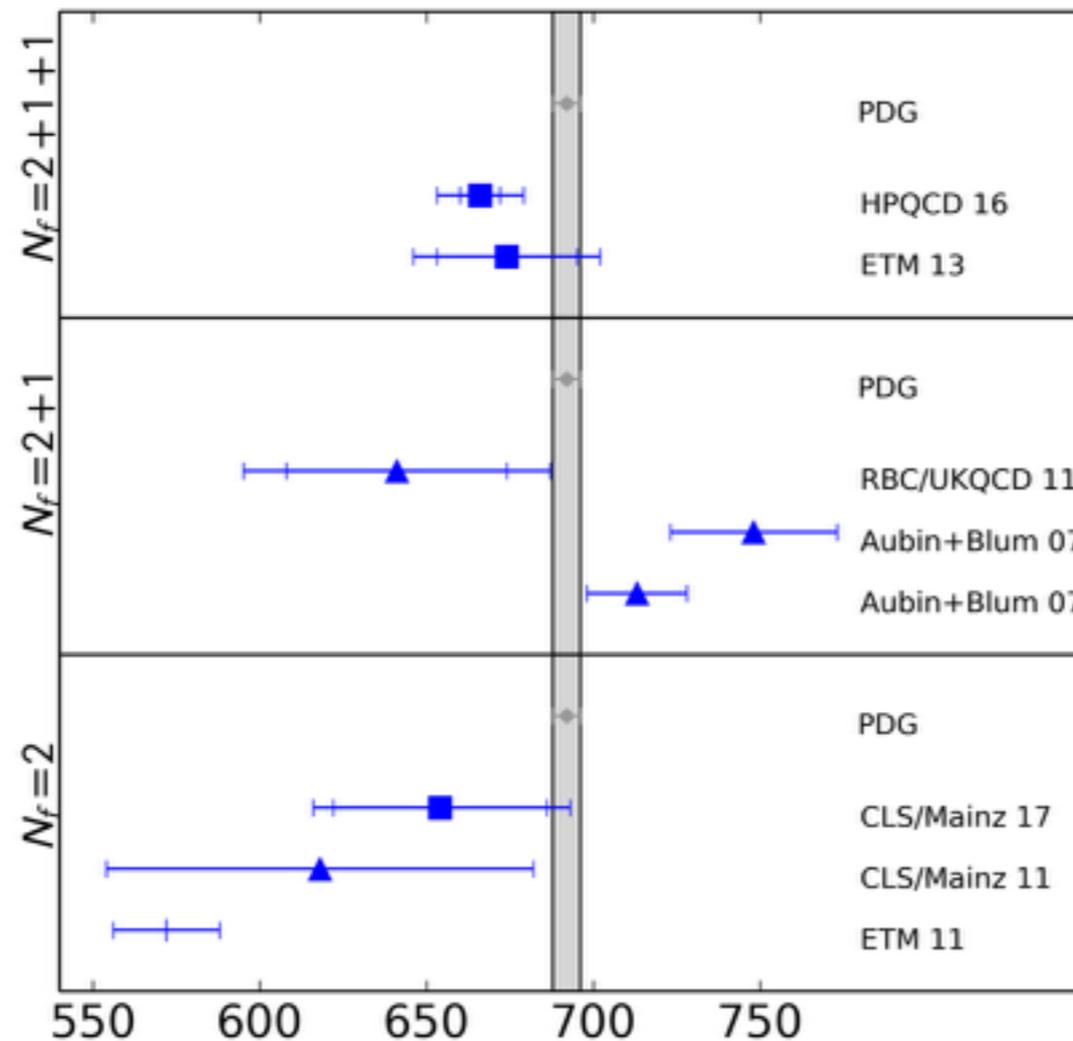
$$a \frac{da_\mu^{\text{hvp}}}{da} = 1.22 \cdot 10^{-7} \quad \Rightarrow \quad \frac{\delta a_\mu^{\text{hvp}}}{a_\mu^{\text{hvp}}} = \underbrace{\frac{1}{a_\mu^{\text{hvp}}} \left| a \frac{da_\mu^{\text{hvp}}}{da} \right|}_{\approx 1.8} \frac{\delta a}{a}$$

$\Rightarrow$  Rather precise determination of lattice spacing  $a$  [fm] required!

# Final result

- \* Estimate from TMR including finite-volume correction

$$a_{\mu}^{\text{hvp}} = (654 \pm 32_{\text{stat}} \pm 17_{\text{syst}} \pm 10_{\text{scale}} \pm 7_{\text{FV}} \pm_{-10}^0_{\text{disc}}) \cdot 10^{-10}$$



# Summary and Outlook

\* Consistency among different methods to determine the HVP contribution to  $(g - 2)$

\* Result of this study:

$$a_{\mu}^{\text{hvp}} = (654 \pm 32^{+21}_{-23}) \cdot 10^{-10}$$

4.8% statistical error

3.3% total systematic error

\* Improvements:

- Process CLS  $N_f = 2+1$  ensembles; employ TMR, time moments
- Determine pion timelike form factor
- Include isospin breaking