Pion transition form factor from dispersion relations

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MH, B. Kubis, D. Sakkas, PRD 86 (2012) 116009

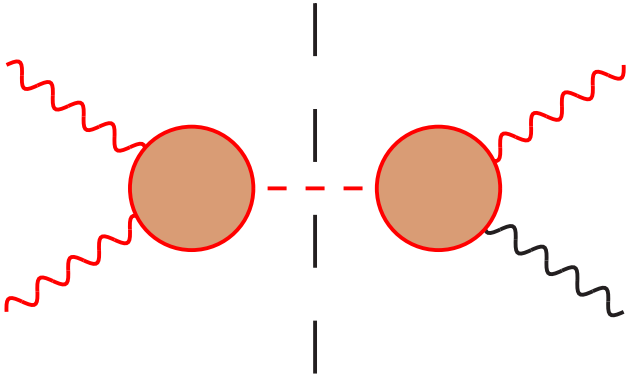
B. Kubis, F. Niecknig, S. Schneider, EPJC 72 (2012) 2014, PRD 86 (2012) 054013

MH, B. Kubis, S. Leupold, F. Niecknig, S. Schneider, EPJC 74 (2014) 3180

L. Bai, MH, B. Kubis, S. Leupold, work in progress

Pion-pole contribution to $g-2$

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Talk by P. Stoffer

$$\Pi_i^{\pi^0\text{-pole}}(s,t,u) = \frac{\rho_{i;s}}{s-M_\pi^2} + \frac{\rho_{i;t}}{t-M_\pi^2} + \frac{\rho_{i;u}}{u-M_\pi^2}$$

$$\rho_{i,s} = \delta_{i1} F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2)$$

$$\rho_{i,t} = \delta_{i2} F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2) F_{\pi^0\gamma^*\gamma^*}(q_2^2,q_4^2)$$

$$\rho_{i,u} = \delta_{i3} F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_4^2) F_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2)$$

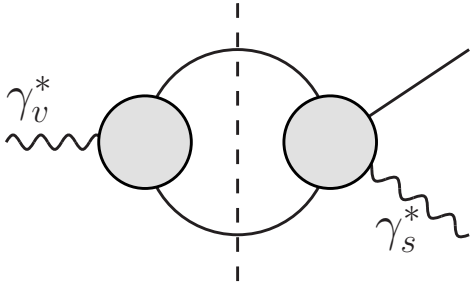
- Determined by on-shell **pion transition form factor** $F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)$
- Pion pole does not, on its own, fulfill OPE constraints for the triangle amplitude

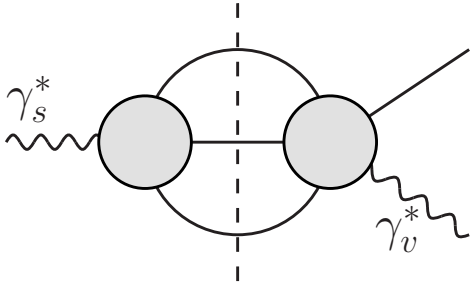
Pion transition form factor: isospin decomposition and unitarity relations

- **Isospin decomposition**

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$

- **Unitarity relations**

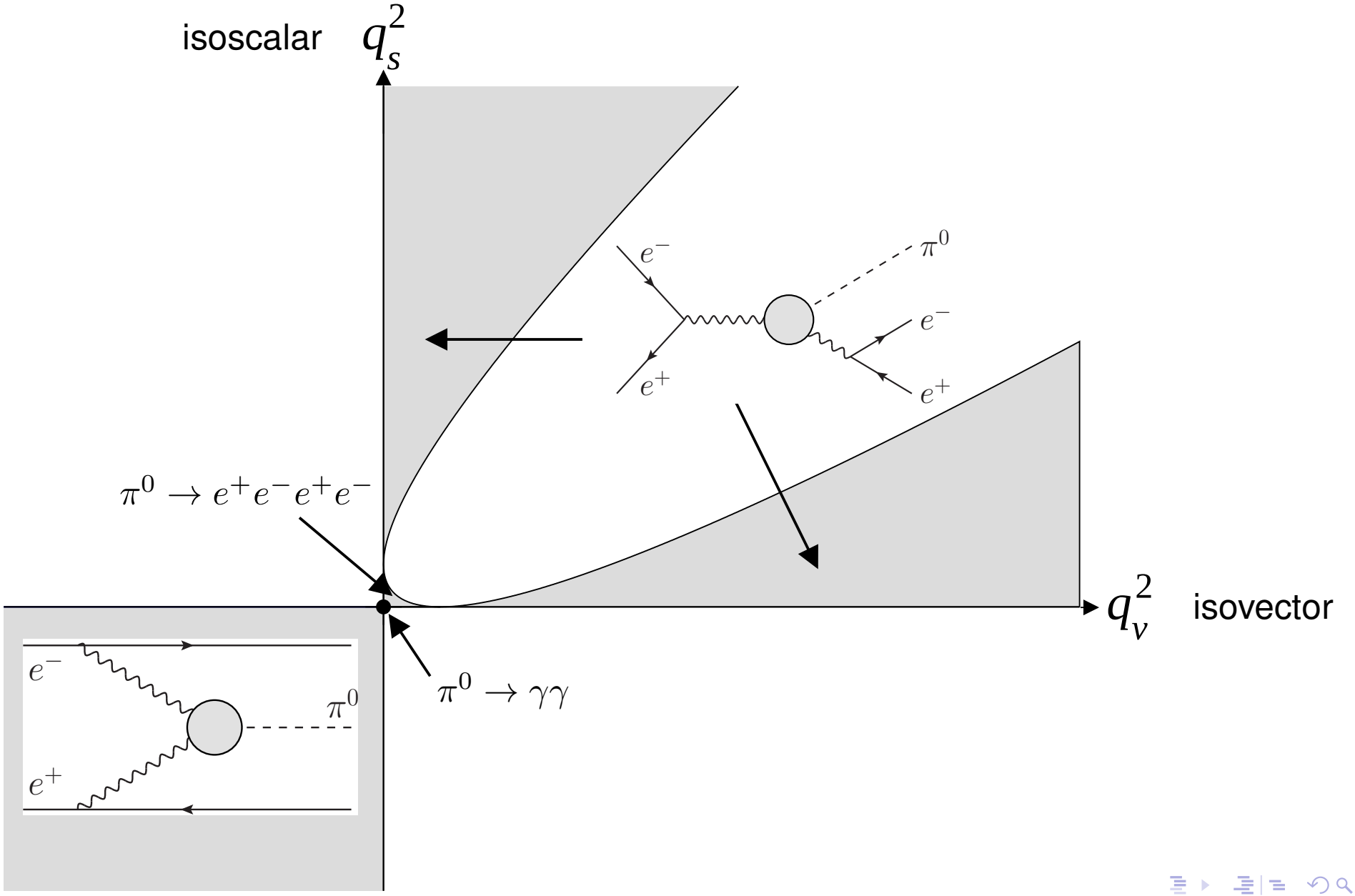




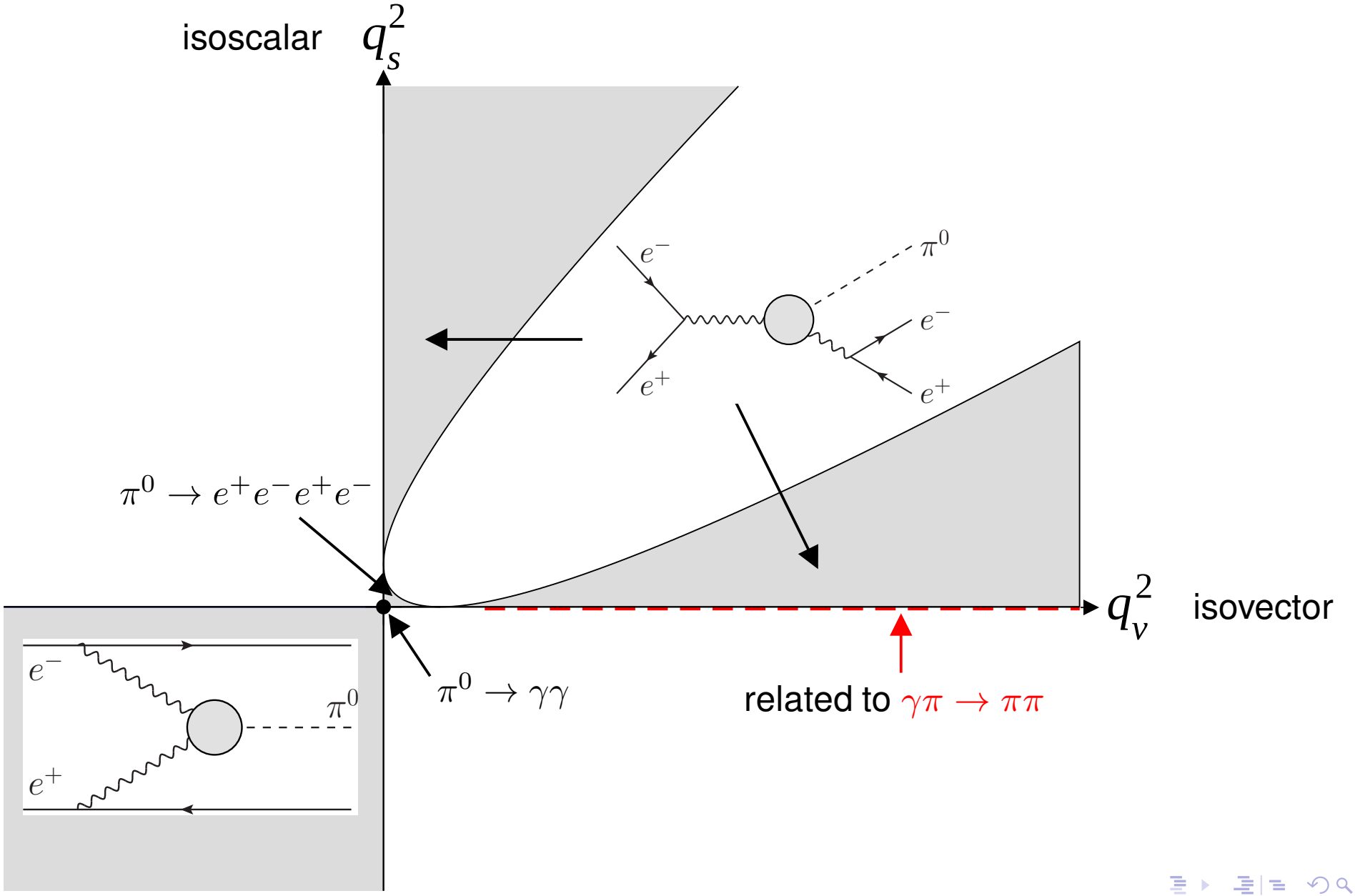
- Isoscalar photon dominated by narrow ω, ϕ

  $\hookrightarrow \omega, \phi \to \pi^0\gamma^*$ transition form factors

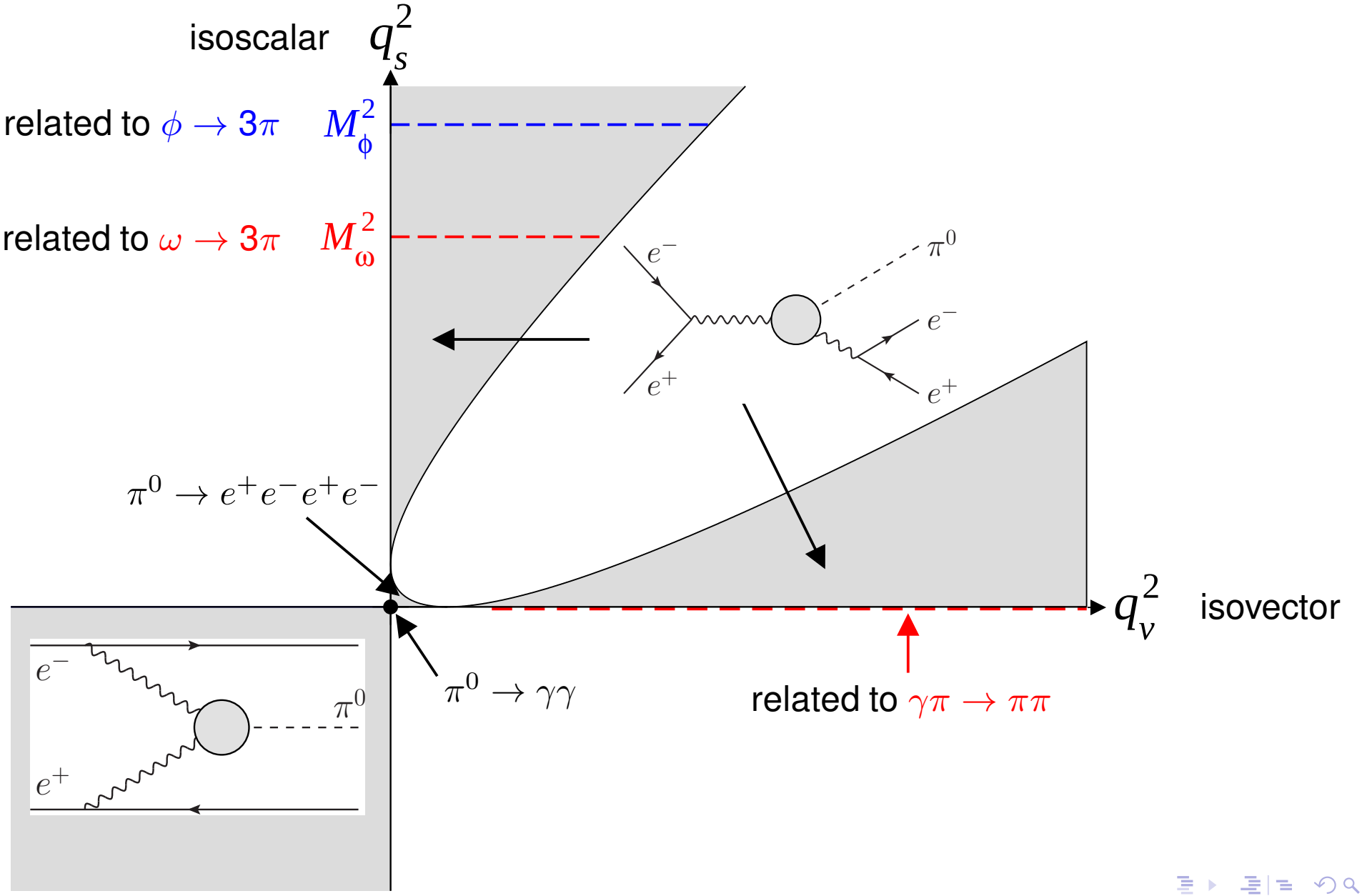
Pion transition form factor: physical regions



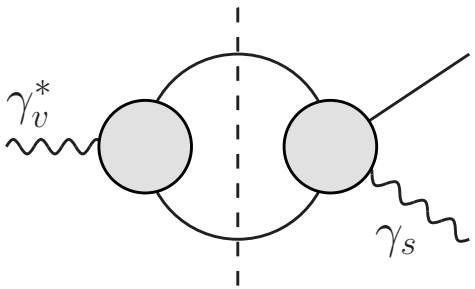
Pion transition form factor: physical regions



Pion transition form factor: physical regions



Step 1: $q_s^2 = 0$



- For on-shell isoscalar photon, $F_{vs}(q_1^2, 0)$, unitarity relation involves
  - Pion vector form factor: F_V^π
  - P-wave for $\gamma\pi \to \pi\pi$: f_1

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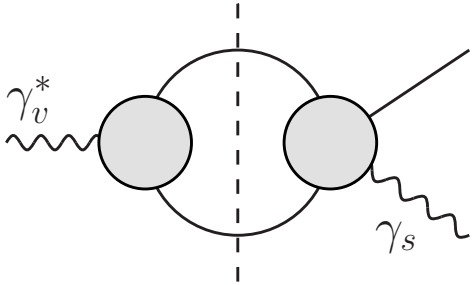
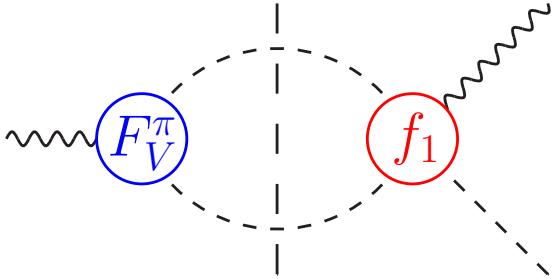


- For on-shell isoscalar photon, $F_{vs}(q_1^2, 0)$, unitarity relation involves
  - Pion vector form factor: F_V^π
  - P-wave for $\gamma\pi \to \pi\pi$: f_1
- Subtracted dispersion relation

$$F_{vs}(s,0) = F_{vs}(0,0) + \frac{s}{12\pi^2} \int_{4M_\pi^2}^{\infty} ds' \, \frac{q_\pi^3(s')\left(F_V^\pi(s')\right)^* f_1(s')}{s'^{3/2}(s'-s)}$$

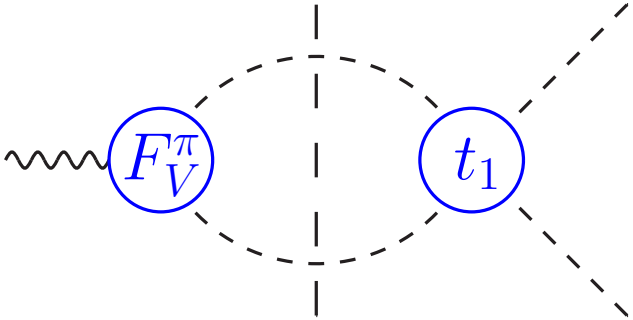


- Subtraction constant: $F_{vs}(0,0) = \frac{F_{\pi\gamma\gamma}}{2} = \frac{e^2}{8\pi^2 F_\pi}$
- Next: how to determine F_V^π and f_1

Step 1: $q_s^2 = 0$, pion vector form factor

- **Unitarity** for **pion vector form factor**

$$\mathrm{Im}\, F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big)\, F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

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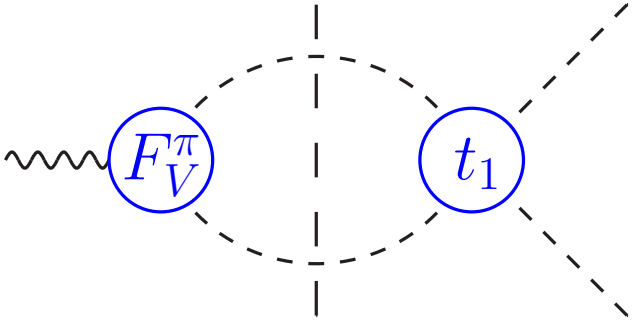


$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

- Solution in terms of **Omnès function** Omnès 1958

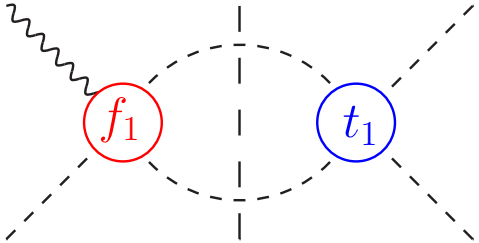
$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow P(s) = 1$

Step 1: $q_s^2 = 0$, $\gamma\pi \to \pi\pi$

- **Unitarity**

$$\mathrm{Im}\, f_1(s) = \theta\left(s - 4M_\pi^2\right) f_1(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ again Watson's theorem, but now **left-hand cut** in $f_1(s)$

Step 1: $q_s^2 = 0$, $\gamma\pi \to \pi\pi$

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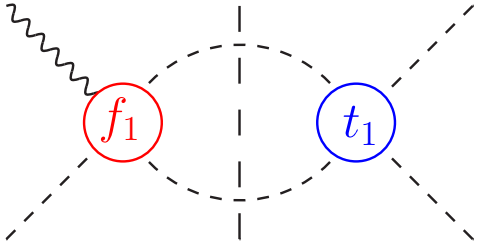


$\hookrightarrow$ again Watson's theorem, but now **left-hand cut** in $f_1(s)$

- Including the left-hand cut

$$\mathrm{Im}\, f_1(s) = \mathrm{Im}\, \mathcal{F}(s) = (\underbrace{\mathcal{F}(s)}_{\text{RHC}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{LHC}})\theta(s - 4M_\pi^2) \sin\delta_1(s) e^{-i\delta_1(s)}$$

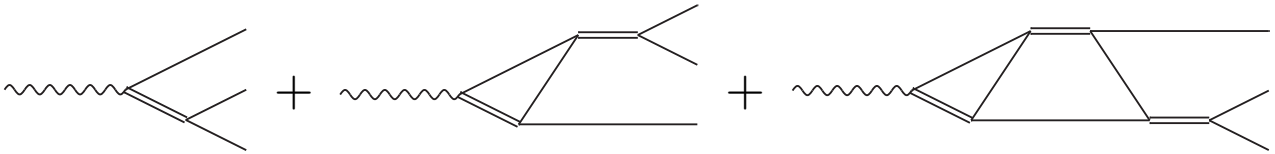
$$f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s) \qquad \hat{\mathcal{F}}(s) = 3\langle (1 - z^2)\mathcal{F}\rangle \qquad \langle z^n \mathcal{F}\rangle = \tfrac{1}{2}\int_{-1}^{1} \mathrm{d}z\, z^n \mathcal{F}(t)$$

Omnès solution for $\mathcal{F}(s)$

$$\mathcal{F}(s) = \Omega_1(s)\left\{ \frac{C_1}{3}(1 - \dot{\Omega}_1(0)s) + \frac{C_2}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\hat{\mathcal{F}}(s')\sin\delta_1(s')}{s'^2(s' - s)|\Omega_1(s')|} \right\}$$

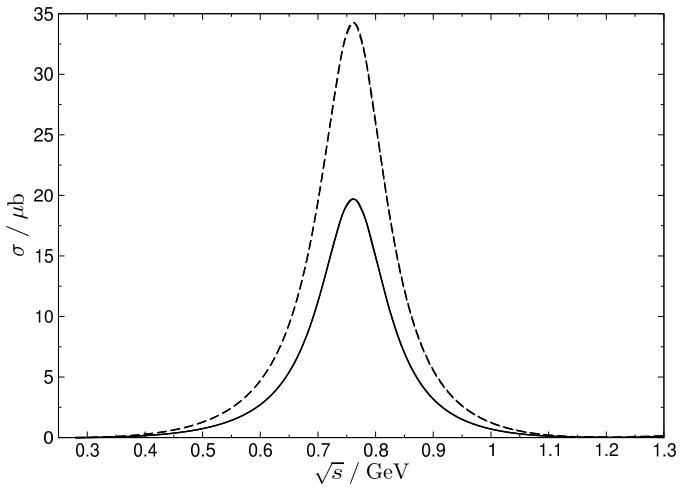
Step 1: $q_s^2 = 0$, $\gamma\pi \to \pi\pi$

- Solve for $\mathcal{F}(s)$ by iteration
- $\hat{\mathcal{F}}(s)$ corresponds to crossed-channel $\pi\pi$ **rescattering**

$\mathcal{F}(s) =$  $+ \cdots$

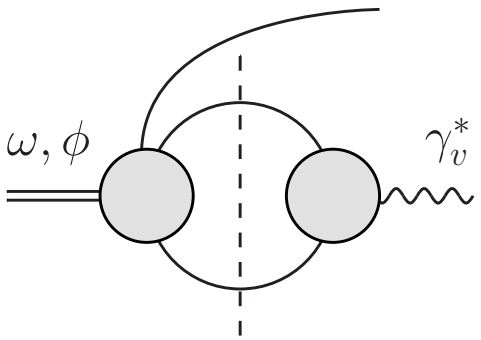
- Representation of the cross section in terms of **two parameters** $\hookrightarrow$ fit C_i to data
  - Test of **chiral anomaly** $F_{3\pi} = e/(4\pi^2 F_\pi^3)$
  - Precise description of f_1
- Looking forward to **COMPASS** result

  $\hookrightarrow$ currently: use chiral prediction

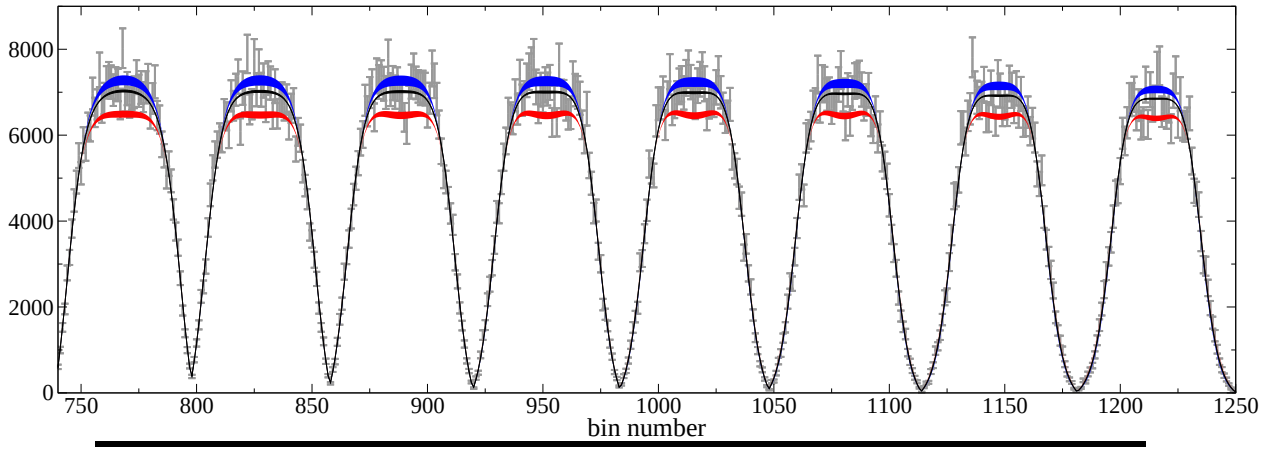


Step 2: $q_s^2 = M_\omega^2, M_\phi^2$

- Input required:
  - Pion vector form factor
  - P-wave for $\omega, \phi \to 3\pi$

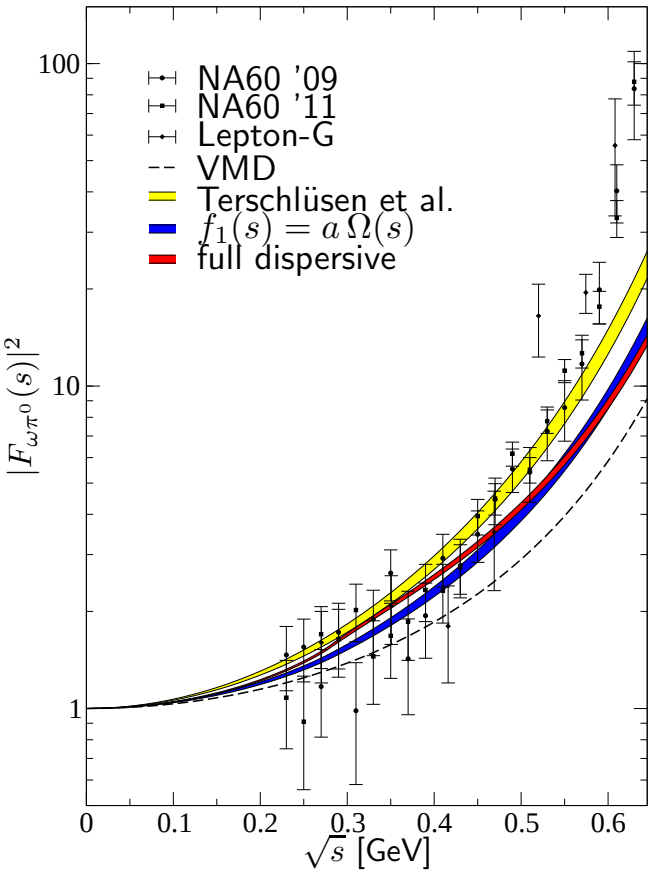


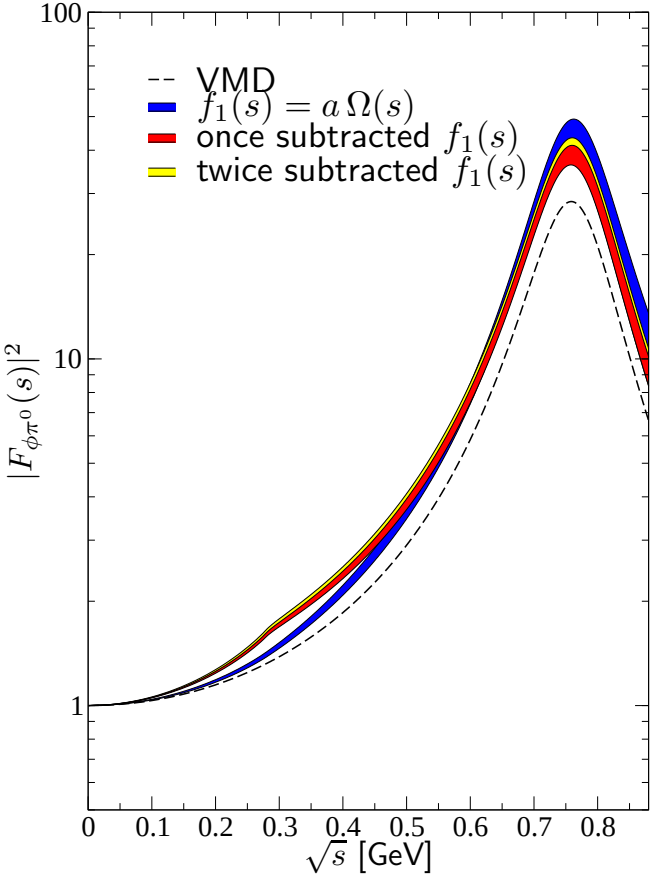
- Additional complications due to **decay kinematics**
- Dalitz plot fit for $\phi \to 3\pi$ KLOE ($\omega \to 3\pi$ recently measured by WASA@COSY)

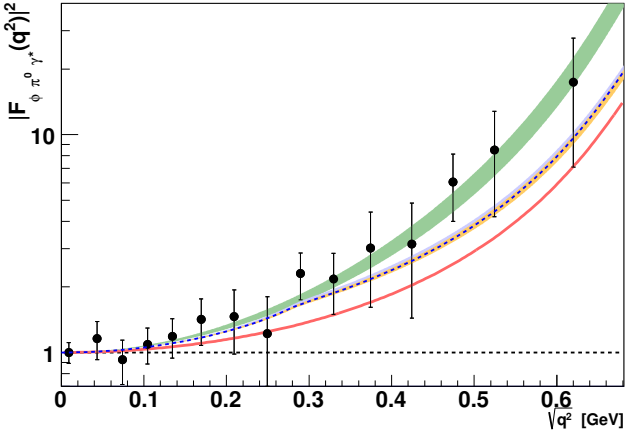


	$\hat{\mathcal{F}} = 0$	once-subtracted	twice-subtracted
χ^2/dof	1.71 . . . 2.06	1.17 . . . 1.50	1.02 . . . 1.03

Step 2: $q_s^2 = M_\omega^2, M_\phi^2$

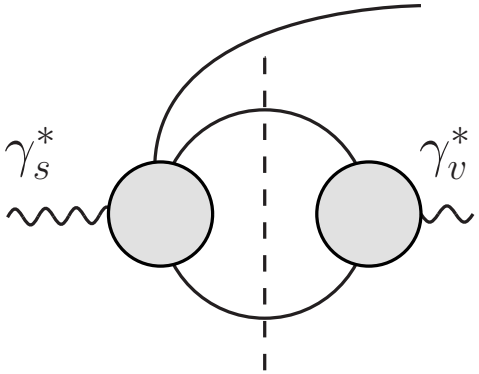






- Puzzle of steep rise in $F_{\omega\pi^0}$
- Test in $F_{\phi\pi^0}$ not yet conclusive KLOE-2 2016

Step 3: arbitrary time-like q_s^2



- General virtualities: how to fix the **normalization**?

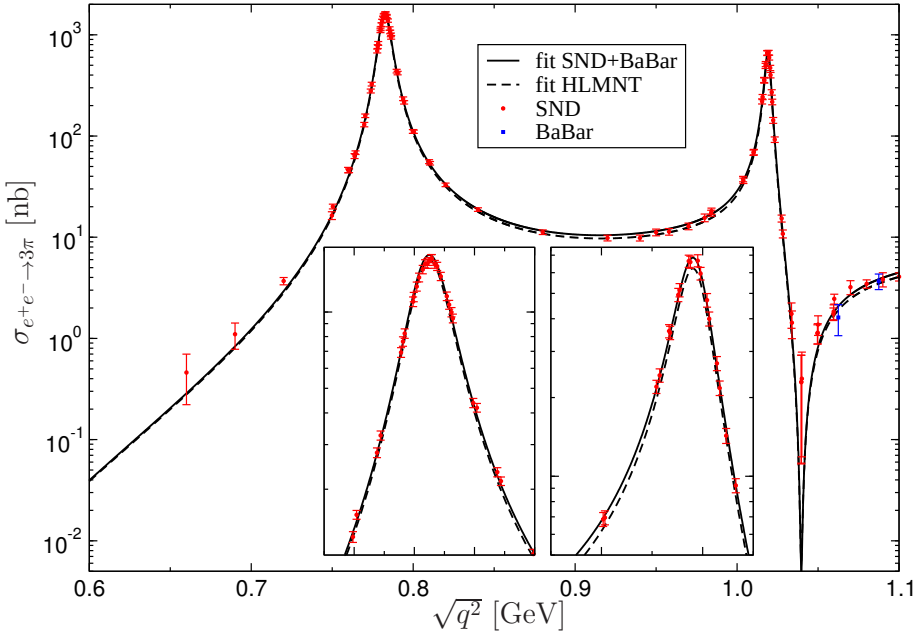
  $\hookrightarrow$ $F_{3\pi}$ for $\gamma\pi \to \pi\pi$, widths for $\omega, \phi \to 3\pi$
- Fit to $e^+e^- \to 3\pi$

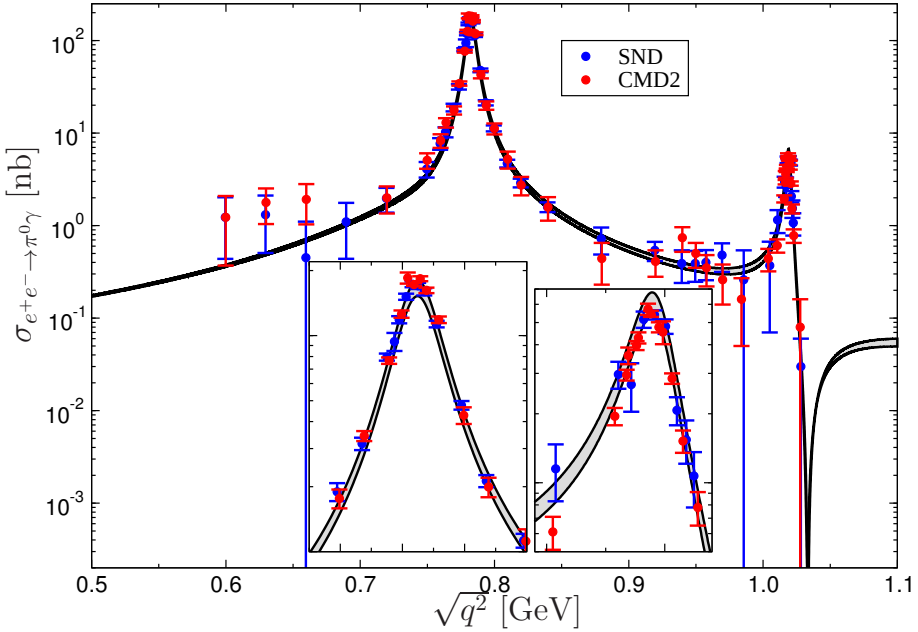
$$a(q^2) = \alpha + \beta q^2 + \frac{q^4}{\pi} \int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \, \frac{\operatorname{Im} \mathcal{A}(s')}{s'^2 (s' - q^2)}$$

$$\mathcal{A}(q^2) = \frac{c_\omega}{M_\omega^2 - q^2 - i\sqrt{q^2}\Gamma_\omega(q^2)} + \frac{c_\phi}{M_\phi^2 - q^2 - i\sqrt{q^2}\Gamma_\phi(q^2)}$$

- α fixed by $F_{3\pi}$, $\Gamma_{\omega/\phi}(q^2)$ include 3π, $K\bar{K}$, $\pi^0\gamma$ channels
- Good analytic properties, free parameters: β, c_ω, c_ϕ
- Valid up to 1.1 GeV, also fit including ω', ω'' to estimate uncertainties

Step 3: predicting $\sigma(e^+e^- \to \pi^0\gamma)$ from $\sigma(e^+e^- \to 3\pi)$





1. Fit dispersive representation to $e^+e^- \to 3\pi$
2. Determines singly-virtual form factor in **time-like region**
3. **Predict** $e^+e^- \to \pi^0\gamma$ as check on the formalism

Step 3: extraction of slope and space-like continuation

- For HLbL need the form factor in the

  space-like region

  $\hookrightarrow$ another dispersion relation

$$F_{\pi^0\gamma^*\gamma}(q^2,0) = F_{\pi\gamma\gamma} + \frac{q^2}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, F_{\pi^0\gamma^*\gamma}(s',0)}{s'(s'-q^2)}$$

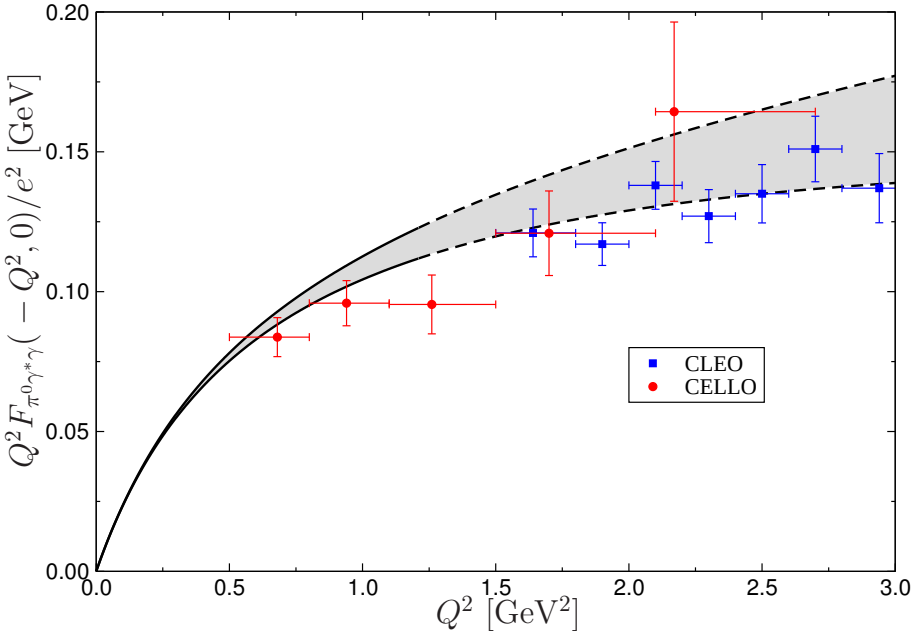
- Sum rules for $F_{\pi\gamma\gamma}$ and **slope parameters**

$$a_\pi = \frac{M_{\pi^0}^2}{F_{\pi\gamma\gamma}} \frac{1}{\pi} \int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, F_{\pi^0\gamma^*\gamma}(s',0)}{s'^2}$$

$$= (30.7 \pm 0.6) \times 10^{-3}$$

$$b_\pi = (1.10 \pm 0.02) \times 10^{-3}$$

- BESIII data Talk by C. Redmer



Step 4: doubly-virtual form factor

- What stops us from tackling the **doubly-virtual case**? In principle nothing:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{\pi^0\gamma^*\gamma}(q_1^2,0) + F_{\pi^0\gamma^*\gamma}(q_2^2,0) - F_{\pi\gamma\gamma} + \frac{e\,q_1^2\,q_2^2}{12\pi^3}\int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{x}\int_{4M_\pi^2}^{\infty}\mathrm{d}y\,\frac{q_\pi^3(y)}{y^{3/2}}\left[\frac{\mathrm{Im}\left\{\left(F_V^\pi(y)\right)^* f_1(y,x)\right\}}{(x-q_2^2)\,(y-q_1^2)} + (q_1^2 \leftrightarrow q_2^2)\right]$$

  where all ingredients are known

- In practice work in progress:
  - description only valid until non-$\pi\pi$ cuts become important

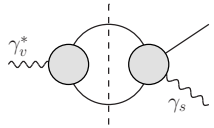
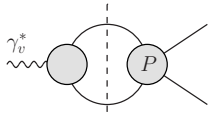
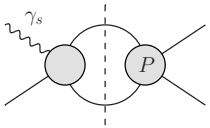
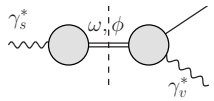
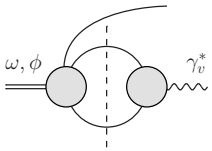
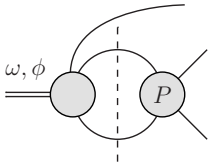
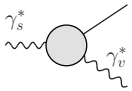
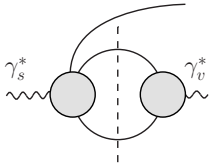
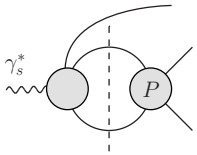
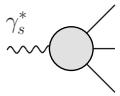
    $\hookrightarrow$ need to study the sensitivity to **high-energy tails**
  - subtracted dispersion relation has bad high-energy behavior

    $\hookrightarrow$ need to study the matching to **pQCD constraints** to obtain useful input for $g-2$

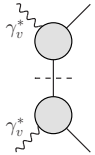
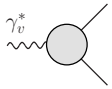
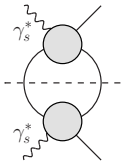
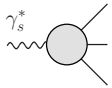
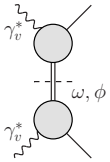
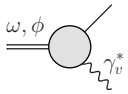
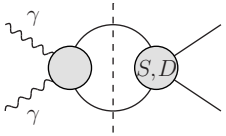
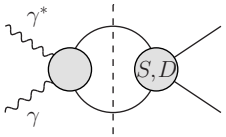
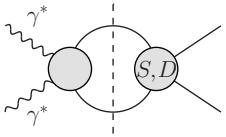
Summary

- Pion transition form factor strongly constrained by $\pi\pi$ cuts, resum $\boldsymbol{\pi\pi}$ **rescattering**
- Valuable input from $\boldsymbol{\gamma\pi \to \pi\pi}$, $\boldsymbol{\omega, \phi \to 3\pi}$, $\boldsymbol{\omega, \phi \to \pi^0\gamma^*}$
- For general virtualities: normalization fixed by $\boldsymbol{e^+e^- \to 3\pi}$, prediction for $\boldsymbol{e^+e^- \to \pi^0\gamma}$
- Singly-virtual form factor analyzed, to be confronted with data
- Doubly-virtual form factor and $g-2$ application to follow soon
- Stay tuned for what's new in the case of $\boldsymbol{\eta, \eta'}$! Talk by B. Kubis

Pion transition form factor: unitarity relations

process	unitarity relations	SC 1	SC 2	
			$F_{\pi^0\gamma\gamma}$	$\gamma\pi \to \pi\pi$
		$F_{3\pi}$	$\sigma(\gamma\pi \to \pi\pi)$	
			$\Gamma_{\pi^0\gamma}$	$\omega \to 3\pi$, $\phi \to 3\pi$
		$\Gamma_{3\pi}$	$\frac{d^2\Gamma}{dsdt}(\omega, \phi \to 3\pi)$	
			$\sigma(e^+e^- \to \pi^0\gamma)$	$\gamma^* \to 3\pi$
		$\sigma(e^+e^- \to 3\pi)$	$\sigma(\gamma\pi \to \pi\pi)$ $\frac{d^2\Gamma}{dsdt}(\omega, \phi \to 3\pi)$	resummation of **$\pi\pi$ rescattering**
		$F_{3\pi}$	$\sigma(e^+e^- \to 3\pi)$	

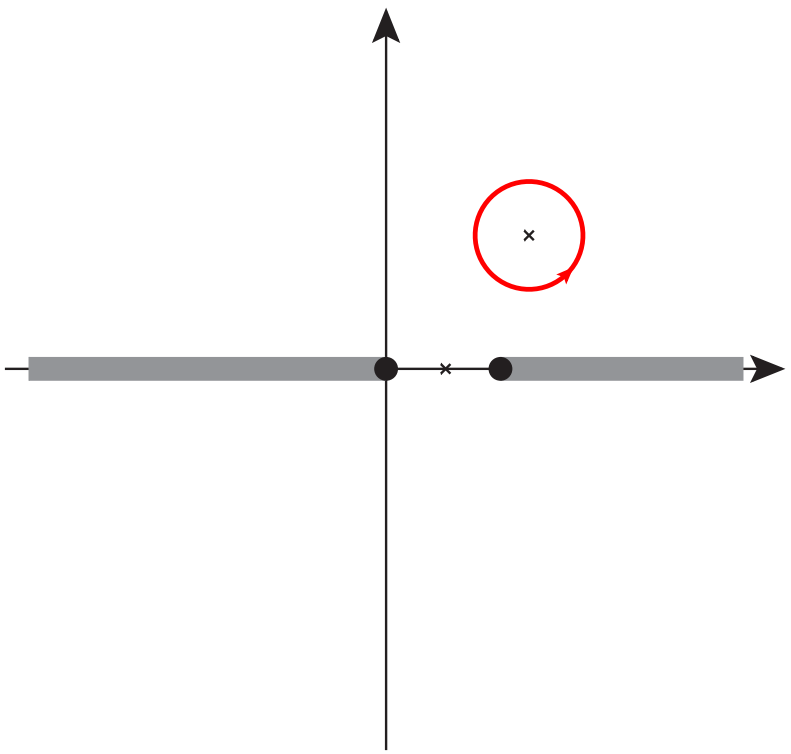
$\gamma^*\gamma^* \to \pi\pi$ partial waves: unitarity relations

process	building blocks and SC	
		**left-hand cut**
		π
		2π
		$3\pi\,(\sim\omega,\phi)$
		unitarity relations
	$\alpha_1 \pm \beta_1,\, \alpha_2 \pm \beta_2$	on-shell
	$\alpha_1(q^2) \pm \beta_1(q^2)$, ChPT $e^+e^- \to \pi\pi\gamma$ $e^+e^- \to e^+e^-\pi\pi$	singly-virtual
	ChPT $(e^+e^- \to \pi\pi\gamma)$ $e^+e^- \to e^+e^-\pi\pi$	doubly-virtual

From Cauchy's theorem to dispersion relations

- **Cauchy's theorem**

$$f(s) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\mathrm{d}s' \, f(s')}{s' - s}$$



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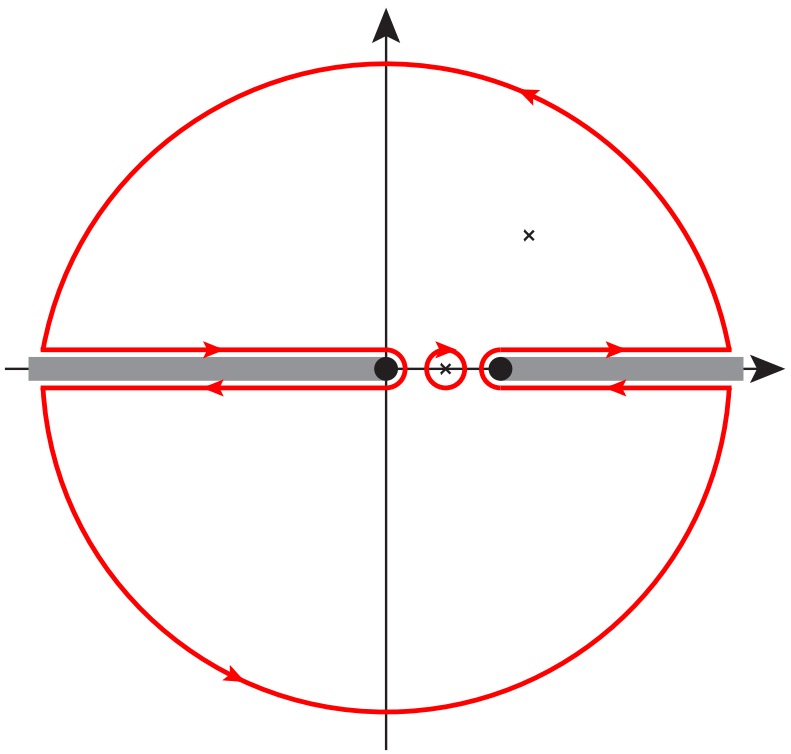
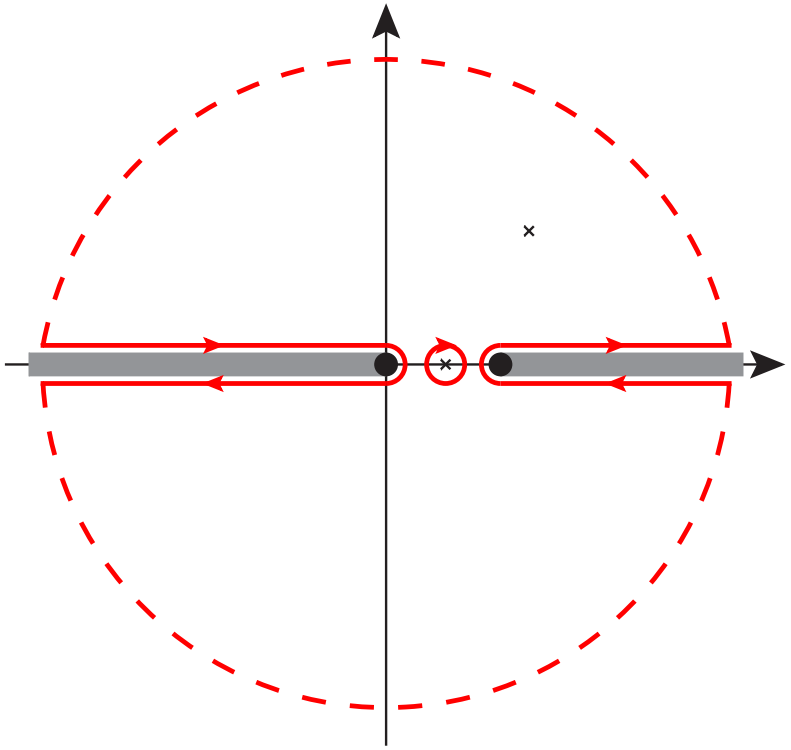


From Cauchy's theorem to dispersion relations

- **Dispersion relation**

$$f(s) = \frac{g}{s - M^2} + \frac{1}{\pi} \int_{\text{cuts}} \frac{\mathrm{d}s' \, \mathrm{Im}\, f(s')}{s' - s}$$

$\hookrightarrow$ **analyticity**



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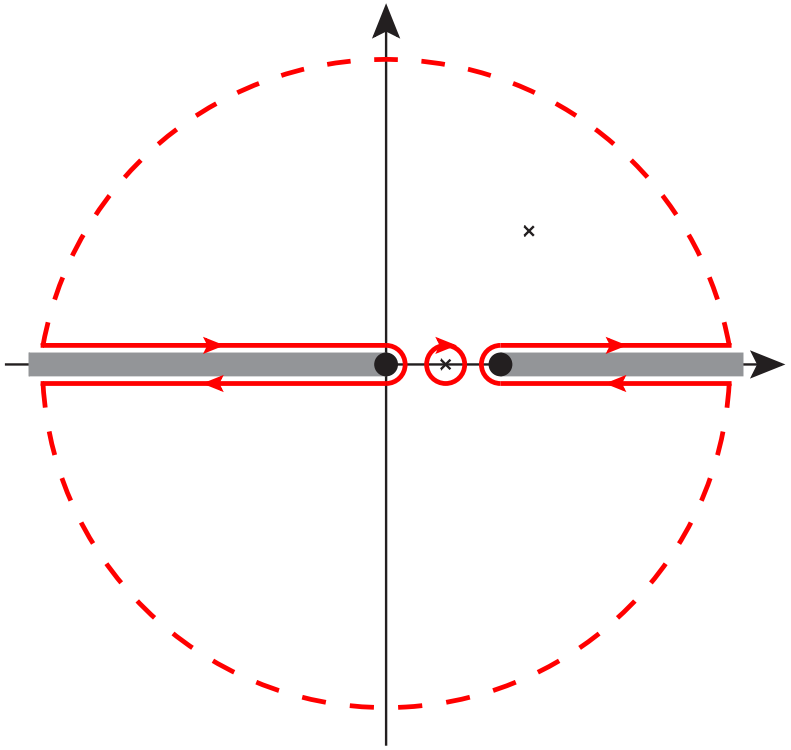

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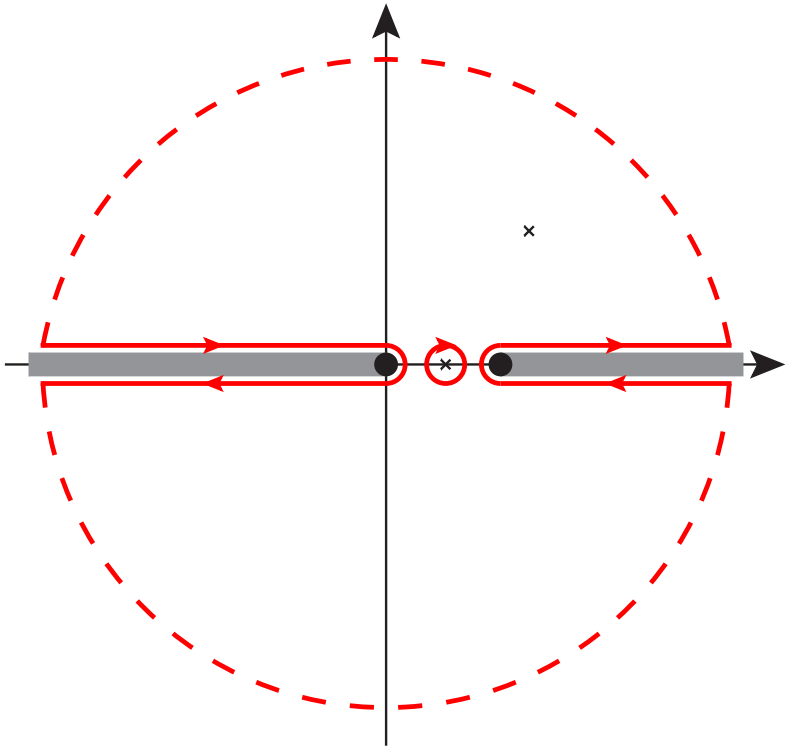

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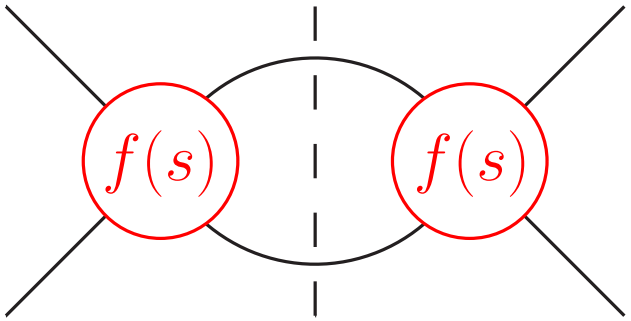
- Imaginary part from **Cutkosky rules**

  $\hookrightarrow$ forward direction: **optical theorem**

  see HVP and $\sigma(e^+e^- \to \text{hadrons})$



- **Unitarity** for partial waves: $\mathrm{Im}\, f(s) = \rho(s)|f(s)|^2$
- Residue g **reaction-independent**