Broadband Localization

Consider two identical radio receivers at the same altitude a distance d in the x direction apart and an unresolved source in direction $\mathrm{Sin}[\theta]\,\hat{\boldsymbol{x}} + \mathrm{Cos}[\theta]\,\hat{\boldsymbol{z}}$ where $\hat{\boldsymbol{z}}$ is the vertical direction. By unresolved we mean that the angular size of the source obeys $\vartheta \ll \lambda / d$ where λ is the wavelength of the radiation one is detecting. If $\boldsymbol{E}_0[t]$ is EM time stream which would be received at the midpoint then the EM wave-train received at the two detectors is $\boldsymbol{E}_\pm[t] = \boldsymbol{E}_0\left[t \pm \frac{d}{2c}\,\mathrm{Sin}[\theta]\right]$. Consider the problem of how to determine θ. If we "stored the voltages" then we would know $\boldsymbol{E}_\pm[t]$ and this becomes a standard problem in signal processing. However we consider more complicated problems below.

Now suppose that instead of having the EM time stream one instead only knows the "visibility", *i.e.* suppose one has a similarly oriented feeds at the two receivers labelled by ± as well as an imaginary feed in the middle labelled by 0 (we consider this fictitious feed to keep the formulae nice and symmetric). The feeds will register a time stream of voltages: $V_0[t]$, $V_\pm[t]$ which one can measure. In the idealized case of infinitely fine sampling and infinite duration of observation on can Fourier transform the voltage time stream

$$\tilde{V}_X[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, V_X[t]$$

where $X = 0, \pm$. Since $V_X[t]$ is real then $\tilde{V}_X[-\omega] = \tilde{V}_X[\omega]^*$. If the voltages all come from the same source position then

$$\tilde{V}_\pm[\omega] = \tilde{V}_0[\omega]\, e^{\pm\frac{1}{2} i\omega \Delta t}.$$

One then computes the *visibility* of the two actual feeds which for this infinite time sample is

$$\hat{C}_{+-}[\omega] = \tilde{V}_+ \tilde{V}_-^* = \left|\tilde{V}_0[\omega]\right|^2 e^{i\omega\Delta t}.$$

Note that

$$\Delta t = \frac{d\,\mathrm{Arg}\left[\hat{C}_{+-}[\omega]\right]}{d\omega}$$

so that a continuous noise free visibility fully determines Δt and hence θ thereby localizing the source exactly.

Now consider the complication of finite bandwidth where we only compute $C_{+-}[\omega]$ for some finite range of ω. We see from the previous formula that finite bandwidth, no matter how small, is not an impediment to localization.

Henceforth let us specify the voltages generated by the source emission in terms of a stochastic function. In particular assume the source emits zero mean Gaussian random noise with Gaussian time profile centered on $t = 0$ *i.e.*

$$\langle V_0[t]\rangle = 0 \qquad .$$

$$\langle V_0[t]\, V_0[t']\rangle = 2\sqrt{\pi}\, \frac{C}{\tau}\, e^{-\frac{1}{2}\left(\frac{t}{\tau}\right)^2}\, \delta[t - t']$$

so that

$$\left\langle \hat{C}_0[\omega]\right\rangle = \left\langle \tilde{V}_0[\omega]\, \tilde{V}_0[\omega']^*\right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \int_{-\infty}^{\infty} dt\, e^{+i\omega' t'}\, \langle V_0[t]\, V_0[t']\rangle$$

$$= \frac{C}{\sqrt{\pi}\,\tau} \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t}\, e^{-\left(\frac{t}{\tau}\right)^2} = e^{-\left(\frac{\omega'-\omega}{2\tau}\right)^2}\, \tau$$

so that

$$C_0[\omega] \equiv \left\langle \hat{C}_0[\omega]\right\rangle = \left\langle \left|\tilde{V}_0[\omega]\right|^2\right\rangle = C$$

and .

$$C_{+-}[\omega] \equiv \left\langle \hat{C}_{+-}[\omega]\right\rangle = C\, e^{i\omega\Delta t}.$$

Next consider the complication of discretization of the visibility from a stochastic source. In particular consider the average channel averaged visibility where we only know

$$C_\alpha = \int_{\omega_\alpha - \frac{1}{2}\delta\omega}^{\omega_\alpha + \frac{1}{2}\delta\omega} \frac{d\omega}{\delta\omega}\, C_{+-}[\omega] = C\, e^{i\omega_\alpha \Delta t}\, j_0\left[\tfrac{1}{2}\,\delta\omega\,\Delta t\right]$$

where $j_0[x] = \mathrm{Sin}[x]/x$ and $\omega_{\alpha+1} - \omega_\alpha = \delta\omega$. In this simple case

$$\Delta t = \frac{\mathrm{Arg}[C_{\alpha+1}] - \mathrm{Arg}[C_{\alpha+1}]}{\delta\omega}$$

so we see that channelization of frequency space is also not necessarily an impediment to localization. A problem may arise because Δt determined by the previous formulae is only defined mod $2\pi/\delta\omega = 1/\delta\nu$. This *sidelobe ambiguity* in Δt corresponds to an ambiguity in $\mathrm{Sin}[\theta]$ of $\frac{2c}{d\,\delta\nu}$. Since $\mathrm{Sin}[\theta]$ only varies between -1 and +1 this is not an ambiguity at all if $\frac{2c}{d\,\delta\nu} > 2$ which requires the channel width to be small enough: $\delta\nu < \frac{c}{d} = 299.792\,\mathrm{kHz}\, \frac{1\,\mathrm{km}}{d}$. Thus requiring completely unambiguous localization is a requirement on channel bandwidth.

While the above requirement on channelization is achievable for km scale baselines, even this requirement is likely more stringent than is necessary. If the two receivers have imaging optics such as a dish or cylinder, then the receivers are "focused" on a limited part of the sky which we call the "main beam". The sensitivity of the receiver to sources outside the main beam falls off rapidly outside of a region of angular size $\Delta\theta_{\mathrm{main}}$. There is little sidelobe ambiguity so long as the ambiguity in θ is much greater than $\Delta\theta_{\mathrm{main}}$ since sources this far away from beam center would need to be much brighter than a source within the main beam and therefore presumably would be less likely to occur. Assuming $\Delta\theta_{\mathrm{main}} \ll 1$ the reduced requirement is that $\frac{2c}{d\,\delta\nu} \gg \Delta\theta_{\mathrm{main}}$ or

$$\delta\nu \ll \frac{2c}{d\,\Delta\theta_{\mathrm{main}}} = 3.43537\,\mathrm{MHz}\, \frac{10\,\mathrm{km}}{d}\, \frac{1°}{\Delta\theta_{\mathrm{main}}}.$$

If we suppose that we are working at $\lambda = 40\,\mathrm{cm}$ $(749.481\,\mathrm{MHz})$ then

$$\delta\theta = \frac{\lambda}{d} = 0.825059''\, \frac{\lambda}{40.\,\mathrm{cm}}\, \frac{100.\,\mathrm{km}}{d}$$

and the channelization requirement is

$$\delta\nu \ll \frac{2c\,\delta\theta}{\lambda\,\Delta\theta_{\mathrm{main}}} = 0.416378\,\mathrm{MHz}\, \frac{40.\,\mathrm{cm}}{\lambda}\, \frac{1°}{\Delta\theta_{\mathrm{main}}}\, \frac{\delta\theta}{1''}.$$

So roughly speaking for 1" resolution with little sidelobe ambiguity one would need 100 km baselines and channel bandwidths no larger than `100 kHz`.

We have already shown the identity

$$\Delta t = \frac{\mathrm{Arg}[C_{\alpha+1}] - \mathrm{Arg}[C_\alpha]}{\delta\omega}$$

but a more useful one is probably

$$\mathrm{Sin}[\Delta t\,\delta\omega] = \mathrm{Sin}[\mathrm{Arg}[C_{\alpha+1}] - \mathrm{Arg}[C_\alpha]] = \frac{\left(\frac{C_{\alpha+1}+C_\alpha}{2}\right)^* \left(\frac{C_{\alpha+1}-C_\alpha}{2}\right) - \left(\frac{C_{\alpha+1}+C_\alpha}{2}\right)^* \left(\frac{C_{\alpha+1}-C_\alpha}{2}\right)}{|C_{\alpha+1}|\,|C_\alpha|}$$

since it is the ratio of two quadratic terms. Noise, when added, will be easier to keep track for these quadratic terms. In particular one can combine channels

$$\mathrm{Sin}[\Delta t\,\delta\omega] = \frac{\Sigma_\alpha \left(\frac{C_{\alpha+1}+C_\alpha}{2}\right)^* \left(\frac{C_{\alpha+1}-C_\alpha}{2}\right) - \left(\frac{C_{\alpha+1}+C_\alpha}{2}\right)^* \left(\frac{C_{\alpha+1}-C_\alpha}{2}\right)}{\Sigma_\alpha |C_{\alpha+1}|\,|C_\alpha|}.$$