

Effective field theories of massive spin 2 fields.
 massive gravity

(1)

Action for a massless spin -2 field takes the form

$$S = \int -\frac{1}{2} \partial_a h_{bb} \partial_A h_{cc} \eta_{DD} \epsilon^{abcd} \epsilon^{ABCD}$$

$$\equiv \int \frac{1}{2} (\partial_a \partial_A h_{bb}) h_{cc} \eta_{DD} \epsilon^{abcd} \epsilon^{ABCD}$$

double ϵ structure

'generalized Kronecker delta'

$$\delta_{ABCD}^{abcd} = \epsilon^{abcd} \epsilon_{ABCD}$$

$$\int_{A_1 \dots A_{4-k}}^{a_1 \dots a_{4-k} \dots} = k! \int_{A_1 \dots A_{4-k}}^{a_1 \dots a_{4-k}}$$

$$\delta_{ABC}^{abc} = \int_{D_1 \dots D_1}^D \delta_{ABCD}^{abcd}$$

$$\text{S.t. } \int_{A_1 \dots A_{4-k}}^{a_1 \dots a_{4-k}} = \int_{A_1}^{a_1} \int_{A_2}^{a_2} \dots \int_{A_{4-k}}^{a_{4-k}}$$

+ permutations

one of the virtues of this formalism, is it makes manifest the linear gauge invariance

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\text{eg } \delta_{AB}^{ab} = \delta_A^a \delta_B^b - \delta_B^a \delta_A^b$$

In fact in this form, if we allow $h_{\mu\nu} \neq h_{\nu\mu}$

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we have invariance under

symmetry. $(\text{Ldiff})^L \otimes (\text{Ldiff})^R$

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Goal is to add a mass term that completes this into 5 PDFs. solution is:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m^2 h_{aA} h_{bB} \delta^{ab; AB}$$

$$\mathcal{L}_{\text{mass}} = +\frac{1}{4} m^2 h_{aA} h_{bB} \eta_{cc} \eta_{dD} \epsilon^{abcd} \epsilon^{ABCD}$$

To see why we write

$$h_{\mu\nu} \Rightarrow h_{\mu\nu} + \frac{1}{2} \partial_\mu A_\nu + \frac{1}{2} \partial_\nu A_\mu$$

$$\mathcal{L}_{\text{mass}} = +\frac{1}{2} m^2 \partial_a A_A (h_{bB} + \partial_b A_B + \partial_B A_b) \delta^{ab; AB} \epsilon^{abcd} \epsilon^{ABCD}$$

$$\Rightarrow \text{usual } m^2 A_A \partial_a h_{bB} \delta^{ab; AB}$$

$$+\frac{1}{2} m^2 \partial_a A_A \partial_B A_b \delta^{ab; AB}$$

$$\begin{aligned}
 \mathcal{L}_{\text{mass}} &= \text{usual} \\
 &- \cancel{m^2} A_A \partial_a h_{bD} \delta^{ab, AB} \\
 &+ \frac{1}{4} m^2 F_{BA} F_{ab} \delta^{ab, AB} \\
 &\underbrace{\hspace{10em}}_{\text{Standard Maxwell } \times \cancel{m^2}}
 \end{aligned}$$

Canonical normalization is achieved with

$$A_\mu \rightarrow \frac{A_\mu}{m} \quad \therefore H_{\mu\nu} = h_{\mu\nu} + \frac{\partial_\mu A_\nu + \partial_\nu A_\mu}{2m}$$

$$\Rightarrow \mathcal{L}_{\text{mass}} = \text{usual} - \cancel{m} A_A \partial_a h_{bB} \delta^{ab, AB} - \frac{1}{4} \cancel{m} F_{\mu\nu}^2$$

A is a massive vector field, with broken U(1) invariance

Stueckelberg again

$$A_\mu \rightarrow \frac{A_\mu}{m} \Rightarrow \frac{\partial_\mu \pi}{m}$$

Putting this together, the mass term is

$$\mathcal{L}_{\text{mass}} = \text{usual} - \frac{1}{4} F_{\mu\nu}^2$$

$$- \cancel{m} A_A \partial_a h_{bB} \delta^{ab, AB} + \underbrace{\partial_A \pi \partial_a h_{bB} \delta^{ab, AB}}$$

$$\cancel{+} \pi \partial_a \partial_A h_{bB} \delta^{ab, AB}$$

$$\underbrace{\frac{1}{2} \partial_a \partial_A h_{bB} \pi \eta_{cc} \delta^{ab, AB}}_{\delta^{abc, ABC}}$$

Now taking $m \rightarrow 0$ to focus on kinetic terms
or focusing on 2 derivative terms

$$\mathcal{L}_{2\text{deriv}} = \left[\frac{1}{2} (\partial_a \partial_A h_{bB}) h_{cc} \cancel{+} \frac{1}{2} (\partial_a \partial_A h_{bB}) \pi \eta_{cc} \right] \delta^{abc, ABC}$$

$$- \frac{1}{4} F_{\mu\nu}^2$$

At this point it is easy to see that if we redefine

$$h_{\mu\nu} \Rightarrow h_{\mu\nu}^c + \frac{1}{2} \eta_{\mu\nu} \Pi$$

$$\Rightarrow \mathcal{L}_{2\text{-deriv}} = \frac{1}{8} (\partial_a \partial_A \Pi) \Pi \delta^{aA} \quad 3 \times 2$$



standard scalar kinetic term.

$$\Pi = \frac{\Pi_c}{\sqrt{2}} \frac{1}{\sqrt{3}}$$

punch line is

$$H_{\mu\nu} = h_{\mu\nu} + \frac{1}{2m^2} \partial_\mu A_\nu + \frac{1}{2m^2} \partial_\nu A_\mu - \frac{2}{\sqrt{3}} \frac{\partial_\mu \partial_\nu \Pi_c}{m^2}$$

$$h_{\mu\nu} = h_{\mu\nu}^c + \frac{1}{\sqrt{6}} \eta_{\mu\nu} \Pi$$

all the sins of massive gravity contained here

$$\frac{2}{M_p} T^{\mu\nu} h_{\mu\nu} = \frac{2}{M_{pl}} T^{\mu\nu} h_{\mu\nu}^c - \frac{1}{M_{pl}} \Pi \Pi \sqrt{\frac{2}{3}}$$

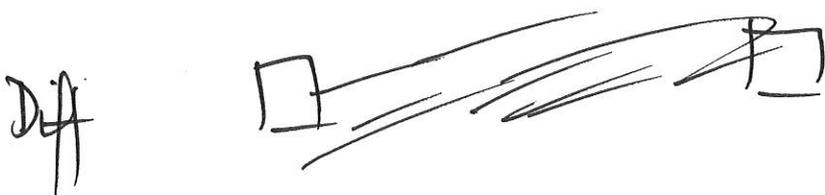
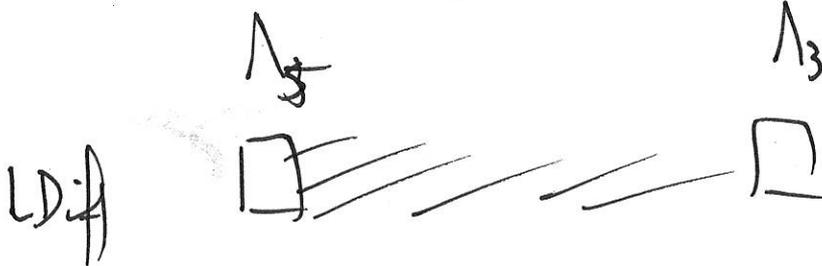
Let us now add interactions.

Approach depends on.

- (A)
1. Is this a 'gravity theory'
(spontaneously broken diffeomorphisms).
 2. Is this a non-gravitational theory
(spontaneously broken linear diffeomorphisms)!

(B) What scale are the interactions.

1. The lowest possible scale (Λ_5)
(in between)
2. The highest possible scale (Λ_3).



Example linear diff LdA theory.

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'Dimensionless metric' $\frac{h_{\mu\nu}}{M_{pl}}$.

'Tree level theory' is unitary gauge.

$$\int -\frac{1}{2} \partial h \partial h \eta \epsilon \epsilon + \frac{1}{2} m^2 h h \delta$$

$$\mathcal{L}_{int} = -m^2 M_{pl}^2 \underbrace{F\left(\frac{h}{M_{pl}}\right)}_{\text{dimensionless function}}$$

N.B. M_{pl} does not have to be the Planck mass here!
It's just an interaction scale in the potential.

eg like inflation $V = -v^4 f\left(\frac{\phi}{\Lambda}\right)$

To view the scales in the EFT, it is better to mark all the degrees of freedom manifest.

$$\mathcal{L}_{int} = -m^2 M_{pl}^2 F \left(\frac{h}{M_{pl}} + \frac{\partial A}{2mM_{pl}} + \frac{\partial A}{2mM_{pl}} - \frac{1}{m^2} \frac{\partial \partial \Pi}{M_{pl}} \right) \quad (8)$$

$$= -m^2 M_{pl}^2 \left(\frac{h}{M_{pl}} \right)^p \left(\frac{\partial A}{mM_{pl}} \right)^q \left(\frac{\partial \partial \Pi}{m^2 M_{pl}} \right)^r \quad c_{pqr}$$

$$\underline{p+q+r \geq 3}$$

If $m \ll M_{pl}$ then dominant interactions are more powers of $\frac{\partial \partial \Pi}{m^2 M_{pl}}$.

Focus on pure $\partial \partial \Pi$ interactions

$$-m^2 M_{pl}^2 \left(\frac{\partial \partial \Pi}{m^2 M_{pl}} \right)^{(r-3)+3} = - \left(\frac{\partial \partial \Pi}{m^2 M_{pl} (mM_{pl})^{\frac{r-3}{2}}} \right)^r$$

In the limit $r \rightarrow \infty$ these interactions are suppressed

by $\Lambda_3^3 = m^2 M_{pl}$.

Defn $\Lambda_N^N = M_{pl} m^{N-1}$

Dominant interaction is

With cubic interactions

$$\Rightarrow \frac{-m^2 M_{pl}^2}{m^6 M_{pl}^3} (\partial\pi)^3 = -\frac{1}{m^4 M_{pl}} (\partial\pi)^3$$

$$= -\frac{1}{\Lambda_5^5} (\partial\pi)^3$$

Dimension 9 operator

Such a theory will be a Λ_5 effective theory.

Given the hierarchy $m \ll M_{pl}$ we can take the decoupling limit $M_{pl} \rightarrow \infty, m \rightarrow 0$ keeping Λ_5 fixed.

In that limit, the tree lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial\mu\partial\mu \delta - \frac{1}{4} F_{\mu\nu}^2 - \pi \partial_a A_{ab} \partial_b B$$

$$- \frac{c}{\Lambda_5^5} (\partial\pi)^3$$

This classical theory admits a galileon symmetry $\pi \rightarrow \pi + \frac{v_\mu x^\mu}{\Lambda_5}$ and has a ghost at the scale Λ_5 .

However, there is nothing wrong with that.

We must now treat this as a Wilsonian effective theory, in which we will generate all Galileon invariant operators from loops at scale Λ_5 .

$$\mathcal{L}_{loop} = \Lambda_5^4 \int \left(\frac{\partial^3 \pi}{\Lambda_5^3}, \frac{\partial}{\Lambda_5} \right)$$

eg. Coleman Weinberg $m^2 \sim \frac{\partial^6 \pi}{\Lambda_5^5}$

$$\Lambda_5^4, \quad \Lambda_5^4 m^2, \quad m^4 \ln \left(\frac{m}{\Lambda_5} \right)$$

$$\int \frac{\partial^6 \pi}{\Lambda_5^5} \quad \int \left(\frac{\partial^6 \pi}{\Lambda_5^5} \right)^4 \ln \left(\frac{\partial^4 \pi / \Lambda_5^5}{\Lambda_5} \right)$$

rewriting back in unitary gauge.

$$\mathcal{L}_{loop} = \Lambda_5^4 \int \left(\frac{\Lambda_3^3}{\Lambda_5^3} \frac{h_{\mu\nu}}{M_{pl}}, \frac{\partial}{\Lambda_5} \right) + \text{sublead.} \dots$$

Just to check...

$$\Lambda_5^4 \left(\frac{\Lambda_3^3}{\Lambda_5^3} \right)^3 h^3$$

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$$= \frac{(m^2 M_{pl})^3}{m^4 M_{pl}} h^3 \sim m^2 M_{pl}^2 h^3$$

~

This was a Λ_5 theory. However, a simple way to increase the scale is to forbid cubic interactions.

if first interaction is quartic, then leading scale comes from

$$- m^2 M_{pl}^2 \left(\frac{\partial \partial \Pi}{m^2 M_{pl}} \right)^4 \sim \frac{(\partial \partial \Pi)^4}{m^6 M_{pl}^2}$$

$$= \frac{(\partial \partial \Pi)^4}{\Lambda_4^8}$$

this is a dimension 12 operator at scale Λ_4 .

loops come in as

$$\Lambda_4^4 H \left(\frac{h \Lambda_3^3}{M_{pl} \Lambda_4^3}, \frac{2}{\Lambda_4} \right)$$

In general of the Nth order interaction dominates.

$$m^2 M_{pl}^2 \left(\frac{\partial \mathcal{L}}{m^2 M_{pl}^2} \right)^N = \frac{(\partial \mathcal{L})^N}{m^{2N-2} M_{pl}^{N-2}}$$

$$= \frac{(\partial \mathcal{L})^N}{\Lambda^{3N-4} \Lambda^{N \left(\frac{3N-4}{N-2} \right)}}$$

Scale is $\Lambda^{\frac{3N-4}{N-2}}$

The other possibility is that all these interactions are zero $N \rightarrow \infty$ however this does not mean no interaction.

It can be achieved if

$$(\partial \mathcal{L})^N = \text{total derivative}$$

Return to

$$= - m^2 M_{pl}^2 \left(\frac{h}{M_{pl}} \right)^p \left(\frac{\partial \mathcal{A}}{m M_{pl}} \right)^2 \left(\frac{\partial \mathcal{L}}{m^2 M_{pl}^2} \right)^r$$

If (A) all interactions with $p=q=0$
one total derivatives.

AND (B) all interactions with $q=1, p=0$
one total derivatives

Then leading non-les interactions are

$$= -m^2 M_{pl}^2 \left(\frac{h}{M_{pl}}\right)^1 \left(\frac{\partial \partial \Pi}{m^2 M_{pl}}\right)^r \left(\frac{\partial A}{m M_{pl}}\right)$$

Eq. 2.1.12

$$- m^2 M_{pl}^2 \left(\frac{\partial A}{m M_{pl}}\right)^2 \left(\frac{\partial \partial \Pi}{m^2 M_{pl}}\right)^r$$

$$\Rightarrow \left[\begin{array}{l} - \Lambda_3^3 h \left(\frac{\partial \partial \Pi}{\Lambda_3^3}\right)^r \\ - (\partial A)^2 \left(\frac{\partial \partial \Pi}{\Lambda_3^3}\right)^r \end{array} \right]$$

→ this will be $FF \left(\frac{\partial \partial \Pi}{\Lambda_3^3}\right)^r$

The unique set of total derivative terms are determined by the double ϵ structure

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$$\epsilon^{abcd} \epsilon^{ABCD} \begin{pmatrix} \eta_{aA} \eta_{bB} \eta_{cC} \partial_a \partial_b \Pi \\ \eta \quad \eta \quad \partial \Pi \quad \partial \Pi \\ \eta \quad \partial \Pi \quad \partial \Pi \quad \partial \Pi \\ \partial \Pi \quad \partial \Pi \quad \partial \Pi \quad \partial \Pi \end{pmatrix}$$

coefficients of expansion of

(characteristic polynomials).

$$\det(\eta_{ab} + \lambda \partial_{ab} \Pi)$$

With this choice mass term is

$$-m^2 M_{pc}^2 \sum_p \alpha_p \eta^{4-p} \left(\frac{h}{M_{pc}} \right)^p$$

decoupling limit is

$$- \cancel{\Lambda_3^3} \sum p \alpha_p \eta^{4-p} \left(\frac{\partial \partial \Pi}{\Lambda_3} \right)^{p-1} \cancel{h} \quad (16)$$

$$- \sum_p p(p-1) \alpha_p \eta^{4-p} \left(\frac{\partial \partial \Pi}{\Lambda_3} \right)^{p-2} \left(\frac{\partial A + \partial \bar{A}}{2} \right)^2$$

$$+ \mathcal{L}_{loop} \Rightarrow \Lambda_3^4 F \left(\frac{h + \frac{\partial A + \partial \bar{A}}{2m} + \frac{\partial \partial \Pi}{m^2}}{M_{pl}}, \frac{\partial}{\Lambda_3} \right)$$

We can now repeat this story verbatim

Before proceeding we note that there is an additional kinetic term allowed (in unitary gauge)

$$\mathcal{L}_{kin} = \Lambda_3^4 \epsilon^{abcd} \epsilon^{ABCD} \left(\frac{\partial_a \partial_b A}{M_{pl}} \frac{h_{cB}}{M_{pl}} \right) \frac{h_{cC}}{M_{pl}} \frac{h_{dD}}{M_{pl}}$$

This new kinetic term is ghost free as can be seen by making the rising the ^{Stück} decomposition (17)

$$\mathcal{L}_{kin} = \frac{1}{3} \epsilon^{abcd} \epsilon^{ABCD} \frac{\partial_a \partial_A h_{bb}}{M_{pl}^3} \left(h_{cc} + \frac{\partial_c V_e}{2m} + \frac{\partial_e V_c}{2m} - \frac{\partial_e \partial_c \Pi}{m^2} \right) \left(h_{dd} + \frac{\partial_d V_d}{2m} + \frac{\partial_D V_d}{2m} - \frac{\partial_d \partial_D \Pi}{m^2} \right)$$

The key point to note is that the equations of motion are second order; since eg if I vary Π , the terms like $\partial \partial \Pi$ can be integrated by parts, but the antisymmetry of the double ϵ symbol ensures $\partial \partial \partial$ terms all vanish.

Since the e.o.m. for all fields are second order, this is enough to guarantee that these terms do not induce a ghost.

Beginning with the unitary gauge theory

$$\mathcal{L} = \frac{1}{2} \epsilon \epsilon (\partial \partial h) h \eta \dots$$

$$- m^2 M_{pl}^2 \sum_{p=0}^4 \epsilon \epsilon \alpha_p \eta^{4-p} \left(\frac{h}{M_{pl}} \right)^p$$

setting vectors to 0 (since they are not saved), the decoupling limit theory is

$$\mathcal{L} = \frac{1}{2} \epsilon \epsilon (\partial \partial h) h \eta \dots$$

$$- \Lambda_3^3 \sum_{p=1}^4 p \alpha_p \epsilon \epsilon \eta^{4-p} \left(\frac{\partial \partial \Pi}{\Lambda_3^3} \right)^{p-1} h$$

Let us assume $\alpha_4 = 0$. If so then there is at least 1 η here.

$$\mathcal{L} = \frac{1}{2} \epsilon \epsilon (\partial \partial h) h \eta \dots$$

$$- \Pi \sum_{p=1}^3 p \alpha_p \epsilon \epsilon \eta^{3-p} \left(\frac{\partial \partial \Pi}{\Lambda_3^3} \right)^{p-2} \partial \partial h$$

It is now clear that defn

$$h = h_c + \pi \sum_{p=1}^3 p \alpha_p \epsilon \epsilon \eta^{3-p} \left(\frac{\partial \partial \pi}{\Lambda^3} \right)^{p-2}$$

gives

$$\mathcal{L} = \frac{1}{2} \epsilon \epsilon \partial \partial h_c h_c \eta$$

$$- \pi \sum_{p, q} p \alpha_p q \alpha_q \epsilon \epsilon \eta^{3-p} \left(\frac{\partial \partial \pi}{\Lambda^3} \right)^{p-1} \epsilon \epsilon \eta^{3-q} \left(\frac{\partial \partial \pi}{\Lambda^3} \right)^{q-2}$$



$$\Rightarrow - \pi \sum_p \beta_p \epsilon \epsilon \eta^{4-p} \left(\frac{\partial \partial \pi}{\Lambda^3} \right)^p$$



Gauche interactors

Full diffs

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The extension of these ideas to full diffs is
realitively straightforward, with only a few minor
changes...

The decomposition...

$$H_{\mu\nu} = h_{\mu\nu} + \frac{1}{2m} (\partial_\mu A_\nu + \partial_\nu A_\mu) - \frac{\partial_\mu \partial_\nu \Pi}{m^2}$$

is replaced by

↖ tensor

$$2 \frac{H_{\mu\nu}}{M_{pl}} = g_{\mu\nu} - \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b$$

where $\phi^a = x^a - \frac{A^a}{2m M_{pl}} + \frac{\partial^a \Pi}{2m^2 M_{pl}}$

Unitary gauge $A^a = \Pi = 0 \Rightarrow \phi^a = x^a$

$$\Rightarrow \boxed{2 \frac{H_{\mu\nu}}{M_{pl}} = g_{\mu\nu} - \eta_{\mu\nu}}$$

The full decomposition is messy, however lets take the Λ_3 decoupling limit of

$$\frac{2H^M}{M_{pl}} = \delta^M{}_\nu - \eta_{ab} g^{mw} \partial_w \phi^a \partial_\nu \phi^b$$

$$\lim_{M_{pl} \rightarrow \infty} \left(\frac{2H^M}{M_{pl}} \right) = \delta^M{}_\nu - \eta_{ab} \eta^{aw} \left(\delta_w^a + \frac{\partial_w \partial^a \pi}{2m^2 M_{pl}} \right) \left(\delta_\nu^b + \frac{\partial_\nu \partial^b \pi}{2m^2 M_{pl}} \right)$$

schematically $\frac{2H}{M_p} = 1 - \left(1 + \frac{\partial \partial \pi}{2m^2 M_{pl}} \right)^2$

So to isolate $\partial \partial \pi$ we do

$$\frac{\partial \partial \pi}{2m^2 M_{pl}} = \sqrt{1 - \frac{2H}{M_{pl}}} - 1$$

stated differently we define

$$K^M{}_\nu = \sqrt{\delta^M{}_\nu - \frac{2H^M{}_\nu}{M_{pl}}} - \delta^M{}_\nu$$

where $\frac{2H^M{}_\nu}{M_{pl}} = g_{\mu\nu} - \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab}$

$$\therefore K^M{}_\nu = \sqrt{g^{\mu\nu} \underbrace{\partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab}}_{f_{\mu\nu}}} - \delta^M{}_\nu$$

The $\sqrt{\quad}$ is well defined, either by rearranging as solution of

$$\eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b = g_{\mu\nu} + 2K_{\mu\nu} + K_{\mu\alpha} K^{\alpha\nu}$$

or by recognizing that locally $g^{\mu\nu} f_{\mu\nu}$ can be diagonalized with a Lorentz transformation

$$g^{\mu\nu} f_{\mu\nu} = \Lambda^\mu{}_\alpha \underbrace{\eta^\alpha{}_\beta}_{\text{diagonal}} \Lambda^\beta{}_\nu$$

10 = 6 + 4

$$\sqrt{g^{\mu\nu}}_{fww} = \Lambda^M \alpha \sqrt{B^\alpha}_\beta (\Lambda^{-1})^\beta_\nu$$

$\sqrt{\text{of diagonal elements with + sign.}}$

eg $B = \begin{pmatrix} b_0^2 & 0 & 0 & 0 \\ 0 & b_1^2 & 0 & 0 \\ 0 & 0 & b_2^2 & 0 \\ 0 & 0 & 0 & b_3^2 \end{pmatrix}$

$$\sqrt{B} = \begin{pmatrix} b_0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_3 \end{pmatrix}$$



$K_{\mu\nu}$ is the non-linear generalization of $\partial_\mu \partial_\nu \pi$
 Following the linear diff discussion, to construct
 a Λ_5 theory of non-linear massive gravity
 we can consider generic interactions of the form

$$m^2 M_{pl}^2 K^3 \sim m^2 M_{pl}^2 \left(\frac{\partial \partial \pi}{\Lambda_3} \right)^3 \sim \frac{(\partial \partial \pi)^3}{\Lambda_5^3}$$

To raise this to Λ_3 we must construct $\epsilon \in \epsilon$ combinations of the k 's. These must also be Diff invariant.

\therefore we are led to

$$\sqrt{-g} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu'\nu'\rho'\sigma'} \left(\begin{array}{l} \delta_{\mu}^{\mu'} \delta_{\nu}^{\nu'} \delta_{\rho}^{\rho'} \delta_{\sigma}^{\sigma'} \\ \delta_{\mu}^{\mu'} \delta_{\nu}^{\nu'} \delta_{\rho}^{\rho'} K^{\sigma'}_{\sigma} \\ \delta_{\mu}^{\mu'} \delta_{\nu}^{\nu'} K^{\rho'}_{\rho} K^{\sigma'}_{\sigma} \\ \delta_{\mu}^{\mu'} K^{\nu'}_{\nu} K^{\rho'}_{\rho} K^{\sigma'}_{\sigma} \\ K^{\mu'}_{\mu} K^{\nu'}_{\nu} K^{\rho'}_{\rho} K^{\sigma'}_{\sigma} \end{array} \right)$$

$\Rightarrow \sqrt{-g} \det (\delta_{\nu}^{\mu} + \lambda K^{\mu}_{\nu})$

↗

terms in expansion in λ .

From an EFT point of view, the generic form of the quantum corrections will be

$$\Lambda_3^4 F \left(K_{\mu\nu}, \frac{\nabla_\mu}{\Lambda_3} \right)$$

where F are the infinite number of local scalar operators constructed out of all powers of K and ~~der~~ covariant derivatives thereof with order unity dimensionless coefficients

eg $\Lambda_3^4 K_{\mu\nu} \frac{\square}{\Lambda_3^2} K^{\mu\nu},$

$$\Lambda_3^4 (K^m_m)^{5-1},$$

$$\Lambda_3^4 \left(\frac{\nabla^\mu K_{\mu\nu}}{\Lambda_3} \right)^2 K^m_m$$

To properly derive the decoupling limit theory
 in the limit $M_{pl} \rightarrow \infty$, $m \rightarrow 0$ kept fixed is slightly complicated (although possible)
 in this form due to difficulty of dealing with $\sqrt{\quad}$ in K .

This can be dealt with by working in vielbein formulation.

GR / metric

Einstein-Cartan.

$Diff(M)$

$Diff(M) \times SO(1,3)_{local}$

(breaks to)



Positional global

Positional global.

In Einstein-Cartan formulation, we need
 4 Stückelberg fields for broken Diffs ϕ^a
 and 6 Stückelberg fields for broken local
 Lorentz transformations. These are defined by
 a local Lorentz transformation Λ_{μ}^a that has
 usual property (N.B. Λ 's will not have kinetic
 term...)

$$\Lambda_{\alpha}^A \Lambda_{\beta}^B \eta_{AB} = \eta_{\alpha\beta}$$

ie $\Lambda \eta \Lambda^T = \eta \equiv \boxed{\Lambda^{-1} = \eta \Lambda^T \eta}$

This is solved by $\Lambda = e^{\eta^{-1} \omega} = e^{\omega_{ab} s^a}$
 where $\omega_{ab}^{(\alpha)} = -\omega_{ba}^{(\alpha)}$ (6 parameters).

The breaking is then described by
 $e_{\mu}^a(x)$ (analogue of $g_{\mu\nu}(x)$)

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AND $f_{\mu}^a(x) = \Lambda^a_{\nu b} \partial_{\mu} \phi^b(x)$
 (analogue of $f_{\mu a}(x) = \eta_{ab} \partial_{\mu} \phi^b$)

To see that this is consistent . . .

$$\eta_{ab} f_{\mu}^a f_{\nu}^b = \eta_{ab} \underbrace{\Lambda^a_A \Lambda^b_B}_{\eta_{AB}} \partial_{\mu} \phi^A \partial_{\nu} \phi^B$$

$$= \eta_{AB} \partial_{\mu} \phi^A \partial_{\nu} \phi^B$$

The double ϵ structure is precisely that of the 'wedge product' combined with an ϵ for the Lorentz indices.

eg

$$e^a \wedge e^b \wedge e^c \wedge f^{cd} \epsilon^{abcd}$$

$$\equiv \frac{1}{4!} \epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} e_{\mu}^a e_{\nu}^b e_{\rho}^c f_{\sigma}^d$$

The Λ_3 mass terms are then

$$\sum_p e^{d-p} f^p$$

over Lorentz indices

meaning

$e \wedge e \wedge e \wedge e$	ϵ
$e \wedge e \wedge e \wedge f$	ϵ
$e \wedge e \wedge f \wedge f$	ϵ
$e \wedge f \wedge f \wedge f$	ϵ
$f \wedge f \wedge f \wedge f$	ϵ

or $\det(e + \lambda f)$ many terms in expansion in powers of λ .

To prove equivalence of the metric formulation with the metric, the key is to look at the equations of motion for the Lorenz Stueckelberg fields.

Consider $\det(e + \lambda \wedge d\phi)$

\nearrow short for e^a \uparrow short for Λ^a_b \nearrow short for $(\partial\phi)^a = \partial_\mu \phi^a$

Varying w.r.t. Λ we use

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$$\delta \det(A) = \det(A) \text{Tr}(A^{-1} \delta A)$$

$$\Rightarrow \text{Tr} \left((e + \lambda \Lambda d\phi)^{-1} \delta \Lambda d\phi \right) = 0$$

$$\text{Tr} \left((e + \lambda \Lambda d\phi)^{-1} (\delta \Lambda \Lambda^{-1}) \eta \Lambda d\phi \right) = 0$$

Now $\delta \Lambda \Lambda^{-1}$ is antisymmetric $(\delta \Lambda \Lambda^{-1})^T = -(\delta \Lambda \Lambda^{-1})$

so this equation amounts to

$$\begin{aligned} \left((e + \lambda f)^{-1} \eta f \right)^T &= (e + \lambda f)^{-1} \eta f \\ &= f^T \eta (e^T + \lambda f^T)^{-1} = (e + \lambda f)^{-1} \eta f \end{aligned}$$

Rearranging we have

$$f^T \eta e = e^T \eta f$$

or in component form

$$\boxed{f_{\mu}^a e_{\nu}^b \eta_{ab} = f_{\nu}^a e_{\mu}^b \eta_{ab}}$$

Deser - van Muehlenbein gauge.

To solve this...

$$(\Lambda d\phi)^T \eta e = e^T \eta \Lambda d\phi$$

$$(d\phi)^T \Lambda^T \eta e = e^T \eta \Lambda d\phi$$

$$\eta e^T \Lambda^{-1} (d\phi)^T \Lambda^T \eta e = \Lambda d\phi e^{-1}$$

$$(\Lambda d\phi e^{-1})^2 = \eta e^T \Lambda^{-1} (d\phi)^T \Lambda^T \eta \Lambda d\phi e^{-1}$$

$$= \eta e^T \Lambda^{-1} F e^{-1}$$

$$\Lambda d\phi e^{-1} = \sqrt{\eta e^T \Lambda^{-1} F e^{-1}}$$

~~det(e + \lambda d\phi e^{-1})~~

$$\det(e + \lambda d\phi) = \det(e) \det(1 + \lambda d\phi e^{-1})$$

$$= \det(e) \det(1 + \lambda \sqrt{\eta e^T \Lambda^{-1} F e^{-1}})$$

$$= \det(e) \det(1 + \lambda e^{-1} \sqrt{\eta e^T \Lambda^{-1} F e^{-1}} e)$$

$$= \det(e) \det(1 + \lambda \sqrt{g^{-1} F}) \quad [g = e^T \eta e]$$

\(\equiv\) structure of mass theory in metric form.

In the vielbein formulation, the derivation of the decoupling limit is extremely simple.

Define $e^a_\mu = \eta^a_\mu + \frac{h^a_\mu}{M_{pl}}$

$\omega_{\mu\nu} \Rightarrow \frac{\omega_{\mu\nu}}{m M_{pl}}$

$\phi^a = x^a + \frac{\partial^a \pi}{2m^2 M_{pl}} - \frac{A^a}{2m M_{pl}}$

Focusing on the special case $A = \omega = 0$.

$f^a_\mu = \eta^a_\mu + \frac{\partial_\mu \partial^a \pi}{2\Lambda_3^3}$

so given interactions $\mathcal{L} = \sum_p \frac{c_p^2}{m^2 M_{pl}^2} \alpha_p e^{d-p} f^p$

d.l. is $\mathcal{L} = \sum_p c_p \Lambda_3^{3(d-p)} \alpha_p h^p \left(\eta + \frac{\partial \partial \pi}{2\Lambda_3^3} \right)^p$

Which can easily be rewritten as

$$\mathcal{L} = \sum_p \binom{3}{p} \beta_p \epsilon \epsilon h \eta^{d-p-1} \left(\frac{\partial \sigma \Pi}{\Lambda^3} \right)^p$$

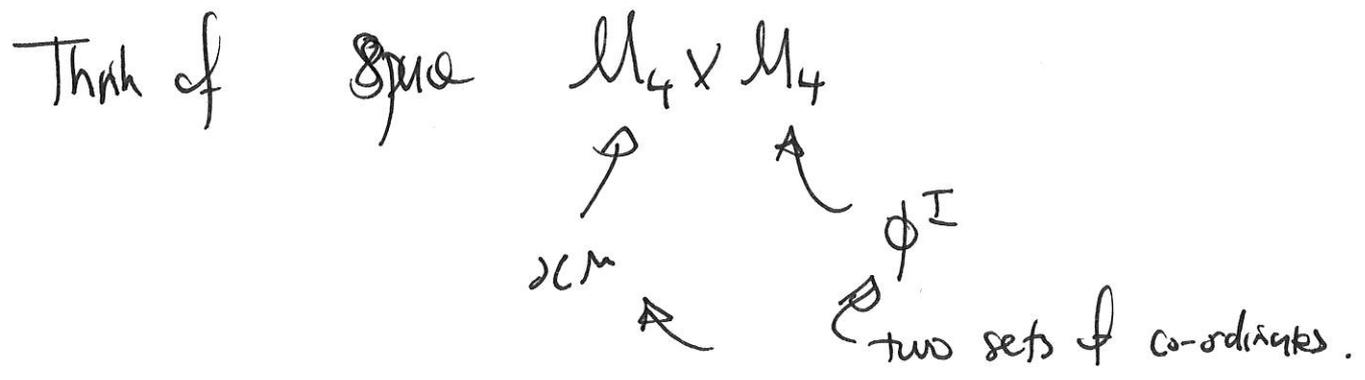
eg $\left(\eta + \frac{\partial \sigma \Pi}{2 \Lambda^3} \right)^p = \sum_{r=0}^p \frac{p!}{r!(p-r)!} \left(\frac{\partial \sigma \Pi}{\Lambda^3} \right)^r \eta^{p-r}$



We can generalize this to arbitrary reference metric geometries (or bi-gravity) as follows.

For an arbitrary reference vierbein:

$$f_{\mu}^a \Rightarrow \Lambda^a_b \tilde{f}_{\mu}^b(\Phi^I) \partial_{\mu} \Phi^I$$



In a locally inertial frame for the background metric

$$f_{\mathbb{I}}^{\sim b} = \delta_{\mathbb{I}}^b + \frac{1}{2} R_{f \mathbb{I} cd} \Phi^c \Phi^d + \dots$$

↑
background curvature

if we scale $R_f \sim m^2 \sim \frac{1}{M_p}$

then we will get an additional contribution in D.L.

$$\mathcal{L} = \sum M_{pc} \Lambda_3^3 \alpha_p \epsilon \epsilon e^{d-p} f^p$$

$$\sim \sum_p \epsilon \epsilon \Lambda_3^3 (d-p) \alpha_p h \eta^{d-p-1} \left(\eta + \frac{d\mathbb{T}}{2\Lambda_3} \right)^p$$

$$+ \sum_p \Lambda_3^3 p \alpha_p \epsilon \epsilon \eta^{d-p} \left(\eta + \frac{d\mathbb{T}}{2\Lambda_3} \right)^{p-1} \frac{M_p R_{f cd} \Phi^c \Phi^d}{2\epsilon}$$

where $\Phi^c = x^c + \frac{d\mathbb{T}}{2\Lambda_3}$

where $M_{pl} R_f$ is kept finite in the decoupling limit. (35)

This generates a new kinetic term for Π .