The pion transition form factor from lattice QCD

Antoine Gérardin

In collaboration with Harvey Meyer and Andreas Nyffeler

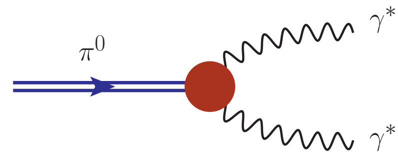
Based on [Phys. Rev. D **94**, 074507 (2016)]

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

First Workshop of the Muon $g-2$ Theory Initiative - Fermilab - June 5, 2017

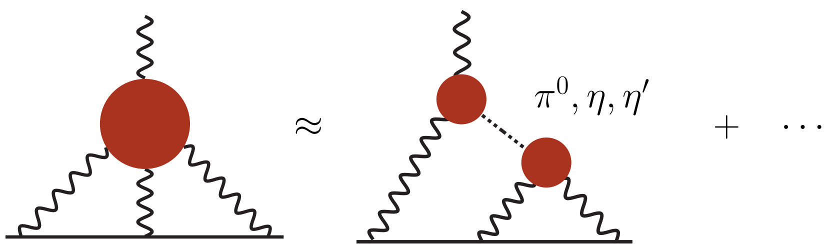
Motivations

The $\pi^0 \to \gamma^*\gamma^*$ transition form factor describe the interaction between a neutral pion and two off-shell photons



- The pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ yields important insights into the dynamics of QCD
  - $\to$ Chiral anomaly
  - $\to$ Brodsky-Lepage behaviour, pion distribution amplitude
  - $\to$ Test the operator product expansion (OPE) in the doubly-virtual case

- Hadronic light-by-light contribution to the $(g-2)_\mu$
  - $\to$ pion-pole contribution (dominant contribution)



Frequent estimates :

$$a_\mu^{\mathrm{HLbL}}(\pi^0) \approx 63.0 \times 10^{-11}$$

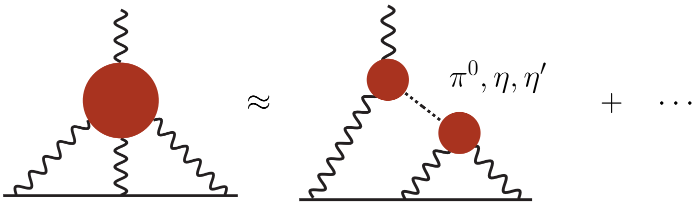
$$a_\mu^{\mathrm{HLbL}}(\eta) \approx 14.5 \times 10^{-11}$$

$$a_\mu^{\mathrm{HLbL}}(\eta') \approx 12.5 \times 10^{-11}$$

Motivations : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha_e}{\pi}\right)^3 \left(a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right)$$



where $(\tau = \cos(\theta)\,, Q_1 \cdot Q_2 = Q_1 Q_2 \cos(\theta))$

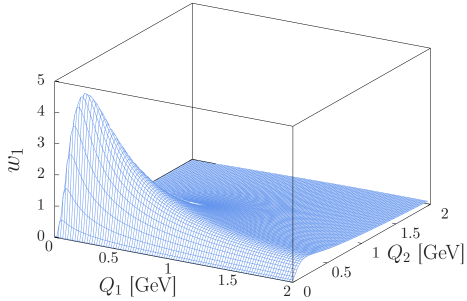
$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)\,,$$

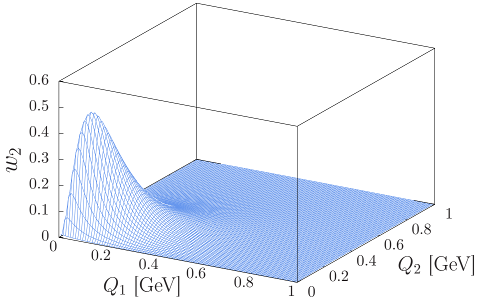
$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)\,.$$

→ Product of one single-virtual and one double-virtual transition form factors

→ $w_{1,2}(Q_1,Q_2,\tau)$ are model-independent weight functions

→ Weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)





Theoretical constraints on $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$

- **Adler-Bell-Jackiw (ABJ) anomaly :**

  ↪ Low virtualities ($Q_1^2 \to 0$, $Q_2^2 \to 0$)

  ↪ Chiral limit

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}$$

- **Brodsky-Lepage behavior :**

  ↪ Single-virtual form factor

  ↪ Off-shell photon : $Q^2 \to \infty$

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,0) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{Q^2}$$

  - The pre-factor depends on the shape on the pion distribution amplitude
  - α_s corrections

- **OPE prediction :**

  ↪ Double-virtual form factor

  ↪ Large virtualities : $Q^2 \to \infty$

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

  - Higher-twist matrix element in the OPE are known : $\frac{2F_\pi}{3Q^2}\left[1 - \frac{8}{9}\frac{\delta^2}{Q^2} + \mathcal{O}\left(\frac{1}{Q^4}\right)\right]$

Experimental status

- Decay width : $\Gamma_{\pi^0\gamma\gamma} = 7.82(22)$ eV $\sim 3\%$ [PrimEx '10]

$$\Gamma_{\pi^0\gamma\gamma} = \frac{\pi\alpha_e^2 m_\pi^3}{4} \mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$$

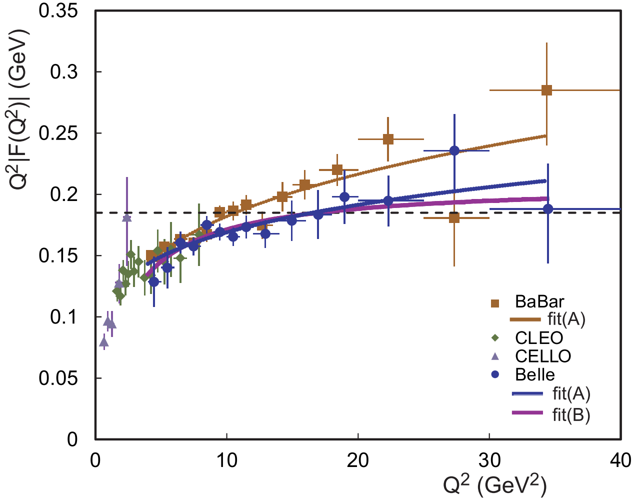
→ Consistent with current theoretical predictions

→ Experimental test of the chiral anomaly

- The single-virtual form factor has been measured (CELLO, CLEO, BaBar, Belle)

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0) = F(Q^2)$$

[Belle '12]



→ Belle data seem to confirm the Brodsky-Lepage behavior $\sim 1/Q^2$.

→ Belle and Babar results are quite different

→ no measurement at low $Q < 0.8$ GeV (dominant contribution)

- No result yet for the double-virtual form factor

↪ but measurement planned at BESIII

Motivations

To estimate the pion pole contribution we need :

- The single and double virtual transition form factor for arbitrary space-like virtualities
- In the kinematical range $Q^2 \in [0-2]\ \mathrm{GeV}^2$ (non-perturbative regime of QCD)

BUT

- Experiments give information on the single-virtual form factor only
- Experimental data are available only for relatively large virtualities $Q^2 > 0.6\ \mathrm{GeV}^2$
- The theory imposes strong constraints for the normalisation and the asymptotic behavior of the TFF

$\hookrightarrow$ Most evaluations of the pion-pole contribution are therefore based on phenomenological models

$\hookrightarrow$ Systematic errors are difficult to estimate

Lattice QCD is particularly well suited to compute the form factor in the energy range relevant to $g-2$!

Lattice computation

Lattice calculation

In Minkowski space-time :

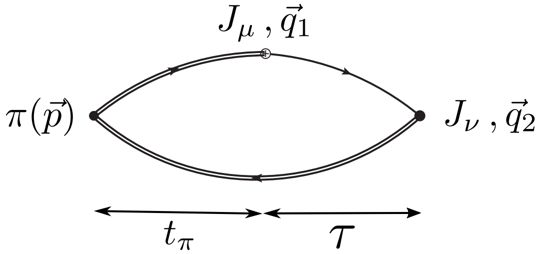
$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2,q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle \Omega | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p)\rangle = M_{\mu\nu}(q_1^2,q_2^2)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \ldots$

In Euclidean space-time : [Ji & Jung '01] [Cohen et al. '08] [Feng et al. '12]

$$M_{\mu\nu}^E(q_1^2,q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1 \tau} \int \mathrm{d}^3z\, e^{-i\vec{q}_1\vec{z}}\, \langle 0 | T\left\{ J_\mu(\vec{z},\tau) J_\nu(\vec{0},0) \right\} | \pi(p)\rangle$$

- Analytical continuation ("$\tau = -i\,t$")
- We must kept $q_{1,2}^2 < M_V^2 = \min(M_\rho^2, 4m_\pi^2)$ to avoid poles
- $q_1 = (\omega_1, \vec{q}_1)$



The main object to compute is the Euclidean three-point correlation function :

$$C_{\mu\nu}^{(3)}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) = \sum_{\vec{x},\vec{z}} \left\langle T\left\{ J_\nu(\vec{0},t_f) J_\mu(\vec{z},t_i) P(\vec{x},t_0) \right\} \right\rangle e^{i\vec{p}\vec{x}}\, e^{-i\vec{q}_1\vec{z}}$$

- We use one local (Z_V computed non perturbatively) and one conserved vector current

Lattice QCD

- Lattice QCD is not a model : specific regularisation of the theory adapted to numerical simulations
- However there are systematic errors that we need to understand :

1) Finite lattice spacing : discretisation errors

$\rightarrow$ 3 lattice spacings ($a = 0.075, 0.065, 0.048$ fm)

$\rightarrow$ Extrapolation to $a = 0$

2) Unphysical quark masses

$\rightarrow$ Different simulations with different pion mass in the range [190:440] MeV

$\rightarrow$ Extrapolation to $m_\pi = m_\pi^{\mathrm{exp}}$

3) Finite volume

$\rightarrow$ Periodic boundary conditions in space, volume effects are $\mathcal{O}(e^{-m_\pi L})$, we use $m_\pi L > 4$

$\rightarrow$ Discrete spatial momenta $\vec{p} = 2\pi/L\vec{n}$

$\rightarrow$ We average over all possible photons spatial momenta $\vec{q}_1$ and $\vec{q}_2$ to increase the statistic

Kinematic reach in the photon virtualities

Photons virtualities :

$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$

$\Rightarrow |\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \omega_1$ is a (real) free parameter

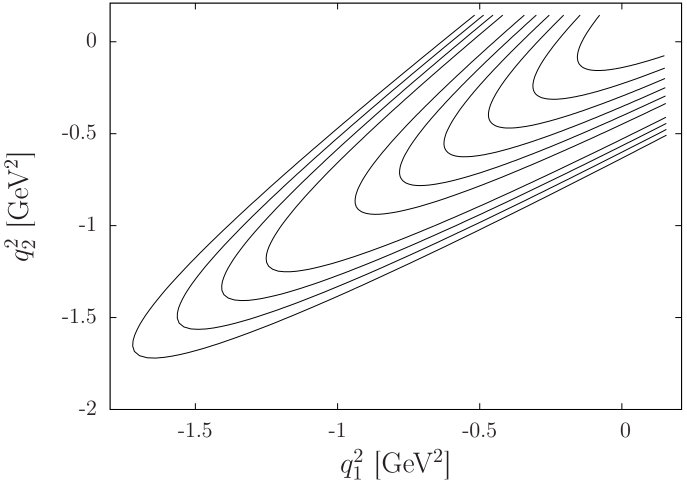


Figure: $L/a = 48$ at $a = 0.065$ fm.

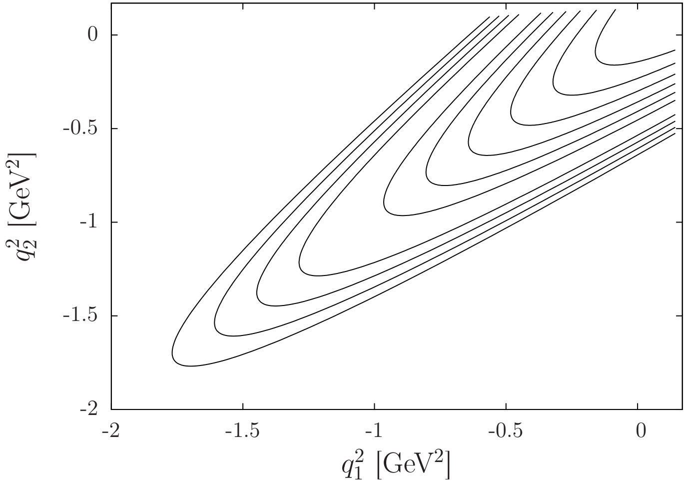


Figure: $L/a = 64$ at $a = 0.048$ fm.

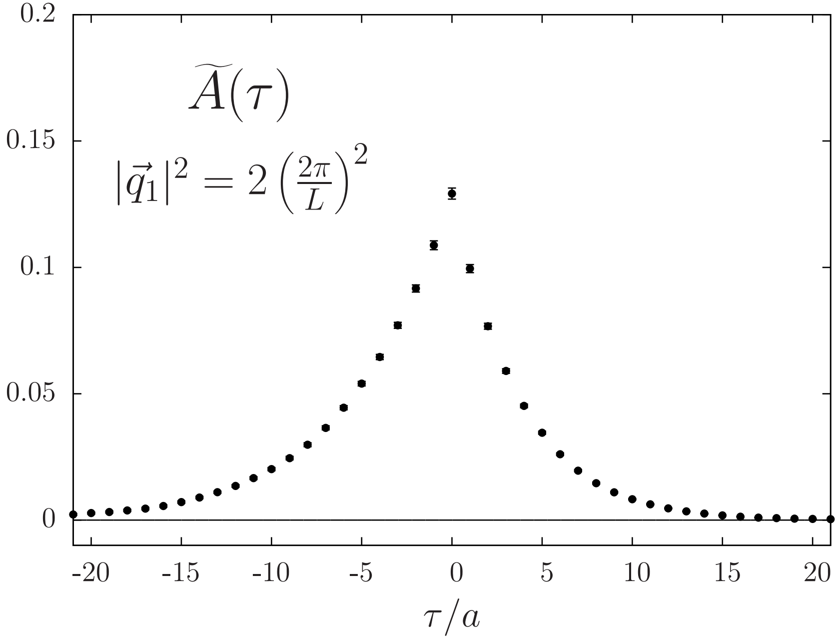
Results

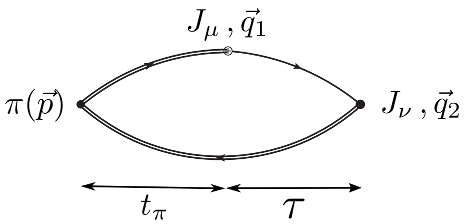
Shape of the integrand for F7 ($a = 0.065$ fm and $m_\pi = 270$ MeV)

$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \widetilde{A}_{\mu\nu}(\tau) \, e^{\omega_1 \tau}$$

$$A_{\mu\nu}(\tau) = \lim_{t_\pi \to \infty} C_{\mu\nu}(\tau, t_\pi) e^{E_\pi t_\pi}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau)\, e^{-E_\pi \tau} & \tau < 0 \end{cases}$$





On the lattice :

- Discrete sum over lattice points :

$$\int \mathrm{d}\tau \to a \sum_\tau$$

- Finite size of the box :

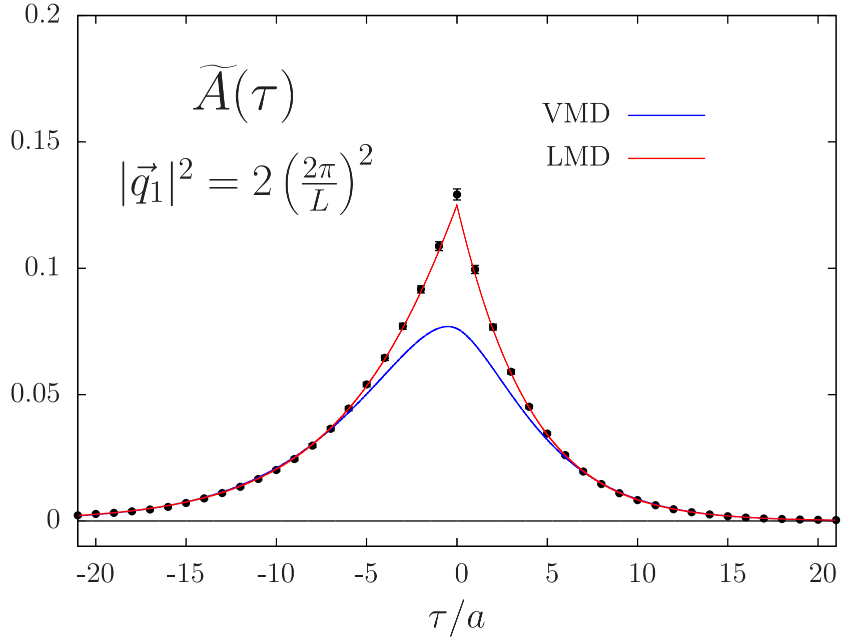
$$|\tau| <= \tau_{\max} \neq \infty$$

Shape of the integrand for F7 ($a = 0.065$ fm and $m_\pi = 270$ MeV)

- The vector meson dominance (VMD) model is expected to give a good description of the data at large τ

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \quad \rightarrow \quad \widetilde{A}(\tau) = \ \ldots \quad \text{(known analytical expression)}$$

- Fit the data at large τ and use the result of the fit for $\tau > \tau_c \gtrsim 1.3$ fm



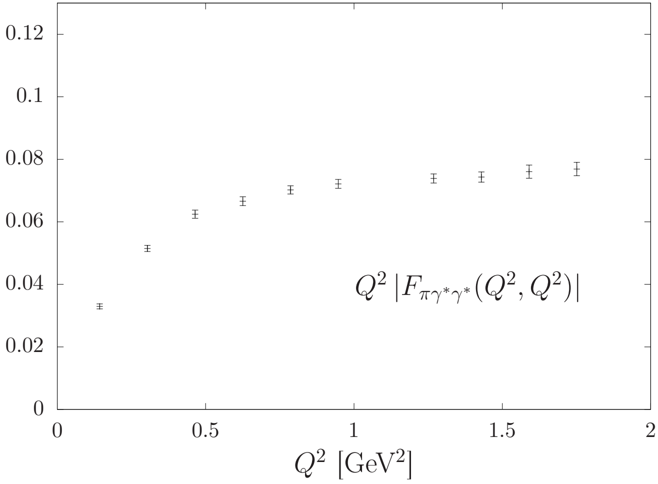
- Check the dependance on the model using LMD rather than VMD:

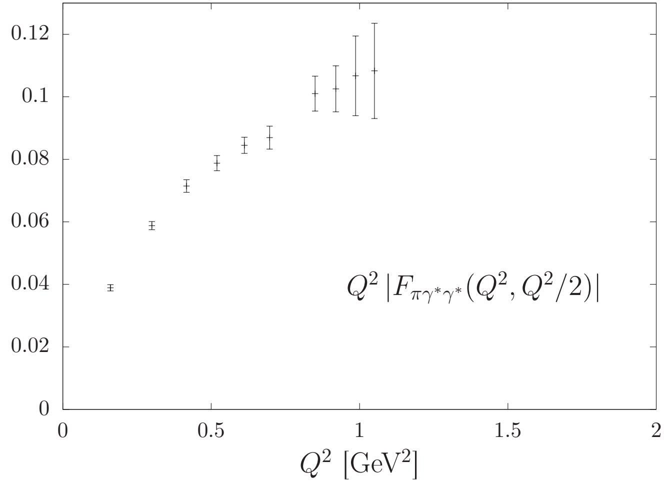
$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- The difference between the two models is included in the systematic error.

Transition form factor : results

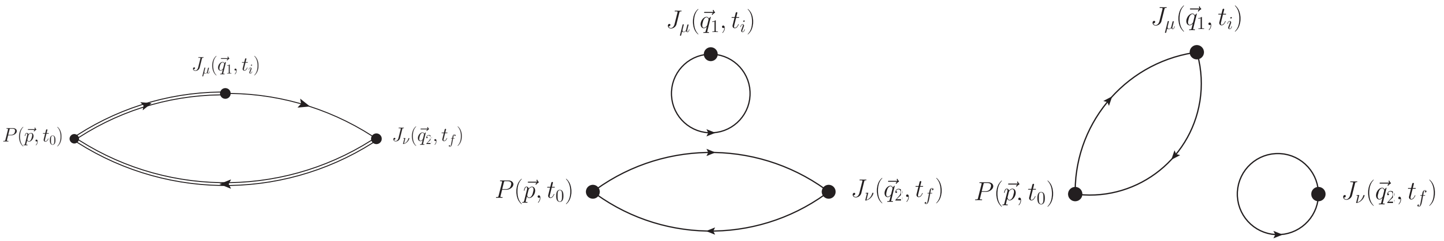
- Results for one of the eight ensembles with $a = 0.048$ fm and $m_\pi = 270$ MeV
- We have access to the single and double-virtual form factor. Two special cases
  - $\rightarrow$ doubly-virtual form factor with $Q_1^2 = Q_2^2$
  - $\rightarrow$ doubly-virtual form factor with $Q_1^2 = 2\,Q_2^2$



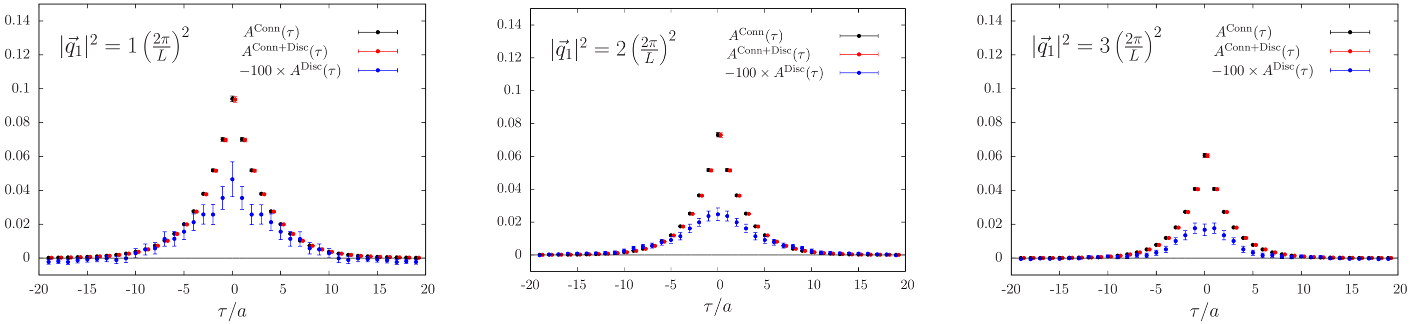


Contrary to the experimental case, the single virtual TFF is more challenging on the lattice

Disconnected contributions



- Disconnected contributions are known to be much more challenging on the lattice
- Computed on E5 only ($a = 0.065$ fm, $m_\pi = 440$ MeV)
- Loops : 75 stochastic sources with full-time dilution and a generalised Hopping Parameter Expansion.
- Two-point functions : 7 stochastic sources with full-time dilution



→ The disconnected contribution is below 1%

→ But the pion mass dependence could be large ...

Continuum and chiral extrapolation

- Lattice results are necessary obtained at finite lattice spacing $a \neq 0$
- Our simulations are also performed at unphysical quark masses

$\hookrightarrow$ Use phenomenological models to describe our data, then extrapolate to the continuum and chiral limit

- VMD model

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- Reproduces the anomaly constraint with $\alpha = 1/4\pi^2 F_\pi$
- And the Brodsky-Lepage in the single-virtual case : $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) = \alpha M_V^2/Q^2$
- But it fails to reproduces the OPE prediction in the double-virtual case

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) = \alpha M_V^4/Q^4 \quad , \quad \mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) \sim 2F_\pi/(3Q^2)$$

Continuum and chiral extrapolation

- Lattice results are necessary obtained at finite lattice spacing $a \neq 0$
- Our simulations are also performed at unphysical quark masses

$\hookrightarrow$ Use phenomenological models to describe our data, then extrapolate to the continuum and chiral limit

- VMD model

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- LMD model (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- Inspired from the large-N_C approximation to QCD
- Reproduces the anomaly constraint with $\alpha = 1/4\pi^2 F_\pi$
- Compatible with the OPE prediction in the double-virtual case with $\beta = -F_\pi/3$
- But this model is not compatible with the Brodsky-Lepage behavior in the single virtual case

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) = -\beta/M_V^2 \quad , \quad \mathcal{F}^{\mathrm{BL}}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) \sim 1/Q^2$$

Continuum and chiral extrapolation

- Lattice results are necessary obtained at finite lattice spacing $a \neq 0$
- Our simulations are also performed at unphysical quark masses

$\hookrightarrow$ Use phenomenological models to describe our data, then extrapolate to the continuum and chiral limit

- VMD model

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- LMD model (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

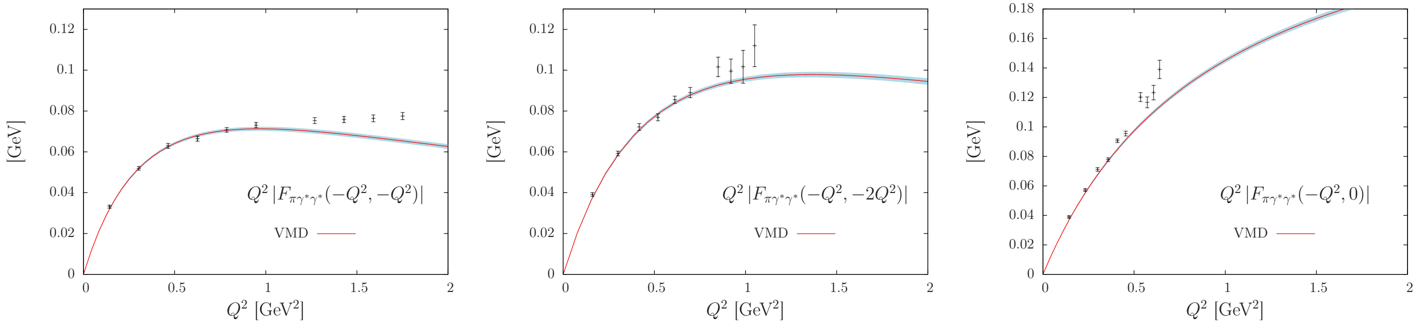
- LMD+V model [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2 + q_2^2) + \widetilde{h}_1 (q_1^2 + q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2 + q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

- Refinement of the LMD model (includes a second vector resonance, $\rho' : M_{V_2}$)
- All the theoretical constraints are satisfied (if one sets $\widetilde{h}_1 = 0$)
- Many more fit parameters

Comparison with phenomenological models : VMD

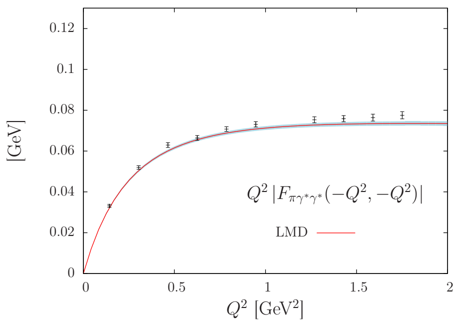
$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

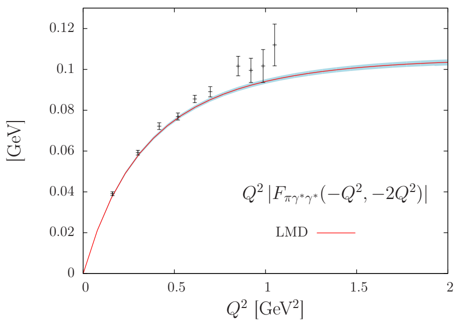


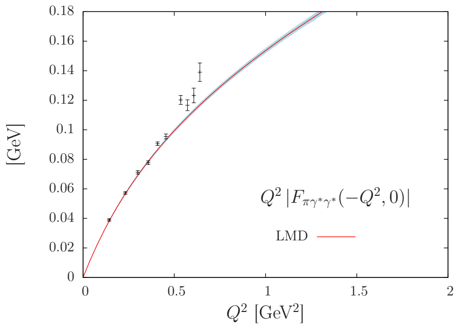
- ↪ α and M_V are fit parameters
- ↪ Global fit (8 ensembles + chiral and continuum extrapolation)
- ↪ The model fails to describe our data ! ($\alpha = 0.243(18)\ \mathrm{GeV}^{-1} \neq \alpha_{\mathrm{th}} = 0.274\ \mathrm{GeV}^{-1}$)
- ↪ The wrong asymptotic behavior of this model (double virtual case) already matters at $Q^2 \sim 1-2\ \mathrm{GeV}^2$

Comparison with phenomenological models : LMD

$$\mathcal{F}^{\text{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$







$$\alpha^{\text{LMD}} = 0.275(18)\ \text{GeV}^{-1} \quad , \quad \beta = -0.028(4)\ \text{GeV} \quad , \quad M_V^{\text{LMD}} = 0.705(24)\ \text{GeV}\,.$$

- α, β and M_V are fit parameters
- The model gives a good description of our data
- α^{LMD} is compatible with the theoretical prediction $\alpha^{\text{th}} = 1/(4\pi^2 F_\pi) = 0.274\ \text{GeV}^{-1} \quad \rightarrow$ (accuracy 7%)
- β^{LMD} is compatible with the OPE prediction $\beta^{\text{OPE}} = -F_\pi/3 = -0.0308\ \text{GeV}$

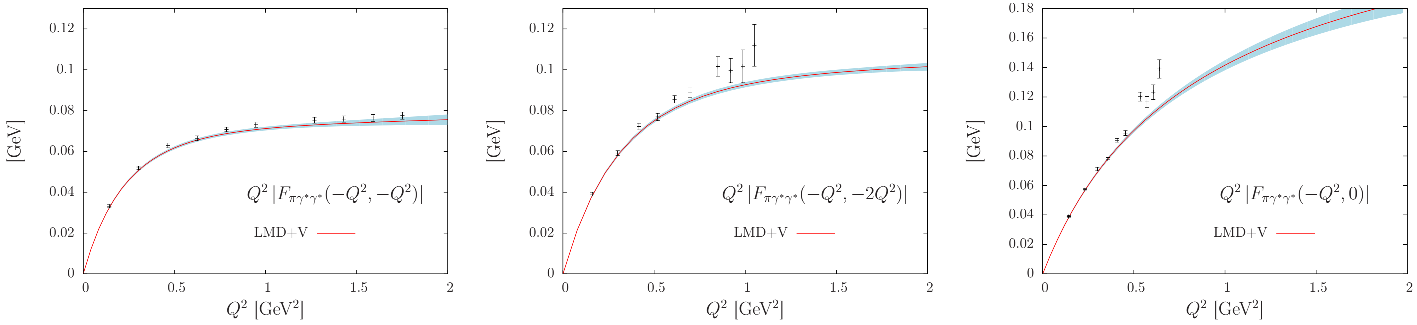
Comparison with phenomenological models : LMD+V

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

- Assumptions:
  - $M_{V_1} = m_\rho^{\mathrm{exp}} = 0.775\ \mathrm{GeV}$ in the continuum and chiral limit (but chiral corrections are taken into account in the fit)
  - Constant shift in the spectrum: $M_{V_2}(\widetilde{y}) = m_{\rho'}^{\mathrm{exp}} + M_{V_1}(\widetilde{y}) - m_\rho^{\mathrm{exp}}$

Comparison with phenomenological models : LMD+V

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\,(q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$



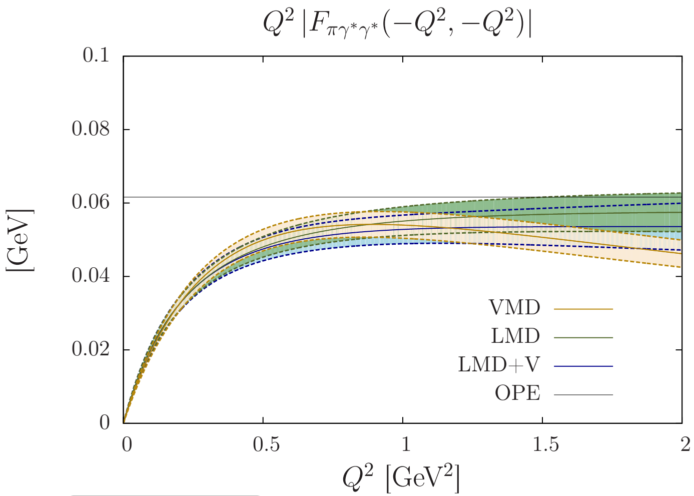
$$\alpha^{\mathrm{LMD+V}} = 0.273(24)\ \mathrm{GeV}^{-1} \quad , \quad \widetilde{h}_2 = 0.345(167)\ \mathrm{GeV}^3 \quad , \quad \widetilde{h}_5 = -0.195(70)\ \mathrm{GeV}\,.$$

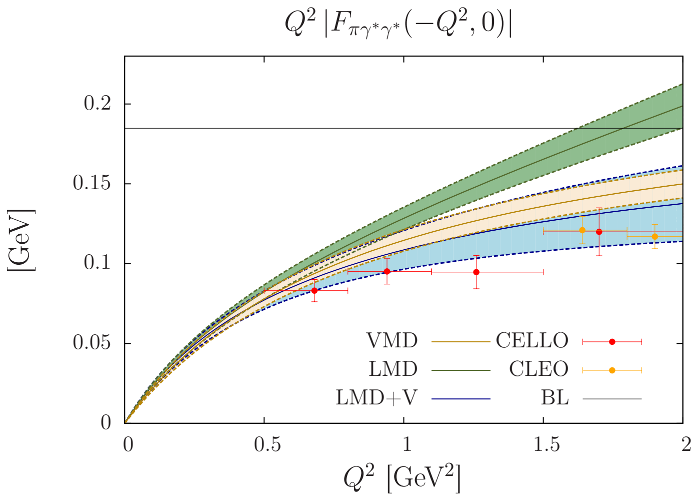
- The data are well describe by this model (same χ^2/d.o.f. as for the LMD model)
- $\alpha^{\mathrm{LMD+V}}$ compatible with the theoretical prediction $\alpha^{\mathrm{th}} = 0.274\ \mathrm{GeV}^{-1}$ (statistical accuracy 9%)
- Fit to CLEO data (single-virtual form factor) : $\widetilde{h}_5 = -0.166(6)\ \mathrm{GeV}$
- $\widetilde{h}_2$ can be fixed by comparing with the subleading term in the OPE [Nesterenko et al '83]

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3}\left[\frac{1}{Q^2} - \frac{8}{9}\frac{\delta^2}{Q^4} + \mathcal{O}\left(\frac{1}{Q^6}\right)\right]$$

- QCD sum rules : $\delta^2 = 0.20(2)\ \mathrm{GeV}^2$ [Novikov et al '84] $\rightarrow \widetilde{h}_2 = 0.327\ \mathrm{GeV}^3$

Final results for the form factor (at the physical point)





LMD model :

$$\alpha^{\mathrm{LMD}} = 0.275(18)(3)\ \mathrm{GeV}^{-1}\,,\quad \beta = -0.028(4)(1)\ \mathrm{GeV}\,,\quad M_V^{\mathrm{LMD}} = 0.705(24)(21)\ \mathrm{GeV}$$

LMD+V model :

$$\alpha^{\mathrm{LMD+V}} = 0.273(24)(7)\ \mathrm{GeV}^{-1}\,,\quad \widetilde{h}_2 = 0.345(167)(83)\ \mathrm{GeV}^3\,,\quad \widetilde{h}_5 = -0.195(70)(34)\ \mathrm{GeV}$$

→ where $\widetilde{h}_0 = -F_\pi/3 = -0.0308$ GeV, $M_{V_1} = 0.775$ GeV and $M_{V_2} = 1.465$ GeV are fixed at the physical point.

Back to phenomenology : the pion-pole contribution

[Jegerlehner & Nyffeler '09] $$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha_e}{\pi}\right)^3 \left(a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right)$$

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau\; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau\; w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ $w_{1,2}(Q_1,Q_2,\tau)$ are some model-independent weight functions (concentrated at small momenta below 1 GeV)

→ The form factor has been extrapolated to the continuum and chiral limit

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

→ most model calculations yield results in the range

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50-80) \times 10^{-11}$$

Model	$a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$
LMD (this work)	68.2(7.4)
LMD+V (this work)	65.0(8.3)
VMD (theory)	57.0
LMD (theory)	73.7
LMD+V (theory + phenomenology)	62.9

Λ [GeV]	LMD		LMD+V	
0.25	14.6	(21.4%)	14.4	(22.1%)
0.5	37.9	(55.5%)	37.2	(57.2%)
0.75	50.7	(74.4%)	49.5	(76.1%)
1.0	57.3	(84.0%)	55.5	(85.4%)
1.5	62.9	(92.3%)	60.6	(93.1%)
2.0	65.1	(95.5%)	62.5	(96.1%)
5.0	67.7	(99.2%)	64.6	(99.4%)
20.0	68.2	(100%)	65.0	(100%)

Perspectives

- Use $N_f = 2+1$ gauge configuration
  $\rightarrow$ dynamical strange quark
  $\rightarrow$ CLS configuration with open boundary conditions

- Include a new kinematical configuration where the pion has one unit of momentum
  $\rightarrow$ allow to probe larger kinematical range

- Using Wilson-Clover fermions, discretisation errors are $\mathcal{O}(a)$.
  $\rightarrow$ Implement full $\mathcal{O}(a)$-improvement to reduce discretization effects
  $\rightarrow$ Requires the $\mathcal{O}(a)$-improvement of the vector current on the lattice

Conclusion

- We have performed a lattice calculation of the pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ with two dynamical quarks and in the momentum region relevant for the $(g-2)_\mu$.

- The VMD model fails to describe our data, especially in the double virtual case.

- However, the LMD and LMD+V models describe our data successfully.

- In particular we recover the anomaly results ($\alpha^{\rm th} = 0.274~\text{GeV}^{-1}$) in the continuum and chiral limit

$$\alpha^{\rm LMD} = 0.275(18)(3)~\text{GeV}^{-1} \quad , \quad \alpha^{\rm LMD+V} = 0.273(24)(7)~\text{GeV}^{-1}$$

  $\rightarrow 7 - 9\%$ accuracy

- Disconnected contributions have been computed on one lattice ensemble.

- Provides a first lattice estimate of the pion-pole contribution to the hadronic light-by-light scattering in the $g-2$ of the muon

$$a_{\mu;\rm LMD+V}^{\rm HLbL;\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$