Pseudoscalar-pole contribution to $(g_\mu - 2)$: a rational approach

Pablo Sanchez-Puertas
sanchezp@ipnp.troja.mff.cuni.cz

Charles University Prague

In collaboration with P. Masjuan

First Workshop of the Muon g-2 initiative
Q Center, 3-6th June 2016

CHARLES UNIVERSITY
Faculty of mathematics
and physics

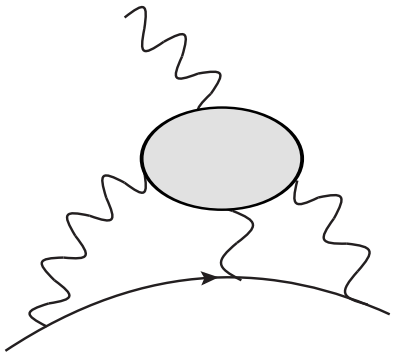
Outline

1. Hadronic light-by-light: pseudoscalar pole contribution
2. Form factor description for the HLbL: rational approach
3. Updated pseudoscalar pole contribution

Section 1

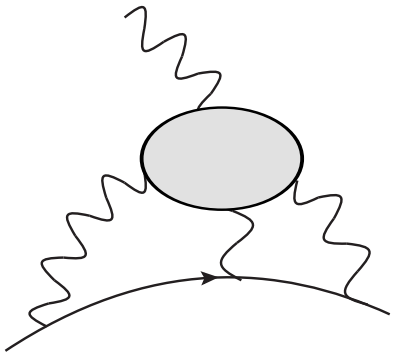
Hadronic light-by-light: pseudoscalar pole contribution

Hadronic Light-by-Light: Brief reminder



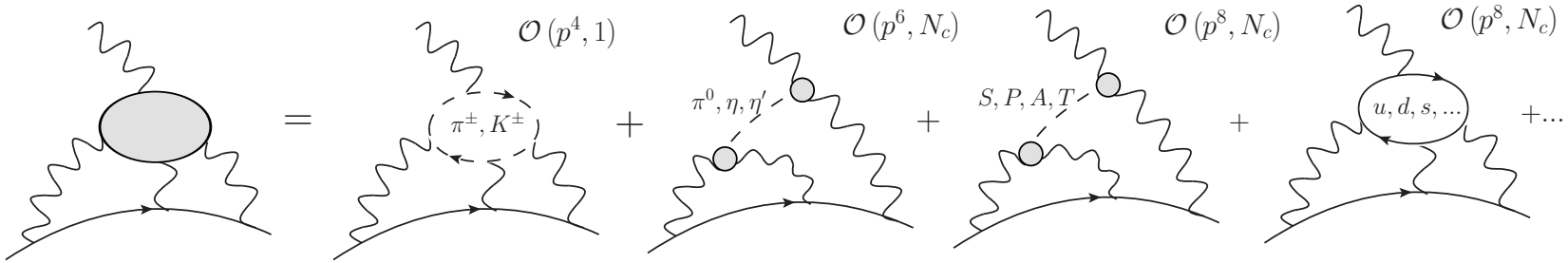
- Not straightforward connection to data
- Dispersive proposals recently (involved)
- Multi-scale problem $\rightarrow$ more difficulties

Hadronic Light-by-Light: Brief reminder

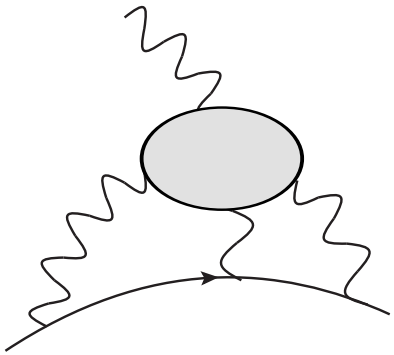


- Not straightforward connection to data
- Dispersive proposals recently (involved)
- Multi-scale problem $\rightarrow$ more difficulties

E. de Rafael (1994): large-N_c + χPT

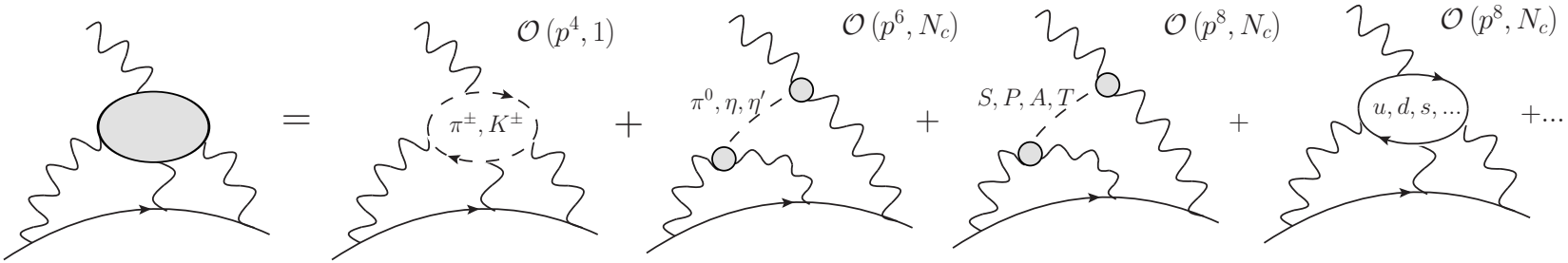


Hadronic Light-by-Light: Brief reminder



- Not straightforward connection to data
- Dispersive proposals recently (involved)
- Multi-scale problem $\rightarrow$ more difficulties

E. de Rafael (1994): large-N_c + χPT

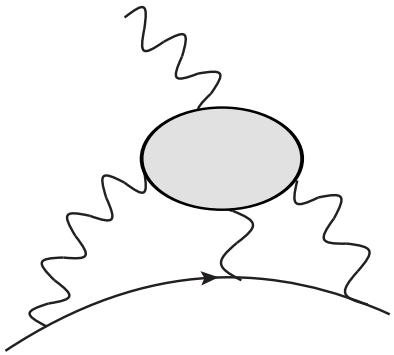


Authors	π^0, η, η'	$\pi\pi, KK$	Resonances	Quark Loop	Total
BPP	85(13)	−19(13)	−4(3)	21(3)	83(32)
HKS	83(6)	−5(8)	2(2)	10(11)	90(15)
KN	83(12)	–	–	–	80(40)
MV	114(10)	–	22(5)	–	136(25)
PdRV	114(13)	−19(19)	8(12)	2.3	105(20)
N/JN	99(16)	−19(13)	15(7)	21(3)	116(39)

Classic Results (10^{-11} units)

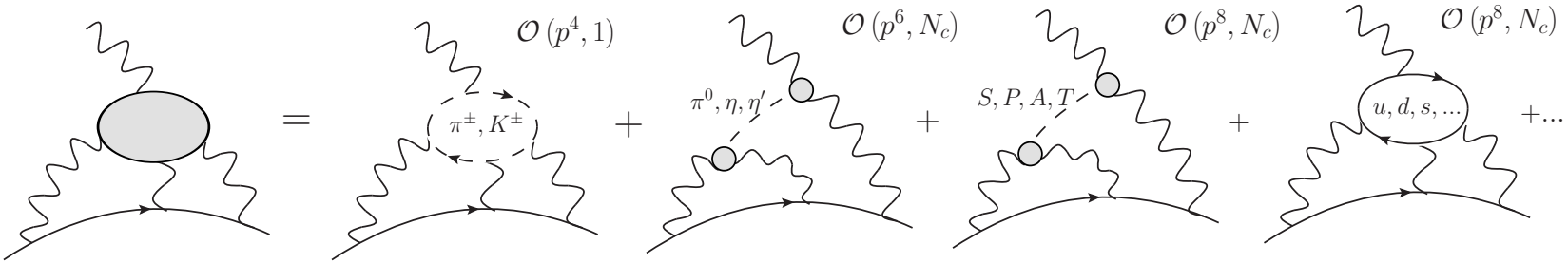
- Enough for $\delta a_\mu = 63 \times 10^{-11}$
- Not in future $\delta a_\mu = 16 \times 10^{-11}$
- Most results circa 15 years old

Hadronic Light-by-Light: Brief reminder



- Not straightforward connection to data
- Dispersive proposals recently (involved)
- Multi-scale problem $\rightarrow$ more difficulties

E. de Rafael (1994): large-N_c + χPT



Authors	π^0, η, η'	$\pi\pi, KK$	Resonances	Quark Loop	Total
BPP	85(13)	−19(13)	−4(3)	21(3)	83(32)
HKS	83(6)	−5(8)	2(2)	10(11)	90(15)
KN	83(12)	–	–	–	80(40)
MV	114(10)	–	22(5)	–	136(25)
PdRV	114(13)	−19(19)	8(12)	2.3	105(20)
N/JN	99(16)	−19(13)	15(7)	21(3)	116(39)

Update
π^0, η, η'
Contributions

HLbL: the pseudoscalar-pole contribution

For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i \cdot x_i} \langle\Omega| T\{j^\mu(x_1) j^\nu(x_2) j^\rho(x_3) j^\sigma(x_4)\} |\Omega\rangle$$

HLbL: the pseudoscalar-pole contribution

For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i\cdot x_i} \langle\Omega| T\{j^\mu(x_1)j^\nu(x_2)j^\rho(x_3)j^\sigma(x_4)\} |\Omega\rangle$$

At low energies insert lowest-lying intermediate states (close to pole):

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i\cdot x_i} \frac{i \langle\Omega| T\{j^\mu(0)j^\nu(x_2)\} |P\rangle \langle P| T\{j^\rho(0)j^\sigma(x_4)\} |\Omega\rangle}{q^2 - m_P^2 + i\epsilon} + \ldots$$

HLbL: the pseudoscalar-pole contribution

For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i\cdot x_i} \langle\Omega| T\{j^\mu(x_1)j^\nu(x_2)j^\rho(x_3)j^\sigma(x_4)\} |\Omega\rangle$$

At low energies insert lowest-lying intermediate states (close to pole):

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = i\epsilon^{\mu\nu\alpha\beta} p_{1\alpha} p_{2\beta} \frac{i|F_{P\gamma^*\gamma^*}(p_1^2, p_2^2)|^2}{(p_1+p_2)^2 - m_P^2} i\epsilon^{\rho\sigma\gamma\delta} p_{1\gamma} p_{2\delta} + \text{ crossed } + \ldots$$

HLbL: the pseudoscalar-pole contribution

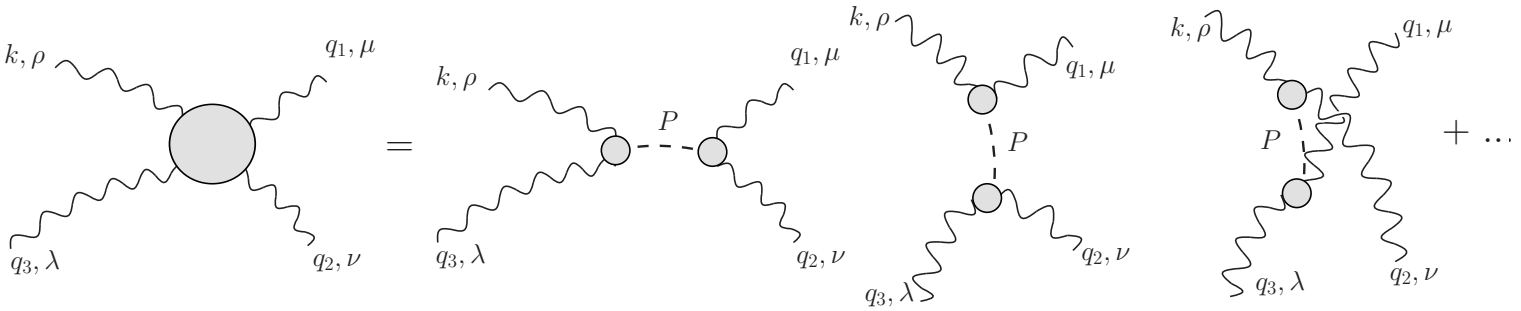
For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i\cdot x_i} \langle\Omega| T\{j^\mu(x_1)j^\nu(x_2)j^\rho(x_3)j^\sigma(x_4)\} |\Omega\rangle$$

At low energies insert lowest-lying intermediate states (close to pole):

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = i\epsilon^{\mu\nu\alpha\beta} p_{1\alpha} p_{2\beta} \frac{i|F_{P\gamma^*\gamma^*}(p_1^2, p_2^2)|^2}{(p_1+p_2)^2 - m_P^2} i\epsilon^{\rho\sigma\gamma\delta} p_{1\gamma} p_{2\delta} + \text{ crossed } + \ldots$$

Related to physical process! Graphically, it looks like



HLbL: the pseudoscalar-pole contribution

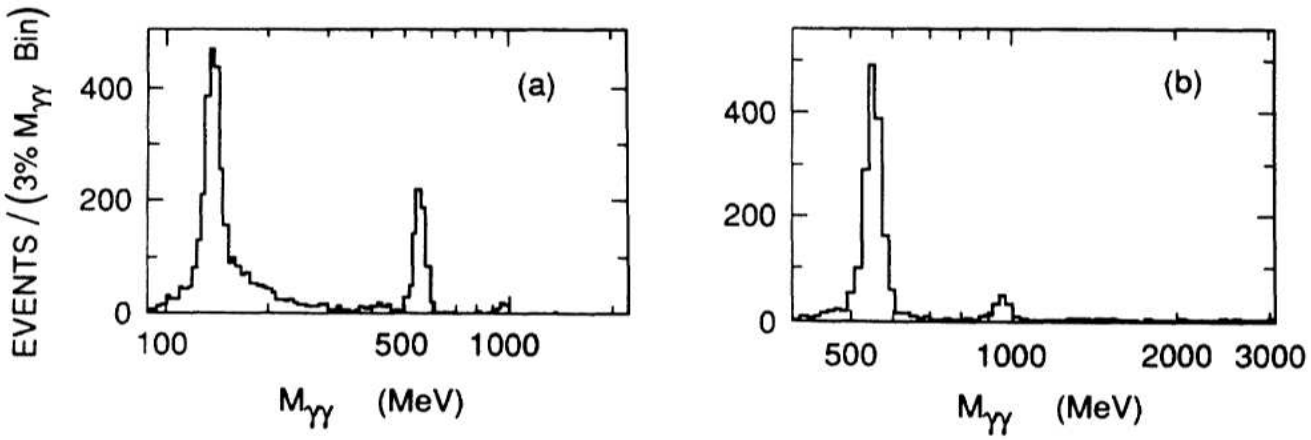
For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{ip_i\cdot x_i} \langle\Omega| T\{j^\mu(x_1)j^\nu(x_2)j^\rho(x_3)j^\sigma(x_4)\} |\Omega\rangle$$

At low energies insert lowest-lying intermediate states (close to pole):

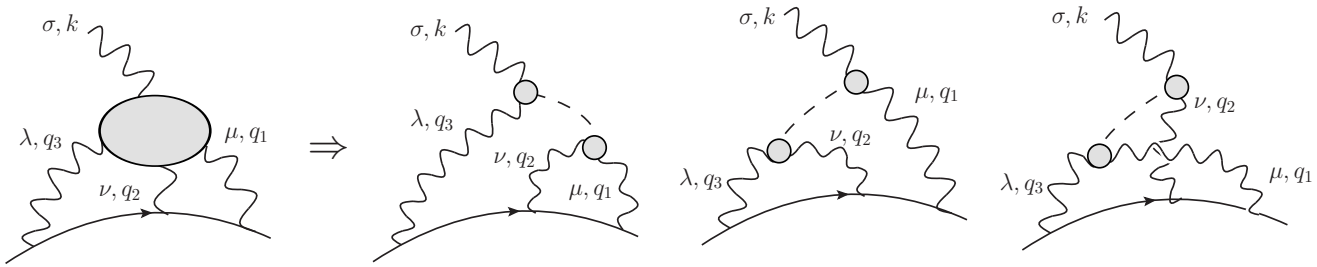
$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = i\epsilon^{\mu\nu\alpha\beta} p_{1\alpha} p_{2\beta} \frac{i|F_{P\gamma^*\gamma^*}(p_1^2, p_2^2)|^2}{(p_1+p_2)^2 - m_P^2} i\epsilon^{\rho\sigma\gamma\delta} p_{1\gamma} p_{2\delta} + \text{ crossed } + \ldots$$

Related to physical process! Experimentally, it looks like



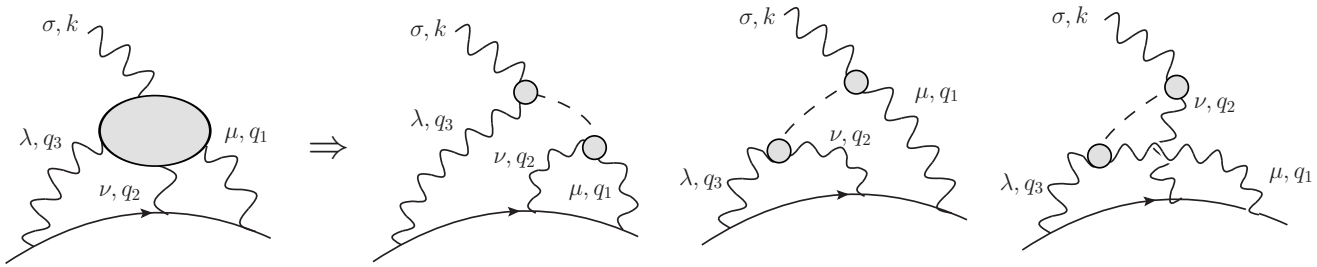
HLbL: the pseudoscalar-pole contribution

- Plug previous result into HLbL $(g-2)$ contribution



HLbL: the pseudoscalar-pole contribution

- Plug previous result into HLbL $(g-2)$ contribution

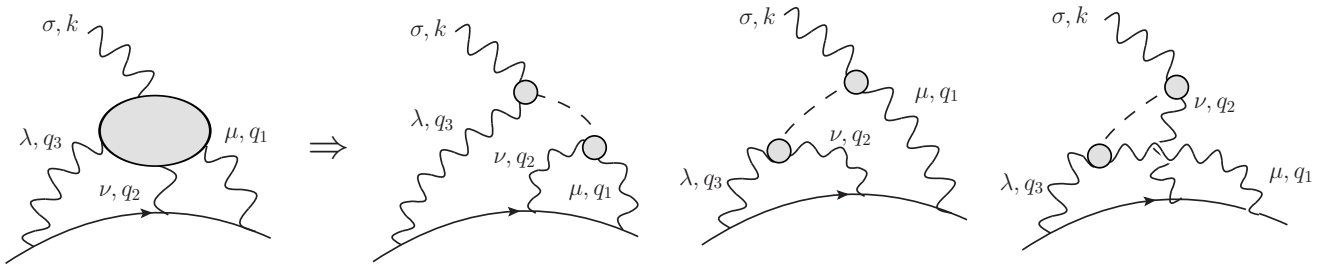


- After some fun with loops and algebra [JN Phys.Rept., 477 (2009)]

$$a_\ell^{\mathrm{HLbL};P} = \frac{-2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1-t^2}\, Q_1^3 Q_2^3$$
$$\times \left[\frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_3^2) F_{P\gamma^*\gamma}(Q_2^2, 0) I_1(Q_1, Q_2, t)}{Q_2^2 + m_P^2} + \frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) F_{P\gamma^*\gamma}(Q_3^2, 0) I_2(Q_1, Q_2, t)}{Q_3^2 + m_P^2} \right]$$

HLbL: the pseudoscalar-pole contribution

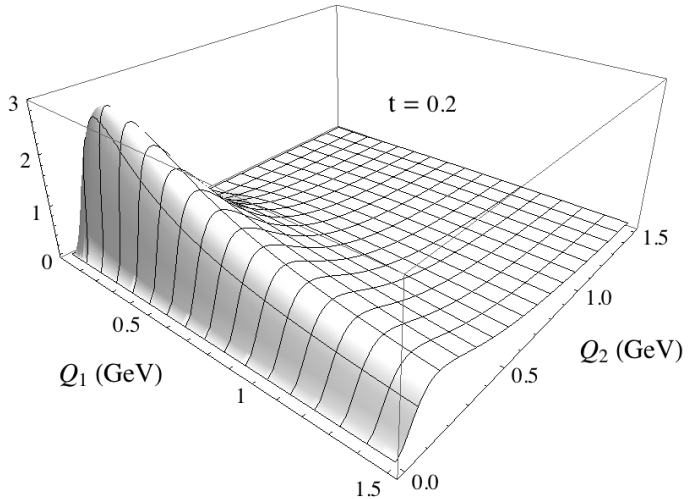
- Plug previous result into HLbL $(g-2)$ contribution

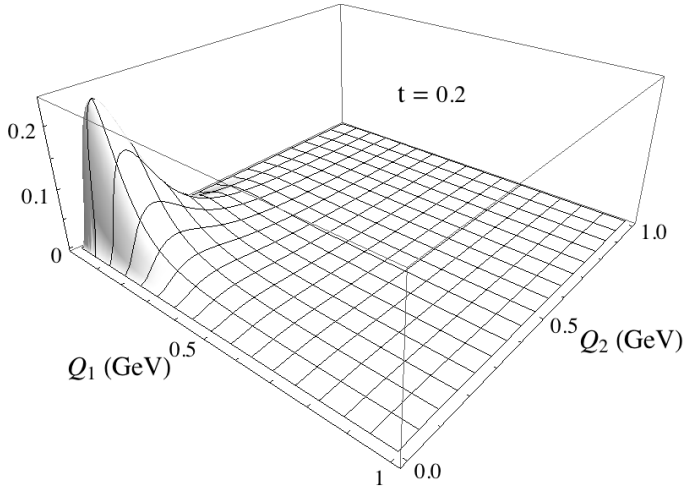


- After some fun with loops and algebra [JN Phys.Rept., 477 (2009)]

$$a_\ell^{\mathrm{HLbL};P} = \frac{-2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1-t^2}\, Q_1^3 Q_2^3$$
$$\times \left[\frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_3^2) F_{P\gamma^*\gamma}(Q_2^2, 0) I_1(Q_1, Q_2, t)}{Q_2^2 + m_P^2} + \frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) F_{P\gamma^*\gamma}(Q_3^2, 0) I_2(Q_1, Q_2, t)}{Q_3^2 + m_P^2} \right]$$

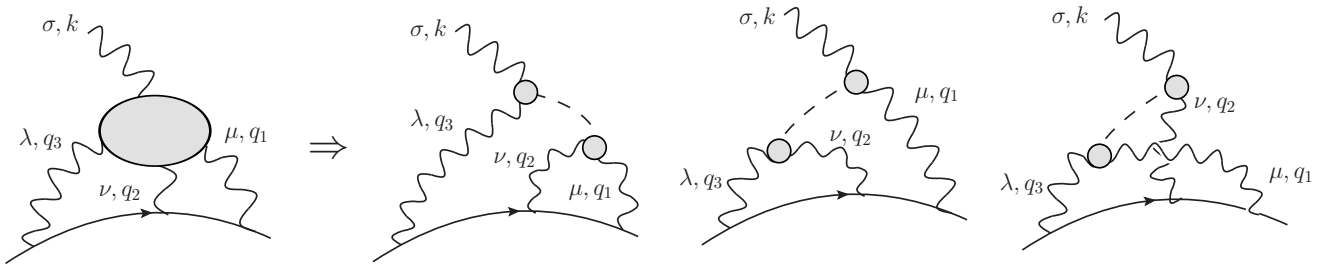
- Without the transition form factors $F_{\pi\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ integrands look like





HLbL: the pseudoscalar-pole contribution

- Plug previous result into HLbL $(g-2)$ contribution



- After some fun with loops and algebra [JN Phys.Rept., 477 (2009)]

$$a_\ell^{\mathrm{HLbL};P} = \frac{-2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1-t^2}\, Q_1^3 Q_2^3$$
$$\times \left[\frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_3^2) F_{P\gamma^*\gamma}(Q_2^2, 0) I_1(Q_1, Q_2, t)}{Q_2^2 + m_P^2} + \frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) F_{P\gamma^*\gamma}(Q_3^2, 0) I_2(Q_1, Q_2, t)}{Q_3^2 + m_P^2} \right]$$

- Without the transition form factors $F_{\pi\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ integrands look like

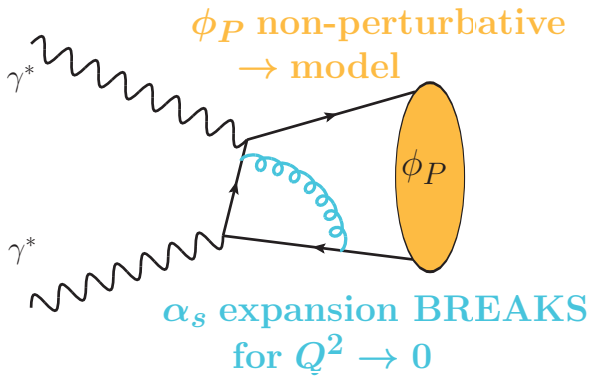
We reduced everything to an integral involving physical input
Description for space-like $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$, specially below 2 GeV
Incorporate high-energy description (otherwise $I_1(Q_1, Q_2, t)$ diverges)

Section 2

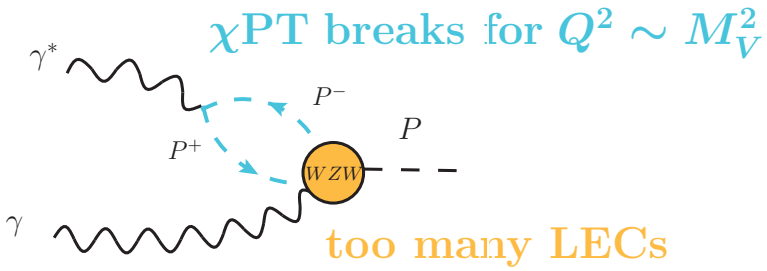
A transition form factor description for the HLbL

Describing the TFF I: First principles

—High Energies: pQCD



—Low Energies: χPT



$$\underline{Q^2 \rightarrow \infty}$$

$$\lim_{Q^2\rightarrow\infty} F_{\pi\gamma\gamma^*}(0, Q^2) = \frac{2F_\pi}{Q^2}$$

$$\lim_{Q^2\rightarrow\infty} F_{\pi\gamma^*\gamma^*}(Q^2, Q^2) = \frac{2F_\pi}{3Q^2}$$

Guarantee convergence!

?

$$\underline{Q^2 \rightarrow 0}$$

$$F_{\pi\gamma\gamma}(0,0) = (4\pi^2 F_\pi)^{-1}$$

Describing the TFF II: Model approaches

—Lagrangian-based

Nambu Jona Lasinio • Hidden Local Symmetry • Resonance chiral th. • ...

- Nice overall picture, but not precision
- Ok, they are models (not full QCD), problem is uncertainty estimate

Describing the TFF II: Model approaches

—Lagrangian-based

Nambu Jona Lasinio • Hidden Local Symmetry • Resonance chiral th. • ...

- Nice overall picture, but not precision
- Ok, they are models (not full QCD), problem is uncertainty estimate

—Phenomenological Data-based

Large-N_c-based + Resonance saturation + Data-fitting

- Experiment is full QCD $\longrightarrow$ Fit it *with a model*
- Data not always available where required $\rightarrow$ extrapolation reliability?
- How to systematically improve to arbitrary *known* precision?

Describing the TFF II: Model approaches

—Lagrangian-based

Nambu Jona Lasinio • Hidden Local Symmetry • Resonance chiral th. • ...

- Nice overall picture, but not precision
- Ok, they are models (not full QCD), problem is uncertainty estimate

—Phenomenological Data-based

Large-N_c-based + Resonance saturation + Data-fitting

- Experiment is full QCD $\longrightarrow$ Fit it *with a model*
- Data not always available where required $\rightarrow$ extrapolation reliability?
- How to systematically improve to arbitrary *known* precision?

—Data-based (not a model)

Dispersive reconstruction

- Data based, in principle full QCD
- In practice most of QCD contributions $\Rightarrow$ Not full Q^2 reconstruction

Objectives and strategies

—What do we need?

A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

Objectives and strategies

—What do we need?

A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

—What is the philosophy?

Toolkit allowing full use of data & QCD constraints on form factors

Objectives and strategies

—What do we need?

A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

—What is the philosophy?

Toolkit allowing full use of data & QCD constraints on form factors

—How to implement for single-virtual case?

We propose to use Padé Approximants

Objectives and strategies

—What do we need?

A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

—What is the philosophy?

Toolkit allowing full use of data & QCD constraints on form factors

—How to implement for single-virtual case?

We propose to use Padé Approximants

—How to implement the double virtual Form Factor?

Generalize our approach to bivariate functions: Canterbury Approximants

Objectives and strategies

—What do we need?
A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

—What is the philosophy?
Toolkit allowing full use of data & QCD constraints on form factors

—How to implement for single-virtual case?
We propose to use Padé Approximants

—How to implement the double virtual Form Factor?
Generalize our approach to bivariate functions: Canterbury Approximants

Padé Approximants: Introduction to the method

Given a function with known series expansion

$$F_{P\gamma\gamma^*}(Q^2) = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + ...) \quad \text{i.e. } \chi\text{PT}$$

Its Padé approximant is defined as

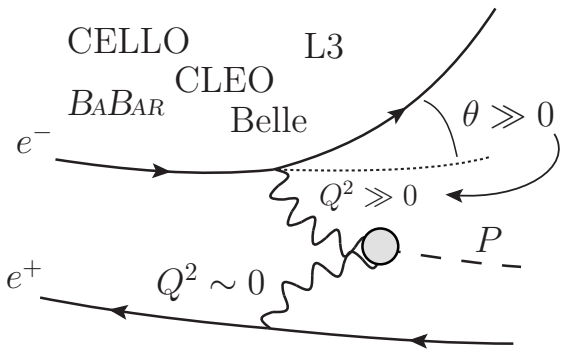
$$P_M^N(Q^2) = \frac{T_N(Q^2)}{R_M(Q^2)} = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + ... + \mathcal{O}(Q^2)^{N+M+1})$$

Convergence th. $\Rightarrow$ Model-independency
Increase $\{N, M\} \Rightarrow$ Systematic error estimation

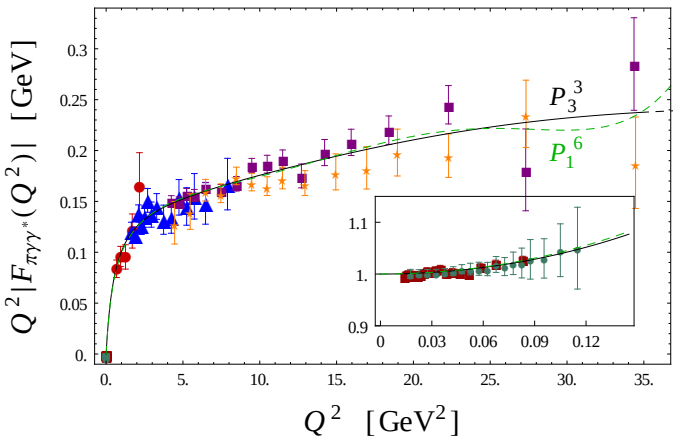
$$P_1^0 = \frac{F_{P\gamma\gamma^*}(0)}{1 - b_P Q^2} = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + \mathcal{O}(Q^4)) \ \cancel{\text{VMD}} \rightarrow \chi\text{PT/DR} + \text{pQCD}$$

Correct low (& high) energy implementation!

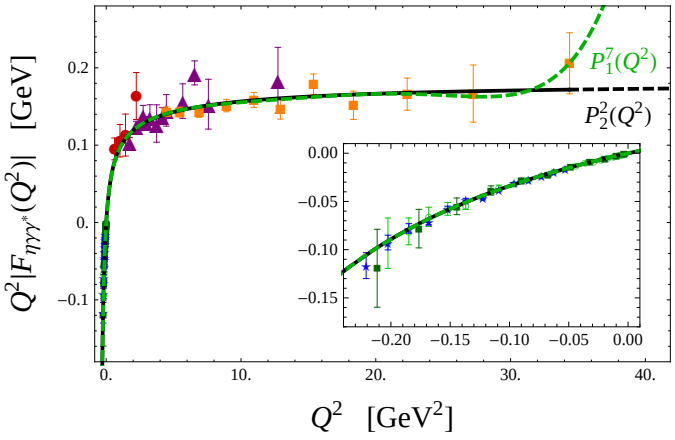
Padé Approximants: Results



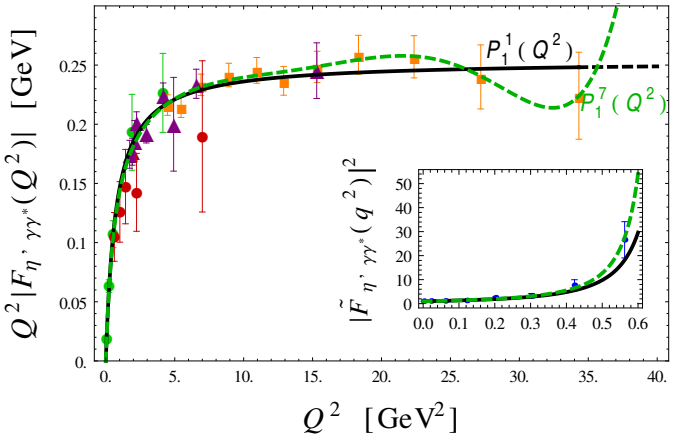
π



η

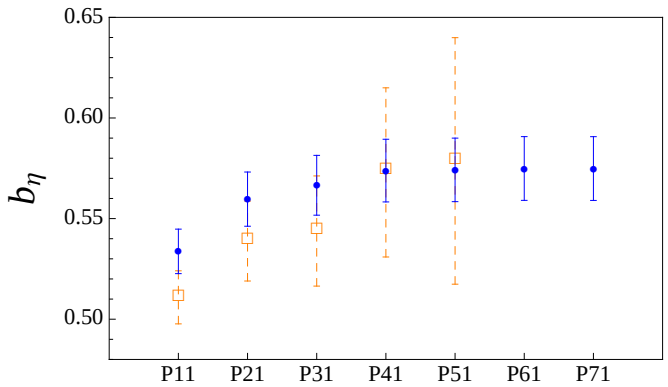


η'

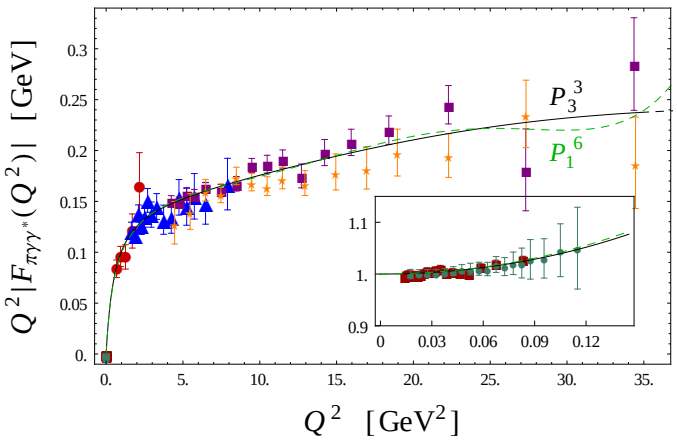


P. Masjuan '12; R. Escribano, P. Masjuan, P. S (& S. Gonzalez) '14 '15 (&16)

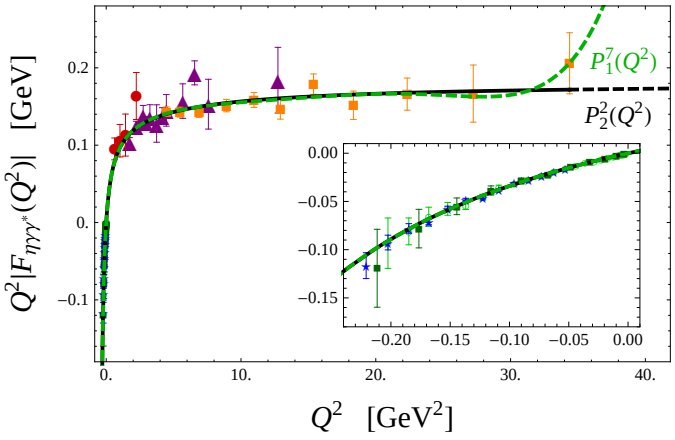
Padé Approximants: Results



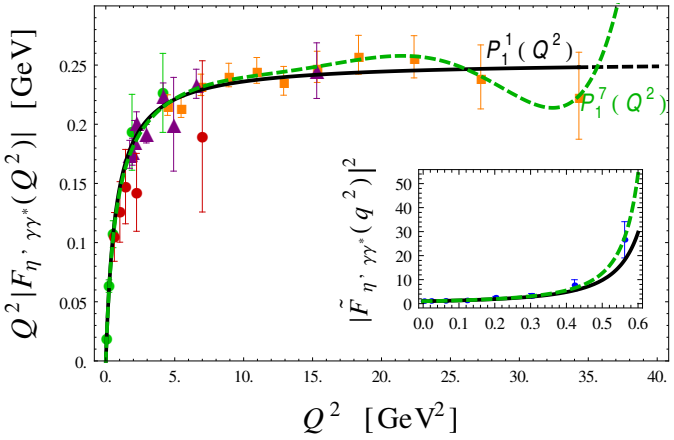
π



η



η'



P. Masjuan '12; R. Escribano, P. Masjuan, P. S (& S. Gonzalez) '14 '15 (&16)

Objectives and strategies

—What do we need?
A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

—What is the philosophy?
Toolkit allowing full use of data & QCD constraints on form factors

—How to implement for single-virtual case?
We proppose to use Padé Approximants

—How to implement the double-virtual Form Factor?
Generalize our approach to bivariate functions: Canterbury Approximants

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

For a symmetric function with Taylor expansion

$$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + c_{1,0}(Q_1^2 + Q_2^2) + c_{2,0}(Q_1^4 + Q_2^4) + c_{1,1}Q_1^2Q_2^2 + ...)$$

Its Canterbury appproximant is defined as

$$C_M^N(Q_1^2, Q_2^2) = \frac{T_N(Q_1^2, Q_2^2)}{Q_M(Q_1^2, Q_2^2)} = \frac{\sum_{i,j}^N a_{i,j} Q_1^{2i} Q_2^{2j}}{\sum_{k,l}^M b_{k,l} Q_1^{2k} Q_2^{2l}}$$

Fulfilling the conditions that

$$\sum_{i,j}^{M} b_{i,j} Q_1^{2i} Q_2^{2j} \sum_{\alpha,\beta}^{\infty} c_{\alpha,\beta} Q_1^{2\alpha} Q_2^{2\beta} - \sum_{k,l}^{N} a_{k,l} Q_1^{2k} Q_2^{2l} = \sum_{\gamma,\delta}^{\infty} d_{\gamma,\delta} Q_1^{2\gamma} Q_2^{2\delta},$$

$$\begin{aligned}
d_{\gamma,\delta} = &\quad 0 \quad && 0 \leq \gamma+\delta \leq M+N \\
d_{\gamma,\delta} = &\quad 0 \quad && 0 \leq \gamma \leq \max(M,N), \\
& && 0 \leq \delta \leq \max(M,N) \\
d_{\gamma,\delta} = &\quad 0 \quad && 1 \leq \gamma \leq \min(M,N), \\
& && \delta = M+N+1-\gamma.
\end{aligned}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

For a symmetric function with Taylor expansion

$$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + c_{1,0}(Q_1^2 + Q_2^2) + c_{2,0}(Q_1^4 + Q_2^4) + c_{1,1}Q_1^2Q_2^2 + ...)$$

Its Canterbury appproximant is defined as

$$C_M^N(Q_1^2, Q_2^2) = \frac{T_N(Q_1^2, Q_2^2)}{Q_M(Q_1^2, Q_2^2)} = \frac{\sum_{i,j}^N a_{i,j} Q_1^{2i} Q_2^{2j}}{\sum_{k,l}^M b_{k,l} Q_1^{2k} Q_2^{2l}}$$

As an example

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2Q_2^2} = F_{P\gamma\gamma}(0,0)(1 + b_P(Q_1^2 + Q_2^2) + a_{P;1,1}Q_1^2Q_2^2 + ...)$$

- Again, similar conclusions/procedure to the previous case
- However, lack of data (desirable; in the meantime, surmountable)

Seeing is believing: toy models and systematics

—a_μ^π: Regge Model—

$$F^{\rm Regge}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	55.2	59.7	60.4	60.6
OPE	65.7	60.8	60.7	60.7
Fit^{OPE}	66.3	62.7	61.1	60.8
Exact		60.7		

—a_μ^π: Logarithmic Model—

$$F^{\rm log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	56.7	64.4	66.1	66.8
OPE	65.7	67.3	67.5	67.6
Fit^{OPE}	79.6	71.9	69.3	68.4
Exact		67.6		

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Seeing is believing: toy models and systematics

—a_μ^π: Regge Model—

$$F^{\rm Regge}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

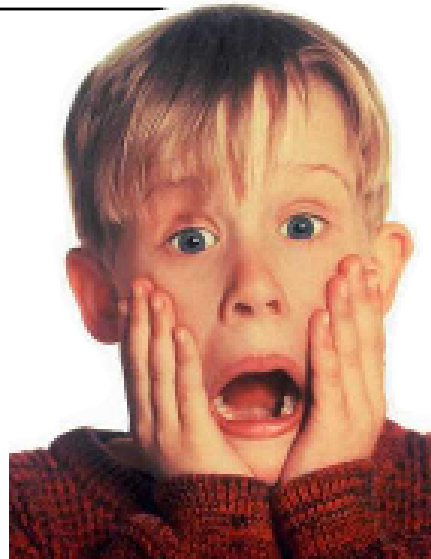
	C_1^0	C_2^1	C_3^2	C_4^3
LE	55.2	59.7	60.4	60.6
OPE	65.7	60.8	60.7	60.7
Fit^{OPE}	66.3	62.7	61.1	60.8
Exact		60.7		

—a_μ^π: Logarithmic Model—

$$F^{\rm log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	56.7	64.4	66.1	66.8
OPE	65.7	67.3	67.5	67.6
Fit^{OPE}	79.6	71.9	69.3	68.4
Exact		67.6		

- The convergence result is excellent!
- The OPE choice seems the best $\rightarrow$ high energy matters
- Still, low energies provide a good performance
- Error $\sim$ difference among elements $\rightarrow$ Systematics!



P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Section 3

Updated pseudoscalar pole contribution

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

$$C_1^0(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + b_P(Q_1^2 + Q_2^2) + a_{P;1,1}Q_1^2 Q_2^2 + ...)$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1. Reproduce original series expansion $\Rightarrow$ low energies
2. Reduce to Padé Approximants

$$C_1^0(Q^2, 0) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P Q^2} = P_1^0(Q^2) \Rightarrow F_{P\gamma\gamma}(0,0) \ \ \& \ \ b_P \ \ \text{determined}$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1. Reproduce original series expansion $\Rightarrow$ low energies
2. Reduce to Padé Approximants (already determined)
3. Systematically implement double virtuality: $a_{P;1,1}$ (LE) (Exp. unknown)

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1. Reproduce original series expansion $\Rightarrow$ low energies
2. Reduce to Padé Approximants (already determined)
3. Systematically implement double virtuality: $a_{P;1,1}$ (LE) (Exp. unknown)

   Constrain OPE $\Rightarrow$ better convergence

$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}; \quad (a_{P;1,1} \equiv 2b_P^2)\ OPE$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1. Reproduce original series expansion $\Rightarrow$ low energies
2. Reduce to Padé Approximants (already determined)
3. Systematically implement double virtuality: $a_{P;1,1}$ (LE) (Exp. unknown)

   Constrain OPE $\Rightarrow$ better convergence

$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}; \quad (a_{P;1,1} \equiv 2b_P^2)\ OPE$$

Simplest rational approximation to $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ Satisfies both low & high energies

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) + \beta_{2,2} Q_1^4 Q_2^4}$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) + \beta_{2,2} Q_1^4 Q_2^4}$$

—Reconstruction

1. Reduce to Padé Approximants

$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \rightarrow$ from PAs

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) + \beta_{2,2} Q_1^4 Q_2^4}$$

—Reconstruction

1. Reduce to Padé Approximants

$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \rightarrow$ from PAs

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1}Q_1^2Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1}Q_1^2Q_2^2 + \beta_{2,1}Q_1^2Q_2^2(Q_1^2 + Q_2^2) + \beta_{2,2}Q_1^4Q_2^4}$$

—Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)

$$F_{\pi\gamma^*\gamma^*} = \frac{1}{3Q^2}(2F_\pi)\left(1 - \frac{8}{9}\frac{\delta^2}{Q^2} + \mathcal{O}(\alpha_s(Q^2))\right) \Rightarrow \beta_{2,2} = 0, \alpha_{1,1}, \beta_{2,1}$$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2)}$$

—Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)
3. Reproduce the low energies ($a_{P;1,1}$)

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2)}$$

—Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)
3. Reproduce the low energies ($a_{P;1,1}$)

   Be generous: all configurations with no poles $\Rightarrow a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2)}$$

—Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)
3. Reproduce the low energies ($a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$)

Reconstructing $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2)}$$

—Reconstruction

1. Reduce to Padé Approximants
2. Reproduce the OPE behavior (high energies)
3. Reproduce the low energies $(a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max})$

Low- and high energies implemented
Full use of data and theory constraints
Double-virtual data for $a_{P;1,1}$ (and δ^2) desirable
Systematization up to required precision $(C_3^2(Q_1^2, Q_2^2) \to C_{N+1}^N(Q_1^2, Q_2^2))$

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\mathrm{min}}$	$a_{P;1,1}^{\mathrm{max}}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t$	$63.0(1.1)_L(0.5)_\delta[1.2]_t$
η	$16.3(0.8)_L(0)_\delta[0.8]_t$	$16.2(0.8)_L(0.6)_\delta[1.0]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
Total	$95.1[1.7]_t$	$93.5[1.7]_t$

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\mathrm{min}}$	$a_{P;1,1}^{\mathrm{max}}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t$	$16.2(0.8)_L(0.6)_\delta[1.0]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
Total	$95.1[1.7]_t$	$93.5[1.7]_t$

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\mathrm{min}}$	$a_{P;1,1}^{\mathrm{max}}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.1[1.7]_t\{3.3\}_{\mathrm{sys}}$	$93.5[1.7]_t\{4.9\}_{\mathrm{sys}}$

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.1[1.7]_t\{3.3\}_{\mathrm{sys}}$	$93.5[1.7]_t\{4.9\}_{\mathrm{sys}}$

—Final Result (combining errors just for clarity)

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	Fact ($a_{P;1,1} = b_P^2$)
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$54.3(1.5)_F(2.2)_{b_\pi}[2.5]_t$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$13.0(0.4)_F(0.2)_{b_\eta}[0.5]_t$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$
Total	$98.4[2.9]_t$	$79.3[2.6]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\mathrm{sys}}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\mathrm{sys}}$
η	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.1[1.7]_t\{3.3\}_{\mathrm{sys}}$	$93.5[1.7]_t\{4.9\}_{\mathrm{sys}}$

—Final Result (combining errors just for clarity)

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

What has been achieved?

P. Masjuan & P. Sanchez Phys.Rev. D95, 054026 (2017)

Final Updated Result

$\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

- Updated value —incl. systematics— meets future exp. precision
- η and η' deserve an appropriate description!
- Full use of experimental dataset

Knecht & Nyffeler: Phys.Rev., D65, 073034 (2002); **RχPT**: Phys.Rev., D89, 073016 (2014)

Final Updated Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

- Updated value —incl. systematics— meets future exp. precision
- η and η' deserve an appropriate description!
- Full use of experimental dataset

Previous Knecht & Nyffeler Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (58(10) + 13(1) + 12(1)) \times 10^{-11} = 83(12) \times 10^{-11}$$

- Intended for $\delta a_\mu = 63 \times 10^{-11}$; no systematics ($N_c \to 30\%$?)
- η and η' oversimplified description (~~OPE~~$\to \delta a_\mu \sim -5 \times 10^{-11}$)
- Old data-base

Knecht & Nyffeler: Phys.Rev., D65, 073034 (2002); **RχPT**: Phys.Rev., D89, 073016 (2014)

Final Updated Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

- Updated value —incl. systematics— meets future exp. precision
- η and η' deserve an appropriate description!
- Full use of experimental dataset

Recent RχPT Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (57.5(0.6) + 14.4(2.6) + 10.8(0.9)) \times 10^{-11} = 82.7(2.8) \times 10^{-11}$$

- Estimation of NLO corrections (syst. error) missing ($N_c \to 30\%$?)
- No data used for the η, η', but mixing+$SU(3)$-symmetry
- Cannot reproduce all HE constraints

Knecht & Nyffeler: Phys.Rev., D65, 073034 (2002); **RχPT**: Phys.Rev., D89, 073016 (2014)

Final Updated Result

$\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

$a_\mu^{\rm HLbL;P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	$a_{P;1,1}^{\rm min}$	$a_{P;1,1}^{\rm max}$
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\rm sys}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\rm sys}$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\rm sys}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\rm sys}$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\rm sys}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\rm sys}$
Total	$98.4[2.9]_t$	$95.1[1.7]_t\{3.3\}_{\rm sys}$	$93.5[1.7]_t\{4.9\}_{\rm sys}$

Future improvements expected

- Main stat. from π^0 $\quad b_\pi \simeq F_{\pi\gamma\gamma} \simeq \mathrm{DV} \gg \mathrm{BL}$
  BES III & GlueX & KLOE-2 — KLOE-2 & PrimEx — BES III — BES III & Belle II
- Then η, η' important $\quad F_{P\gamma\gamma} \gtrsim b_P \simeq \mathrm{DV} \simeq \mathrm{BL}$
  GlueX — BES III & A2 — — Belle II

Knecht & Nyffeler: Phys.Rev., D65, 073034 (2002); RχPT: Phys.Rev., D89, 073016 (2014)

Final Updated Result

$\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.6(2.7) + 16.3(1.3) + 14.5(1.8)) \times 10^{-11} = 94.3(5.3) \times 10^{-11}$$

$a_\mu^{\rm HLbL;P} \times 10^{11}$	OPE ($a_{P;1,1} = 2b_P^2$)	$a_{P;1,1}^{\rm min}$	$a_{P;1,1}^{\rm max}$
π^0	$65.3(1.4)_F(2.4)_{b_\pi}[2.8]_t$	$64.1(1.3)_L(0)_\delta[1.3]_t\{1.2\}_{\rm sys}$	$63.0(1.1)_L(0.5)_\delta[1.2]_t\{2.3\}_{\rm sys}$
η	$17.1(0.6)_F(0.2)_{b_\eta}[0.6]_t$	$16.3(0.8)_L(0)_\delta[0.8]_t\{0.8\}_{\rm sys}$	$16.2(0.8)_L(0.6)_\delta[1.0]_t\{0.9\}_{\rm sys}$
η'	$16.0(0.5)_F(0.3)_{b_{\eta'}}[0.6]_t$	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\rm sys}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\rm sys}$
Total	$98.4[2.9]_t$	$95.1[1.7]_t\{3.3\}_{\rm sys}$	$93.5[1.7]_t\{4.9\}_{\rm sys}$

Future improvements expected

- Main stat. from π^0 $\quad b_\pi \simeq F_{\pi\gamma\gamma} \simeq \mathrm{DV} \gg \mathrm{BL}$
  BES III & GlueX & KLOE-2 — KLOE-2 & PrimEx — BES III — BES III & Belle II
- Then η, η' important $\quad F_{P\gamma\gamma} \gtrsim b_P \simeq \mathrm{DV} \simeq \mathrm{BL}$
  GlueX — BES III & A2 — — Belle II

Interesting paths

- $P \to \bar{\ell}\ell$ & $P \to \bar{\ell}\ell\bar{\ell}'\ell'$
  NA62 for π — LHCb & Redtop for η, η' ?

Knecht & Nyffeler: Phys.Rev., D65, 073034 (2002); RχPT: Phys.Rev., D89, 073016 (2014)

Summary & Outlook

- Systematic data-driven TFF description [Canterbury approximants]
- Full use of SL and low-energy TL data + theory constraints
- New value $a_\mu^{HLbL;\pi,\eta,\eta'} = 94.3(5.3) \times 10^{-11}$ including systematics
- Error meets future experiments $\delta a_\mu \sim 16 \times 10^{-11}$ requirements
- Forthcoming improvements: BESIII, KLOE2, GlueX, PrimEx, BelleII
- $P \to \bar{\ell}\ell, \bar{\ell}\ell\bar{\ell}'\ell'$ of help too (the only η, η' double-virtual probes?)

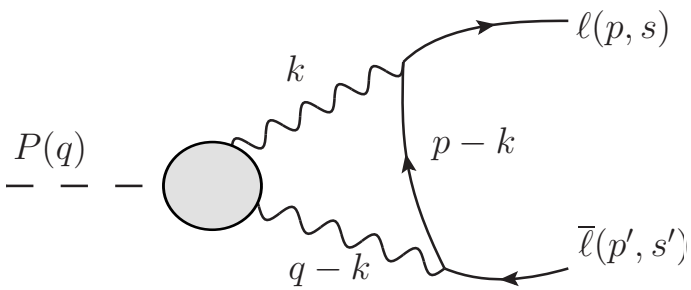
Section 4

Backup

Section 5

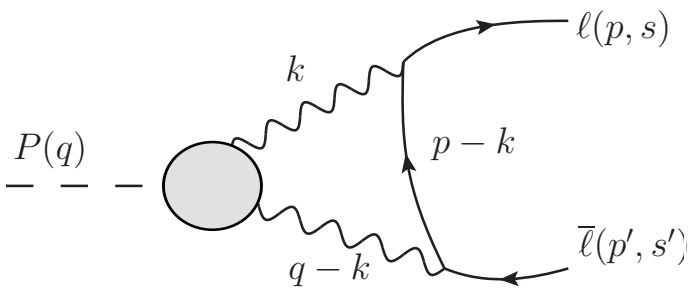
$P \to \bar{\ell}\ell$ decays: further information and new physics

$P \to \bar{\ell}\ell$ decays: a brief introduction



- Probes the (double virtual) TFF
- Clean check assuming no NP
- Alternatively, deviation $\to$ NP

$P \to \bar{\ell}\ell$ decays: a brief introduction

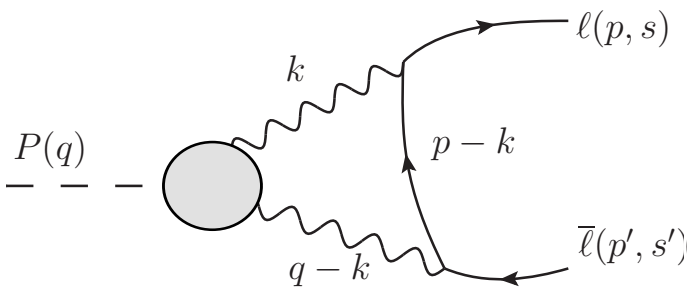


$$\frac{\mathrm{BR}(P \to \bar{\ell}\ell)}{\mathrm{BR}(P \to \gamma\gamma)} = 2\left(\frac{\alpha m_\ell}{m_P}\right)^2 |\mathcal{A}(m_P^2)|^2$$

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2}\int d^4k \frac{\left(k^2q^2 - (k\cdot q)^2\right)\tilde{F}_{P\gamma^*\gamma^*}(k^2,(q-k)^2)}{k^2(q-k)^2\left((p-k)^2 - m_\ell^2\right)}.$$

- The process is low-energy dominated
- UV divergent for a constant TFF

$P \to \bar{\ell}\ell$ decays: a brief introduction



$$\frac{\mathrm{BR}(P \to \bar{\ell}\ell)}{\mathrm{BR}(P \to \gamma\gamma)} = 2\left(\frac{\alpha m_\ell}{m_P}\right)^2 |\mathcal{A}(m_P^2)|^2$$

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2}\int d^4k \frac{\left(k^2q^2 - (k\cdot q)^2\right)\tilde{F}_{P\gamma^*\gamma^*}(k^2,(q-k)^2)}{k^2(q-k)^2\left((p-k)^2 - m_\ell^2\right)}.$$

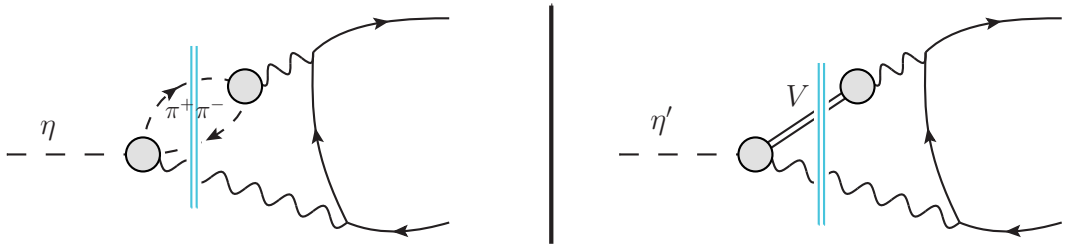
Ideal case for our approach

Previous comments apply to this case, but novelties ...

- $-m_P^2 \leq Q^2 \leq \infty$: care with η and η'
- Frequent employed approximations do not apply to the η, η'

Additional systematics

We must deal with a new feature: hadronic thresholds

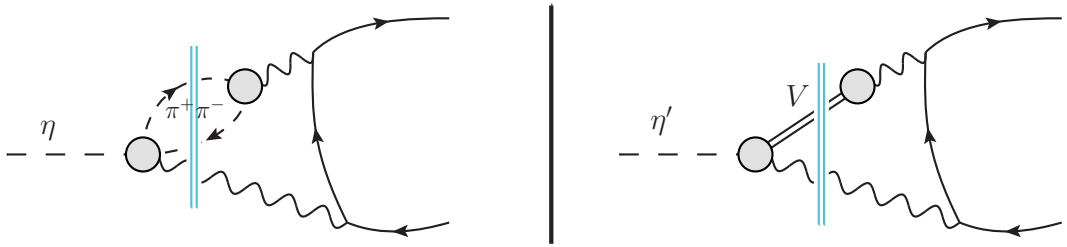


No previous studies about threshold effects on $P \to \bar{\ell}\ell$ decays

(1) Can our approach deal with it?
(2) Associated systematic error?

Additional systematics

We must deal with a new feature: hadronic thresholds



No previous studies about threshold effects on $P \to \bar{\ell}\ell$ decays

(1) Can our approach deal with it?
(2) Associated systematic error?

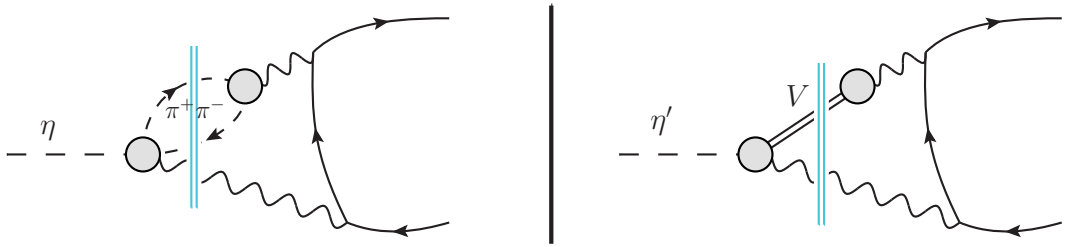
Factorized ansatz $\tilde{F}_{P\gamma^*\gamma}(q_1^2, q_2^2) = \tilde{F}_{P\gamma^*\gamma}(q_1^2) \times \tilde{F}_{P\gamma^*\gamma}(q_2^2)$

$$\tilde{F}_{P\gamma^*\gamma}(s) = c_{P\rho} G_\rho(s) + c_{P\omega} G_\omega(s) + c_{P\phi} G_\phi(s)$$

With $G_V(s)$ fulfilling appropriate analytic and unitary constraints

Additional systematics

We must deal with a new feature: hadronic thresholds



No previous studies about threshold effects on $P \to \bar{\ell}\ell$ decays

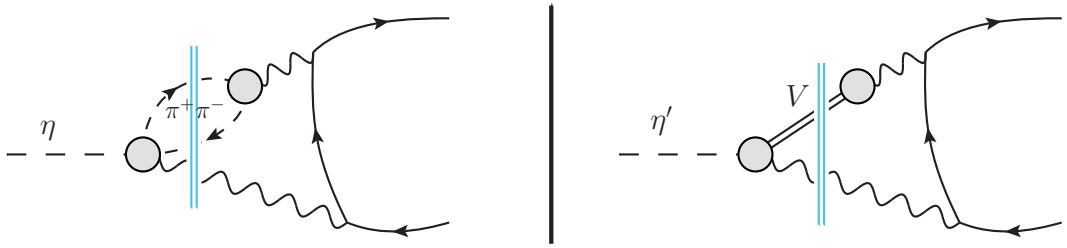
(1) Can our approach deal with it?
(2) Associated systematic error?

$$G_\rho(s) = \frac{M_\rho^2}{M_\rho^2 - s + \frac{sM_\rho^2}{96\pi^2 F_\pi^2}\left(\ln\left(\frac{m_\pi^2}{\mu^2}\right) + \frac{8m_\pi^2}{s} - \frac{5}{3} - \sigma(s)^3 \ln\left(\frac{\sigma(s)-1}{\sigma(s)+1}\right)\right)}$$

D. Gomez Dumm, A. Pich, J. Portoles '00

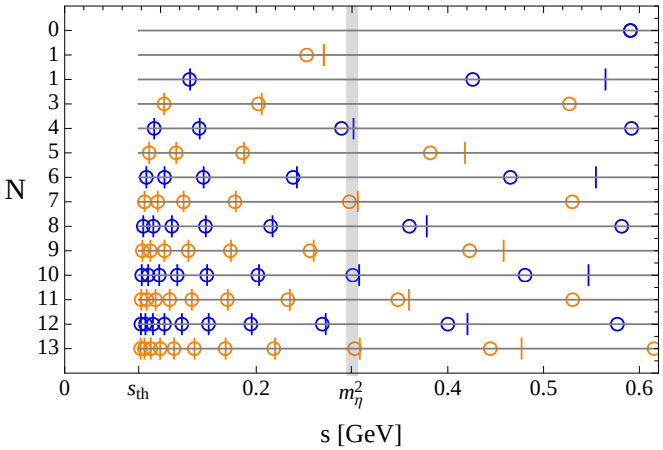
Additional systematics

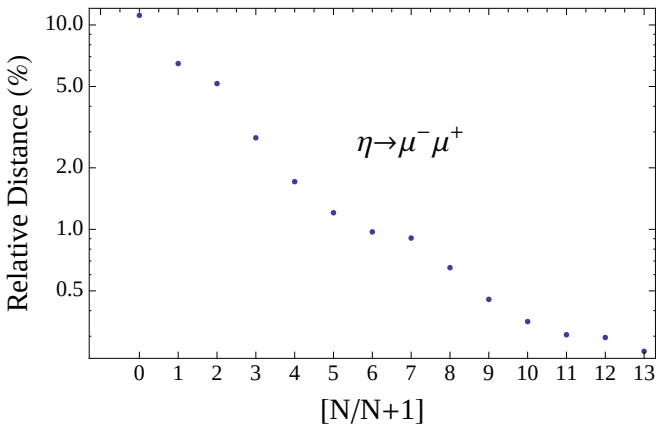
We must deal with a new feature: hadronic thresholds



No previous studies about threshold effects on $P \to \bar{\ell}\ell$ decays

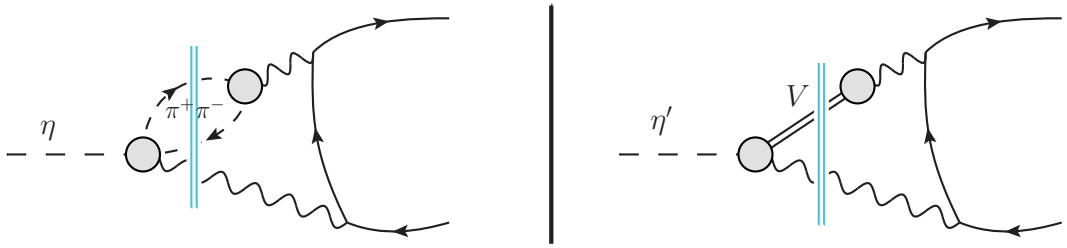
(1) Can our approach deal with it? **It works for the loop integral**
(2) Associated systematic error?





Additional systematics

We must deal with a new feature: hadronic thresholds



No previous studies about threshold effects on $P \to \bar{\ell}\ell$ decays

(1) Can our approach deal with it? **It works for the loop integral**
(2) Associated systematic error? **From realistic unitary model**

$BR(P \to \ell\ell)$	toy model	C_1^0	Error (%)
$(\eta \to ee) \times 10^{-9}$	5.410	5.418	0.16
$(\eta \to \mu\mu) \times 10^{-6}$	4.494	4.527	0.74
$(\eta' \to ee) \times 10^{-10}$	1.705	1.883	9
$(\eta' \to \mu\mu) \times 10^{-7}$	1.195	1.461	18

(3) Final systematic eror: Stat + Syst + **Threshold**

Final Results

BR	Our result (OPE÷Fact)	Approx	Exp
$\pi^0 \to e^+e^- \times 10^8$	$(6.20 \div 6.35)(4)$	$(6.17 \div 6.31)$	$7.48(38)$
$\eta \to e^+e^- \times 10^9$	$(5.31 \div 5.44)(4)$	$(4.58 \div 4.68)$	$\leq 2.3 \times 10^3$
$\eta \to \mu^+\mu^- \times 10^6$	$(4.72 \div 4.52)(5)$	$(5.16 \div 4.88)$	$5.8(8)$
$\eta' \to e^+e^- \times 10^{10}$	$(1.82 \div 1.87)(18)$	$(1.22 \div 1.24)$	≤ 56
$\eta' \to \mu^+\mu^- \times 10^7$	$(1.36 \div 1.49)(26)$	$(1.42 \div 1.41)$	-

P. Masjuan, P. Sanchez, JHEP 1608 (2016) 108

Final Results

BR	Our result (OPE÷Fact)	Approx	Exp
$\pi^0 \to e^+e^- \times 10^8$	$(6.20 \div 6.35)(4)$	$(6.17 \div 6.31)$	$7.48(38)$
$\eta \to e^+e^- \times 10^9$	$(5.31 \div 5.44)(4)$	$(4.58 \div 4.68)$	$\leq 2.3 \times 10^3$
$\eta \to \mu^+\mu^- \times 10^6$	$(4.72 \div 4.52)(5)$	$(5.16 \div 4.88)$	$5.8(8)$
$\eta' \to e^+e^- \times 10^{10}$	$(1.82 \div 1.87)(18)$	$(1.22 \div 1.24)$	≤ 56
$\eta' \to \mu^+\mu^- \times 10^7$	$(1.36 \div 1.49)(26)$	$(1.42 \div 1.41)$	-

- Approximate results $\Rightarrow$ large systematics; similar for LO χPT

P. Masjuan, P. Sanchez, JHEP 1608 (2016) 108

Final Results

BR	Our result (OPE÷Fact)	Approx	Exp
$\pi^0 \to e^+e^- \times 10^8$	$(6.20 \div 6.35)(4)$	$(6.17 \div 6.31)$	$7.48(38)$ 3σ
$\eta \to e^+e^- \times 10^9$	$(5.31 \div 5.44)(4)$	$(4.58 \div 4.68)$	$\leq 2.3 \times 10^3$
$\eta \to \mu^+\mu^- \times 10^6$	$(4.72 \div 4.52)(5)$	$(5.16 \div 4.88)$	$5.8(8)$ 1.3σ
$\eta' \to e^+e^- \times 10^{10}$	$(1.82 \div 1.87)(18)$	$(1.22 \div 1.24)$	≤ 56
$\eta' \to \mu^+\mu^- \times 10^7$	$(1.36 \div 1.49)(26)$	$(1.42 \div 1.41)$	-

- Approximate results $\Rightarrow$ large systematics; similar for LO χPT

P. Masjuan, P. Sanchez, JHEP 1608 (2016) 108

Final Results

BR	Our result (OPE÷Fact)	Approx	Exp
$\pi^0 \to e^+e^- \times 10^8$	$(6.20 \div 6.35)(4)$	$(6.17 \div 6.31)$	$6.87(36)$ 1.8σ
$\eta \to e^+e^- \times 10^9$	$(5.31 \div 5.44)(4)$	$(4.58 \div 4.68)$	$\leq 2.3 \times 10^3$
$\eta \to \mu^+\mu^- \times 10^6$	$(4.72 \div 4.52)(5)$	$(5.16 \div 4.88)$	$5.8(8)$ 1.3σ
$\eta' \to e^+e^- \times 10^{10}$	$(1.82 \div 1.87)(18)$	$(1.22 \div 1.24)$	≤ 56
$\eta' \to \mu^+\mu^- \times 10^7$	$(1.36 \div 1.49)(26)$	$(1.42 \div 1.41)$	-

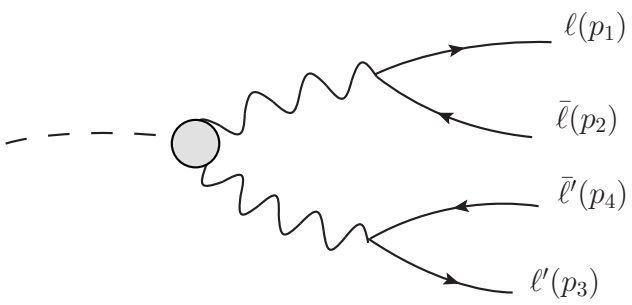
- Approximate results $\Rightarrow$ large systematics; similar for LO χPT
- Recent RC studies imply lower BR for π^0 [T. Husek, K. Kampf, J. Novotny, '14; P. Vasko, J. Novotny '11]

P. Masjuan, P. Sanchez, JHEP 1608 (2016) 108

Section 6

Double Dalitz Decays

Double Dalitz decays



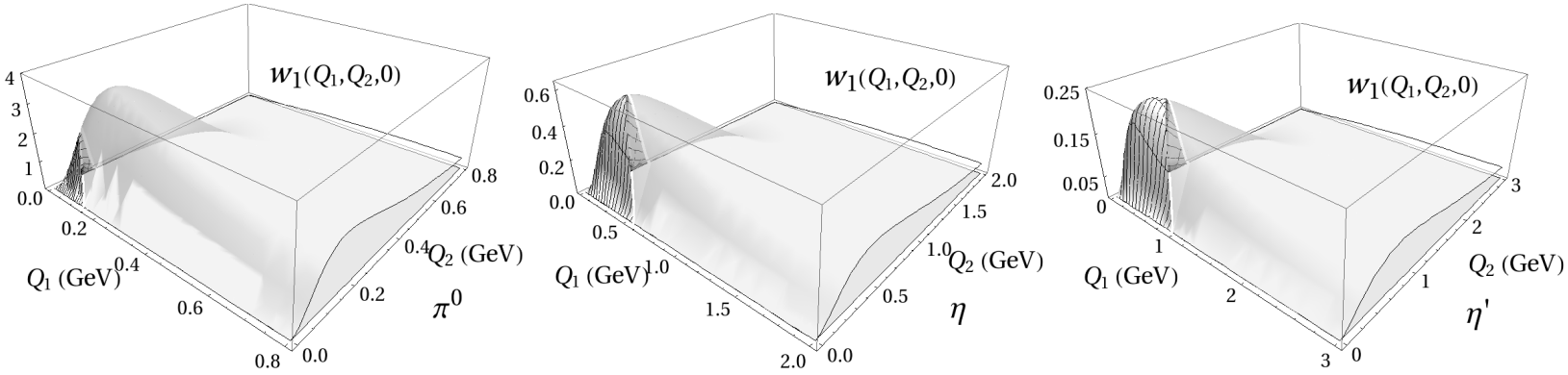
Phase space probed (timelike)

$$2m_\ell \leq q_1 \leq m_P - 2m_{\ell'}$$
$$2m_{\ell'} \leq q_2 \leq m_P - q_1$$

Form factor from Exp/QED

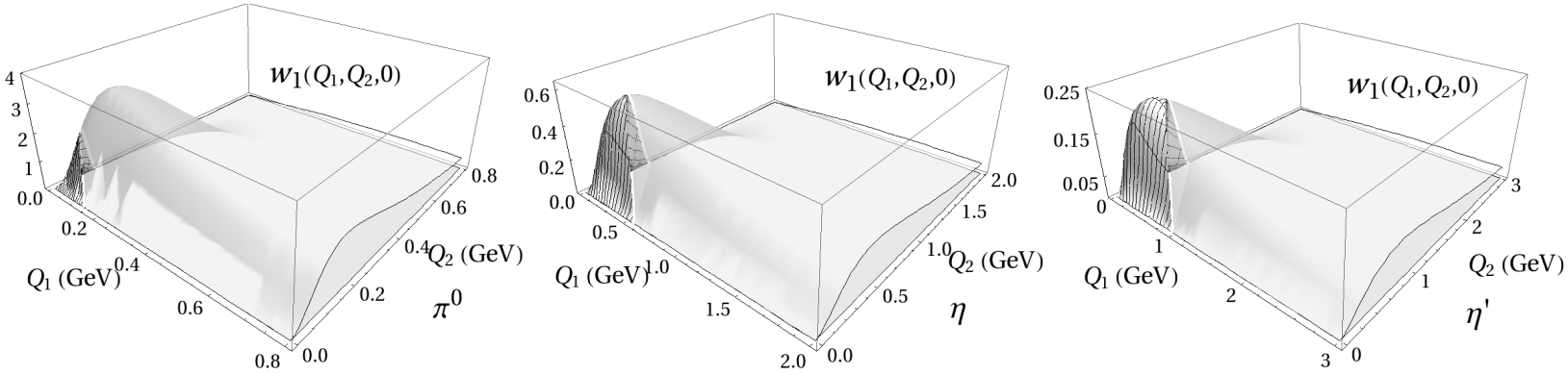
Double Dalitz decays

Naive SL extrapolation



Double Dalitz decays

Naive SL extrapolation



Previous results

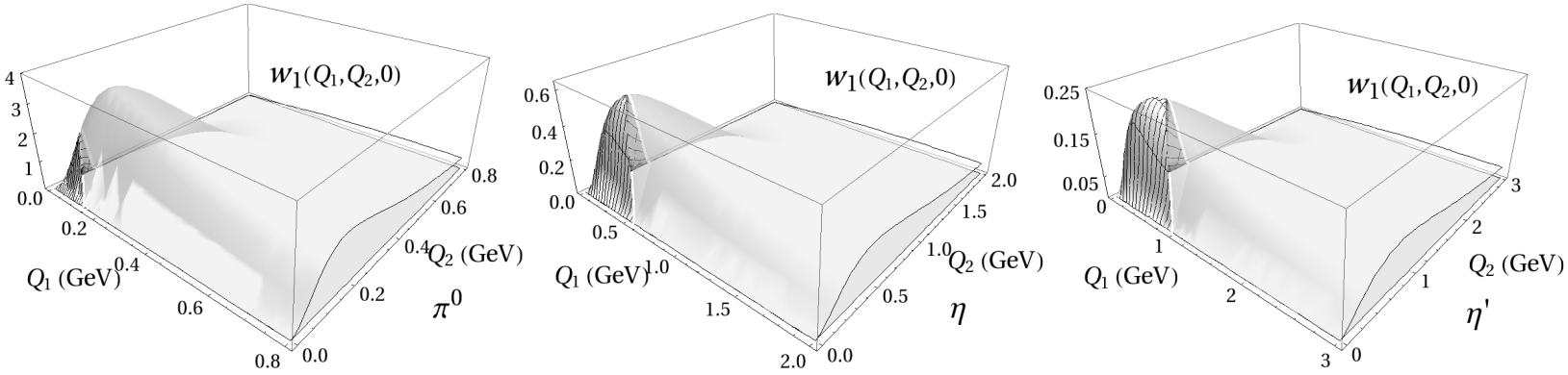
Perrson 0106130 (2001) **VMD**, T. Petri 1010.2378 (2010) **VMD**, C.C. Lih JPhys G38 (2011) **LF pQCD**, C. Terschlüssen *et al* EPJ A49 (2013) ***R*χ*PT***, R. Escribano *et al* 1511.04916 (2015) **PAs**, Weil *et al* 1704.06046 (2017) **DSE**

Essentially the FF effect is ...

$\pi^0 \to 2e^+2e^-$	$\Rightarrow 0.5\%$	$\eta \to 2e^+2e^-$	$\Rightarrow 5\%$	$\eta' \to 2e^+2e^-$	$\Rightarrow 25\%$
		$\eta \to e^+e^-\mu^+\mu^-$	$\Rightarrow 40\%$	$\eta' \to e^+e^-\mu^+\mu^-$	$\Rightarrow 130\%$
		$\eta \to 2\mu^+2\mu^-$	$\Rightarrow 60\%$	$\eta' \to 2\mu^+2\mu^-$	$\Rightarrow 130\%$

Double Dalitz decays

Naive SL extrapolation



Previous results

Perrson 0106130 (2001) **VMD**, T. Petri 1010.2378 (2010) **VMD**, C.C. Lih JPhys G38 (2011) **LF pQCD**, C. Terschlüssen *et al* EPJ A49 (2013) ***R*χ*PT***, R. Escribano *et al* 1511.04916 (2015) **PAs**, Weil *et al* 1704.06046 (2017) **DSE**

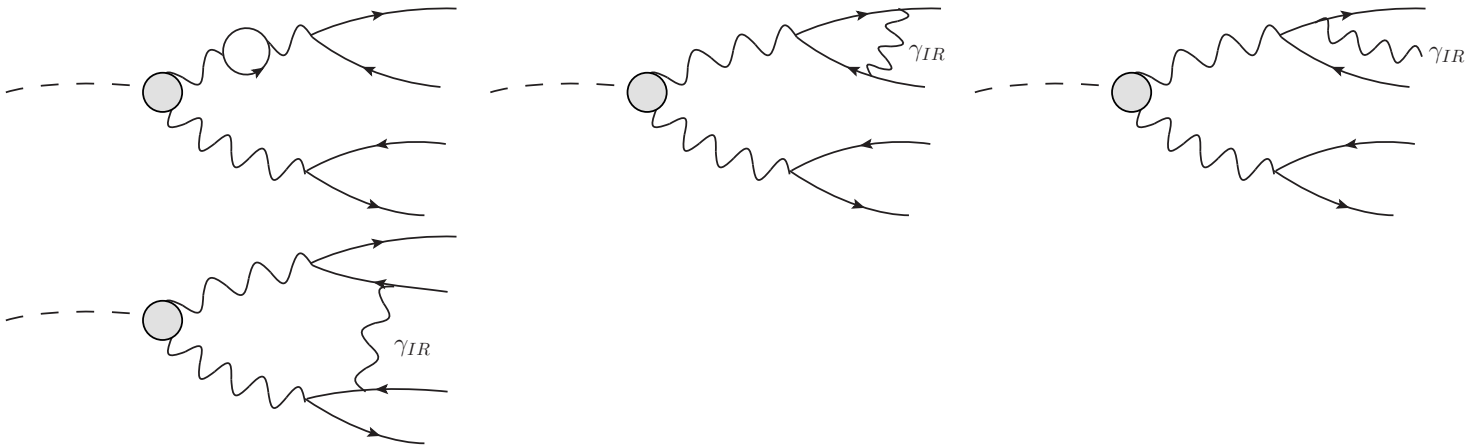
Essentially the FF effect is ... **provided RC under control!**

$\pi^0 \to 2e^+2e^-$	$\Rightarrow 0.5\%$	$\eta \to 2e^+2e^-$	$\Rightarrow 5\%$	$\eta' \to 2e^+2e^-$	$\Rightarrow 25\%$
		$\eta \to e^+e^-\mu^+\mu^-$	$\Rightarrow 40\%$	$\eta' \to e^+e^-\mu^+\mu^-$	$\Rightarrow 130\%$
		$\eta \to 2\mu^+2\mu^-$	$\Rightarrow 60\%$	$\eta' \to 2\mu^+2\mu^-$	$\Rightarrow 130\%$

Double Dalitz decays

Radiative corrections so far

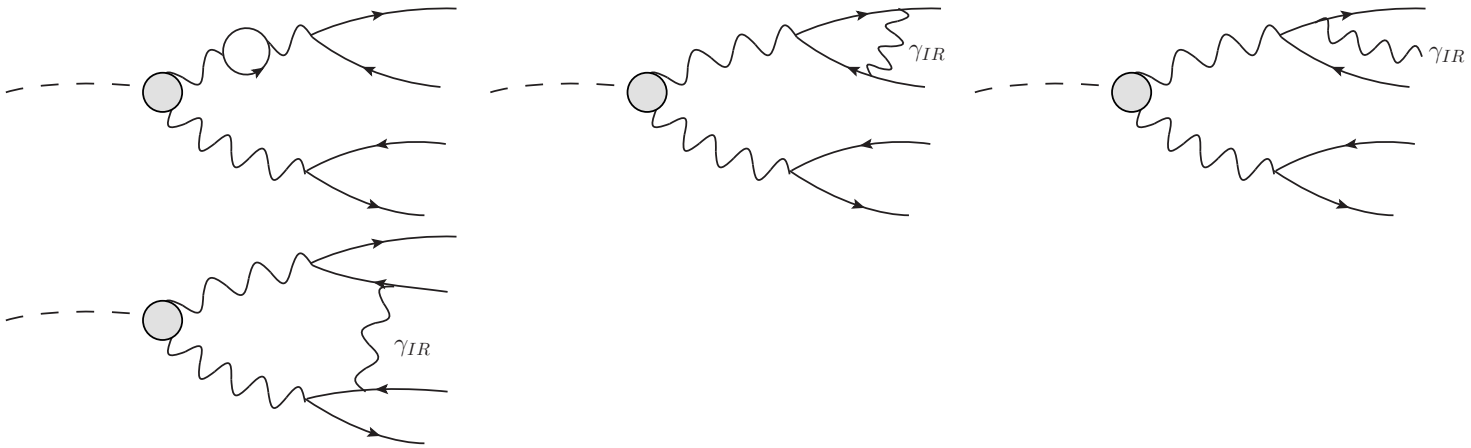
A.R. Barker *et al* PRD 67 033008 (2003), including



Double Dalitz decays

Radiative corrections so far

A.R. Barker *et al* PRD 67 033008 (2003), including



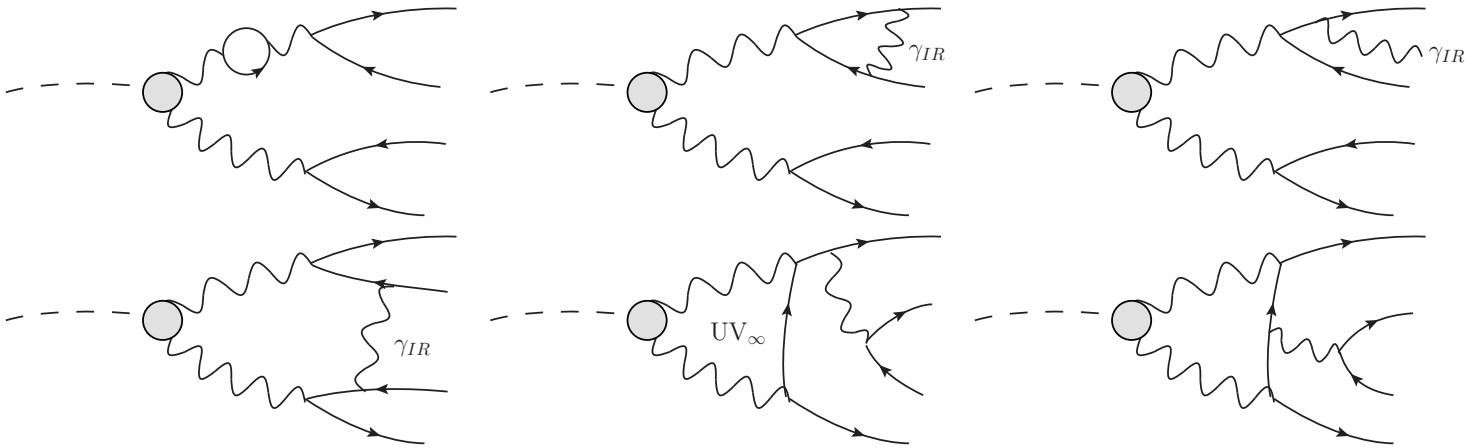
Motivations for new study

- Cross-check always welcomed
- Incorrect F_2 "factorization" in $\bar{\ell}\Gamma^\mu\ell$
- Bremsstrahlung typo/mistake?

Double Dalitz decays

Radiative corrections so far

A.R. Barker *et al* PRD 67 033008 (2003), including



Motivations for new study

- Cross-check always welcomed
- Incorrect F_2 "factorization" in $\bar{\ell}\Gamma^\mu\ell$
- Bremsstrahlung typo/mistake?
- Additional diagrams missing

Double Dalitz decays

Current status (in coll. with K. Kampf & J. Novotny)

- F_2 "factorization" $\Rightarrow$ Negligible (F_2-correction tiny itself)
- Analytical results (all but 5-point) $\Rightarrow$ Ok (but bremsstrahlung)
- Numerical checks very similar
- 3-point NEW contribution ready: small if $\ell = e$; FF important
- 4-point NEW contribution in progress

Future

- Prepare a MC generator (NA62)?
- Interest in crossed process $e^+e^- \to e^+e^-P$?

Double Dalitz decays (PRELIMINARY!)

Current status (in coll. with K. Kampf & J. Novotny)

- F_2 "factorization" $\Rightarrow$ Negligible (F_2-correction tiny itself)
- Analytical results (all but 5-point) $\Rightarrow$ Ok (but BS)
- Bremsstrahlung: typo/mistake? $\Rightarrow$ Numerically ($K_L \to e^+e^-\mu^+\mu^-$)

Premiliminary results for const. FF and $x_{4e}^{cut} = 0.9985$

	$\pi^0 \to 2e^-2e^+$	$K_L \to 2e^-2e^+$	$K_L \to e^-e^+\mu^+\mu^-$	$K_L \to 2\mu^+2\mu^-$
$\overline{\delta}$(Old)	-0.1948	-0.2618	-0.0788	0.0805
δ(New)	$-0.1716(2)$	$-0.2286(2)$	$-0.0767(1)$	0.0705
$\overline{\delta}$(New)?	$-0.2072(3)$	$-0.2964(4)$	$-0.0831(1)$	0.0657

Relation $\overline{\delta}$ and δ; is it $\overline{\delta} \simeq \delta(1+\delta)^{-1}$?? (or at least lower bound)

Additional diagrams

- 3-point $\Rightarrow$ Negligible for $\ell = \ell' = e$ ($\sim$helicity-suppression):
  $\delta_{0.77} \to \{-0.0003, -0.0004, 0.0005, 3 \times 10^{-5}\}$ (but FF-dep.!)
- 4-point $\Rightarrow$ Work in progress

TAKE IT WITH CAUTION: TOUGH REVISION AHEAD!