Theoretical Description of Lepton-Nucleus Scattering

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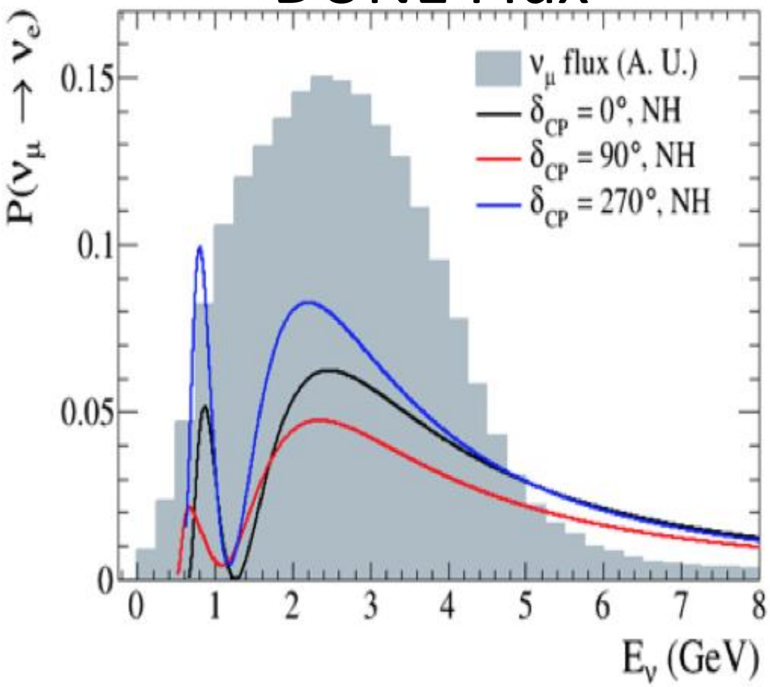
The Problem

DUNE Flux



The width of the energy-distribution implies that kinematics exist where the energy and momentum transfers to the nucleus are small, through the quasielastic and Delta regions to Deep Inelastic Scattering.

The dynamics of neutrino-nuclear scattering must be understood to a sufficient level of accuracy to allow the initial neutrino energy to be determined.

It is not clear at this point what level of accuracy needs to be attained.

It is also not clear what level of accuracy can be obtained by theoretical calculations of neutrino scattering reactions on the nuclei used in the detectors.

The Basic Model of Nuclear Physics

This model is:

- nonrelativistic,
- assumes that the explicit degrees of freedom are nucleons,
- and is described by that Hamiltonian operator

$$\hat{H} = \sum_{i}^{A} T_i + \sum_{i<j}^{A} \hat{V}_{ij} + \sum_{i<j<k}^{A} \hat{V}_{ijk} + \ldots.$$

The two-body potential is obtained

- Phenomenologically
- Using one-boson exchange models
- Using χEFT

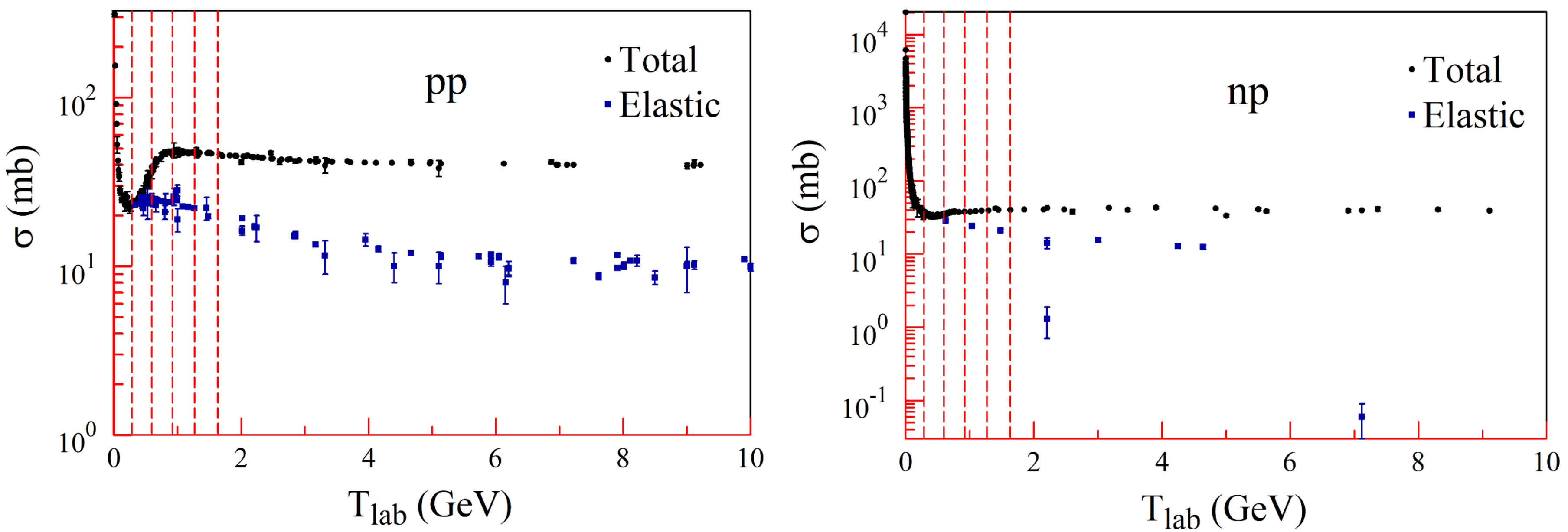
In all cases, the long-range part of the potential is given by 1-pion exchange.

Model parameters are fit to NN scattering for incident kinetic energies < 350 MeV.

The three-body potential is phenomenological or from χEFT.

Parameters obtained by fitting few-body energies.

NN Total Cross Sections



The two-body potentials use angular momentum operators of the form

$$\left\{1,\ L\cdot S,\ \sigma_1\cdot\sigma_2,\ S_{12},\ L^2,\ (L\cdot S)^2\ ,\ L^2\sigma_1\cdot\sigma_2\right\}$$

and isospin operators of the form

$$\{1,\ \tau_1\cdot\tau_2,\}$$

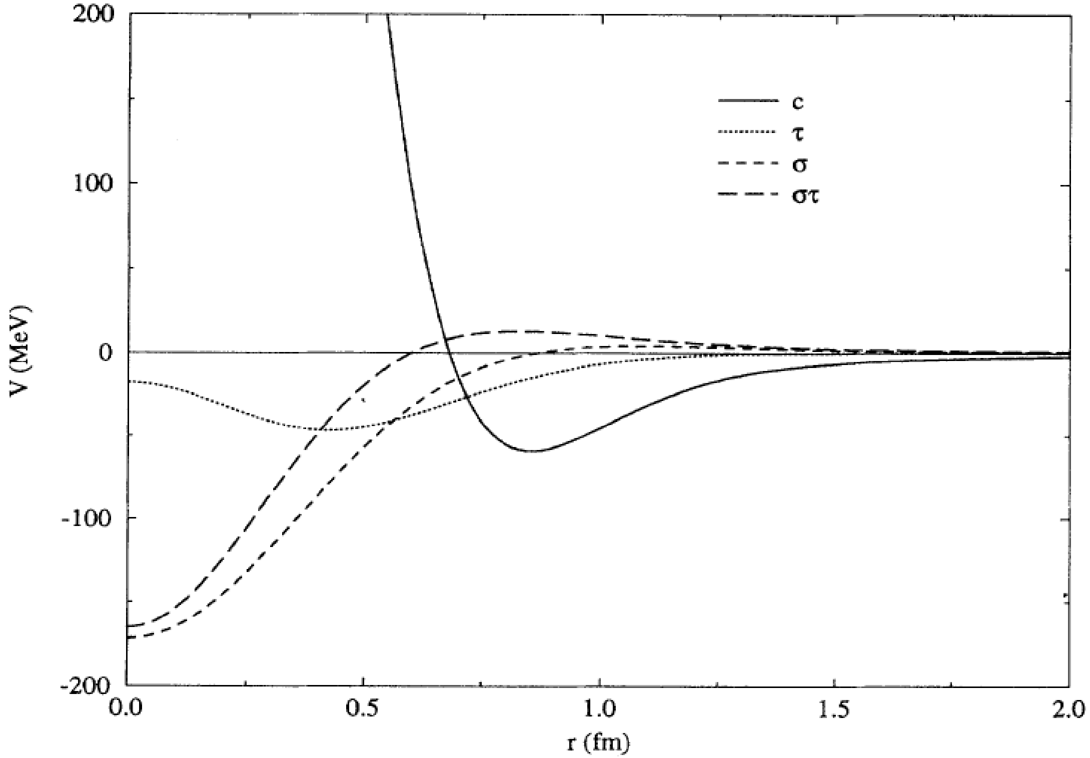
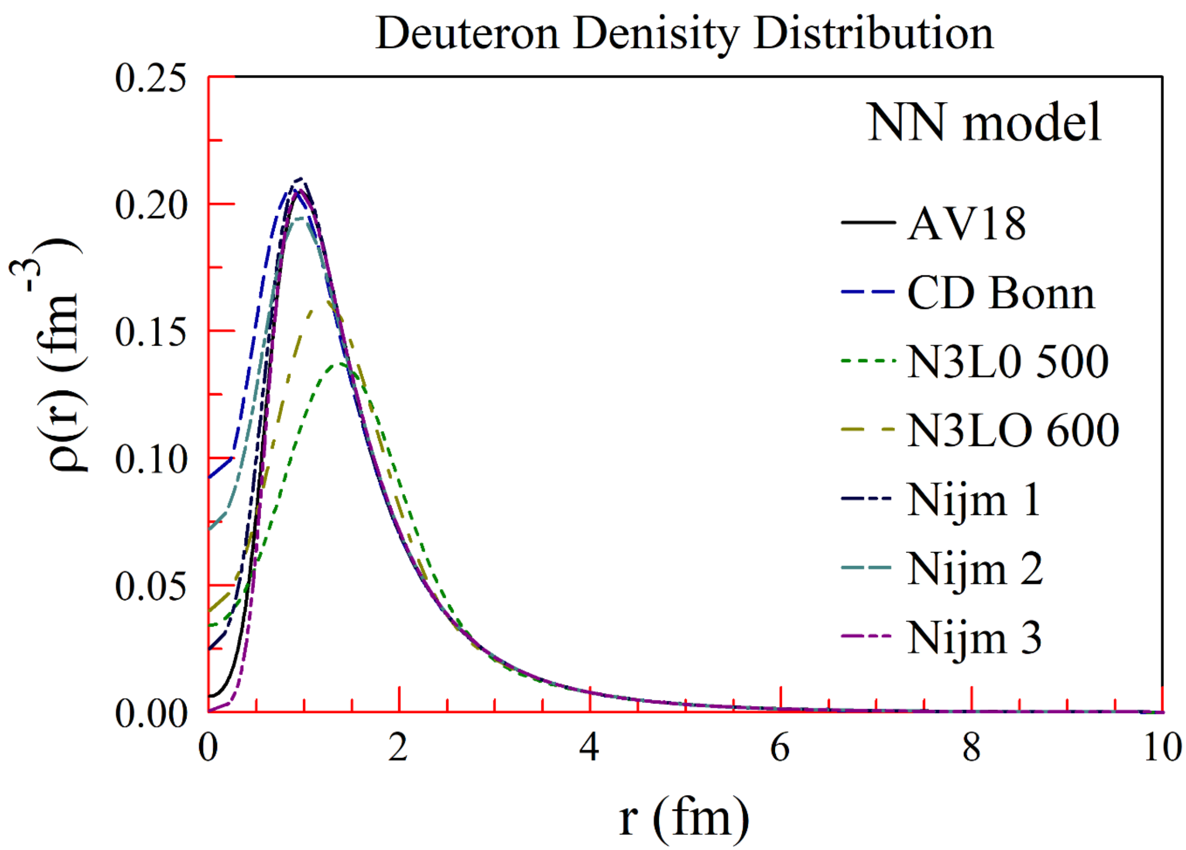


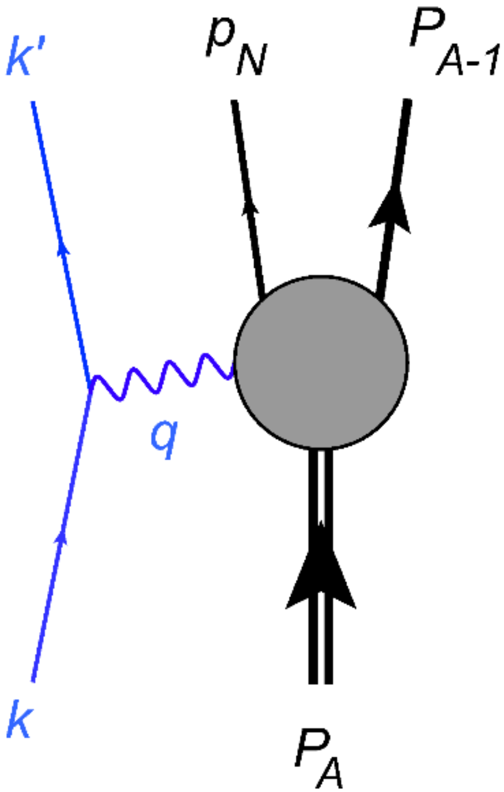
FIG. 6. Central, isospin, spin, and spin-isospin components of the potential. The central potential has a peak value of 2031 MeV at $r = 0$.



Deuteron Denisity Distribution
NN model
AV18
CD Bonn
N3L0 500
N3LO 600
Nijm 1
Nijm 2
Nijm 3
ρ(r) (fm^-3)
r (fm)
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10

Independent Particle Model
(Simple Shell Model)

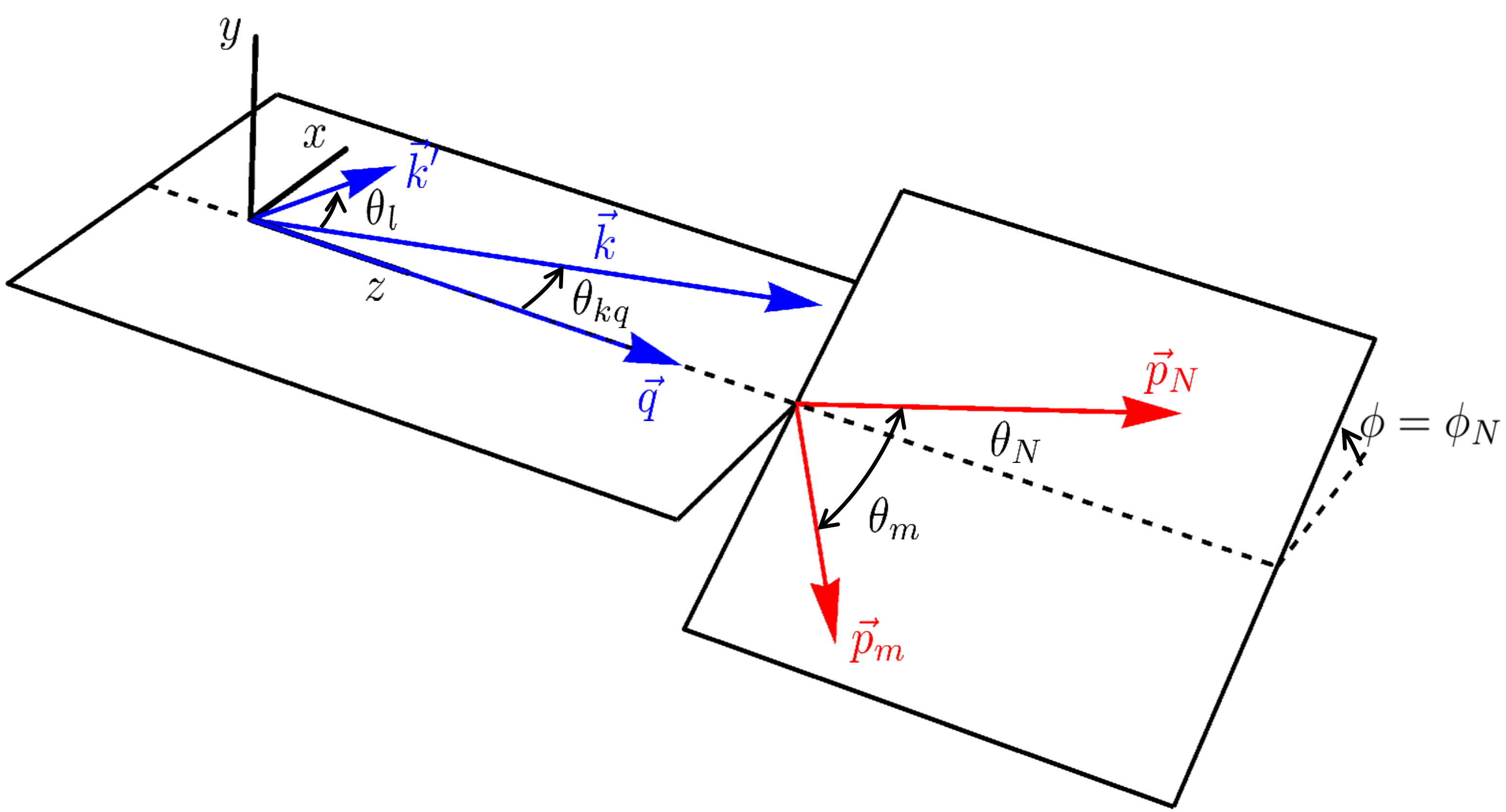
Electron Scattering



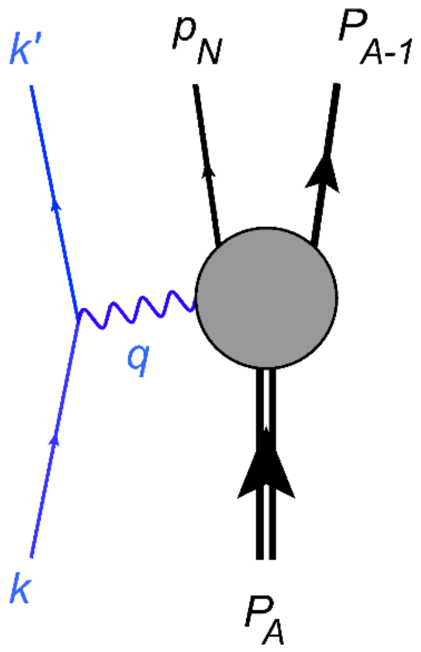
Current conservation

Kinematics

Fixed *q* frame variables



The Number of Free Kinematical Variables



5 four-momenta	+20
On-shell conditions	-5
Four-momentum conservation	-4
Choose z-axis	-2
Choose scattering plane	-1
Choose rest frame	-3
Independent degrees of freedom	**5**

Cross Sections

Inclusive Cross Section

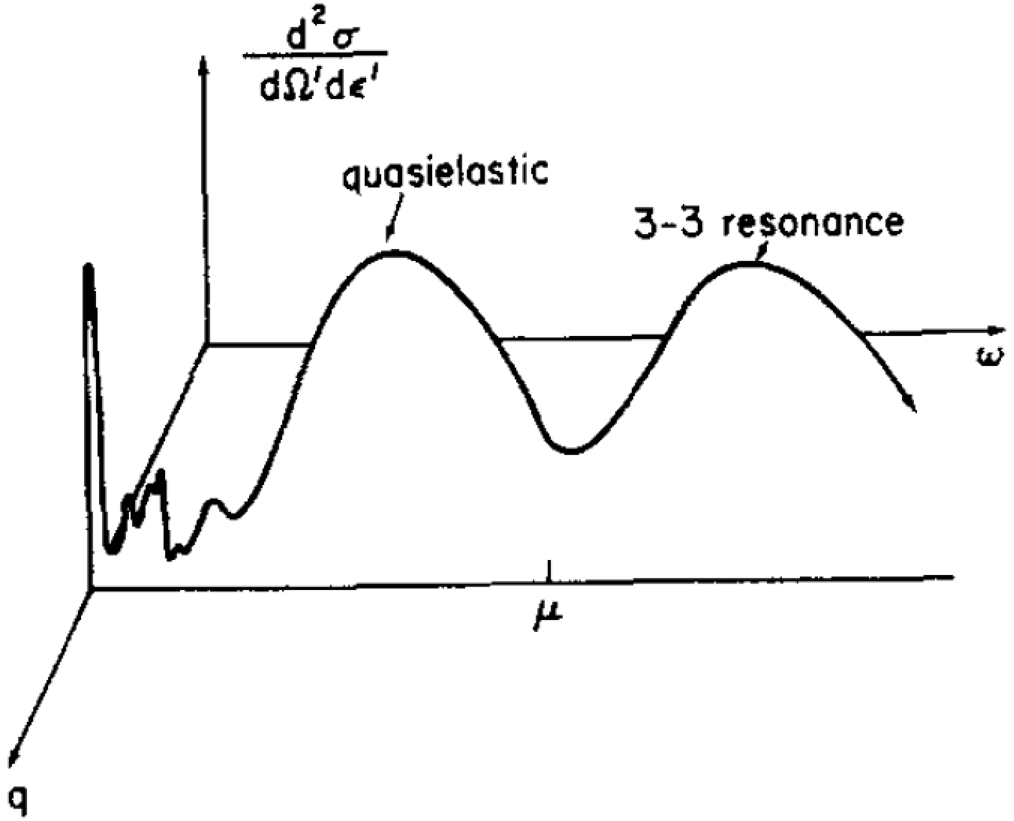
$$\frac{d\sigma^2}{dk' d\Omega_{k'}} = \frac{m_N}{4\pi^2} \sigma_{Mott} \left(v_L R_L + v_T R_T \right)$$

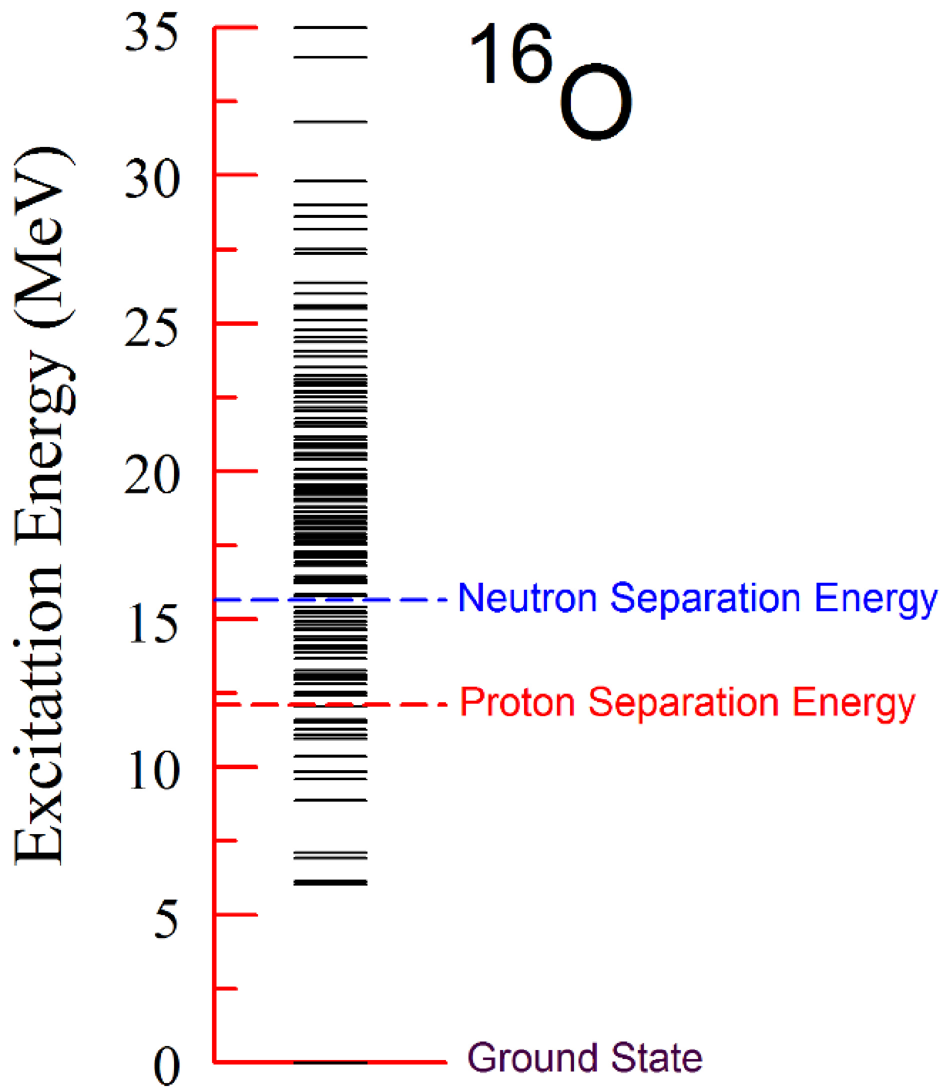
$$\sigma_{Mott} = \frac{\alpha \cos^2 \frac{\theta_l}{2}}{4k^2 \sin^4 \frac{\theta_l}{2}}$$

$$v_L = \frac{Q^4}{q^4}$$

$$v_T = \frac{Q^2}{2q^2} + \tan^2 \frac{\theta_l}{2}$$

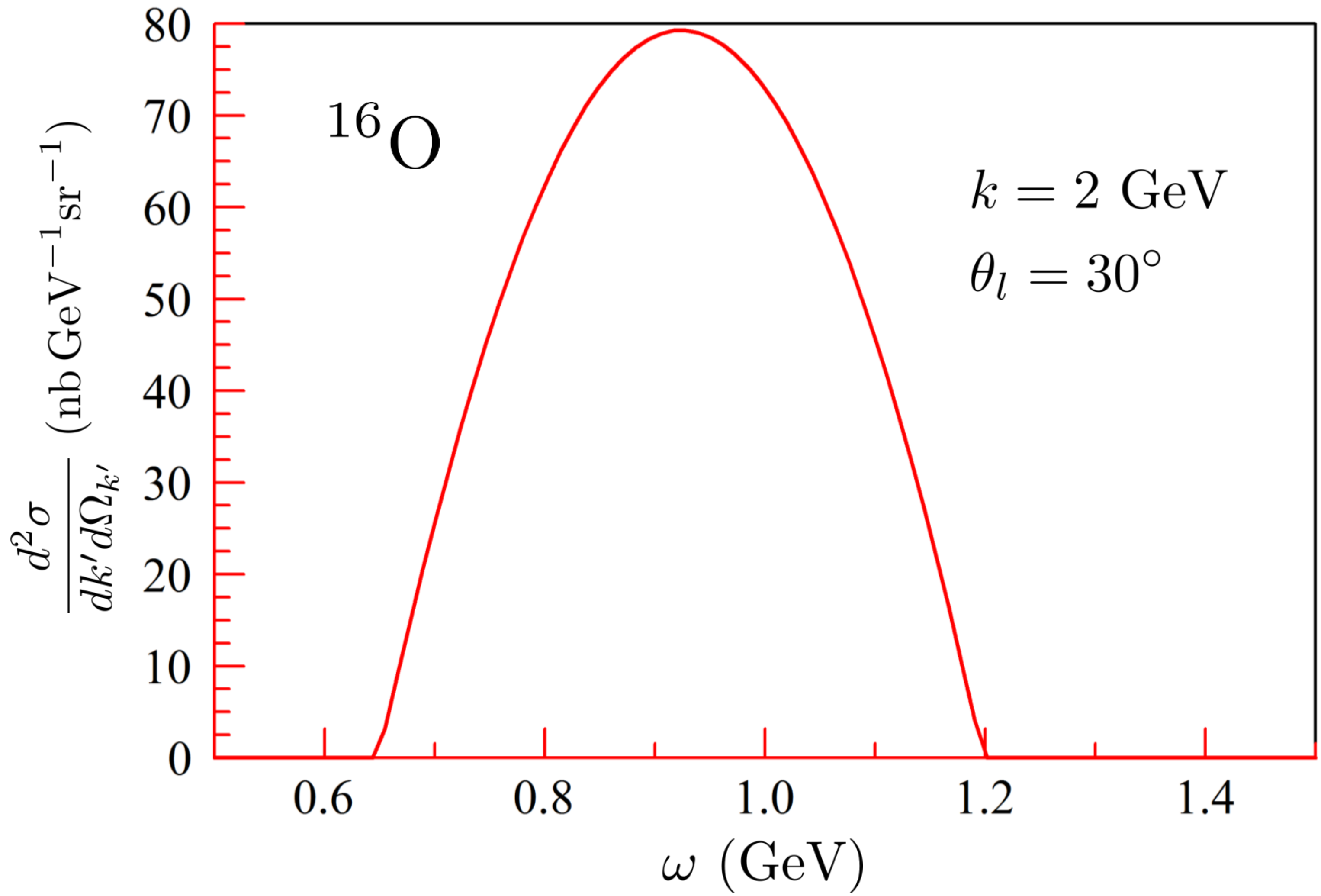
Inclusive "Quasielastic" Scattering



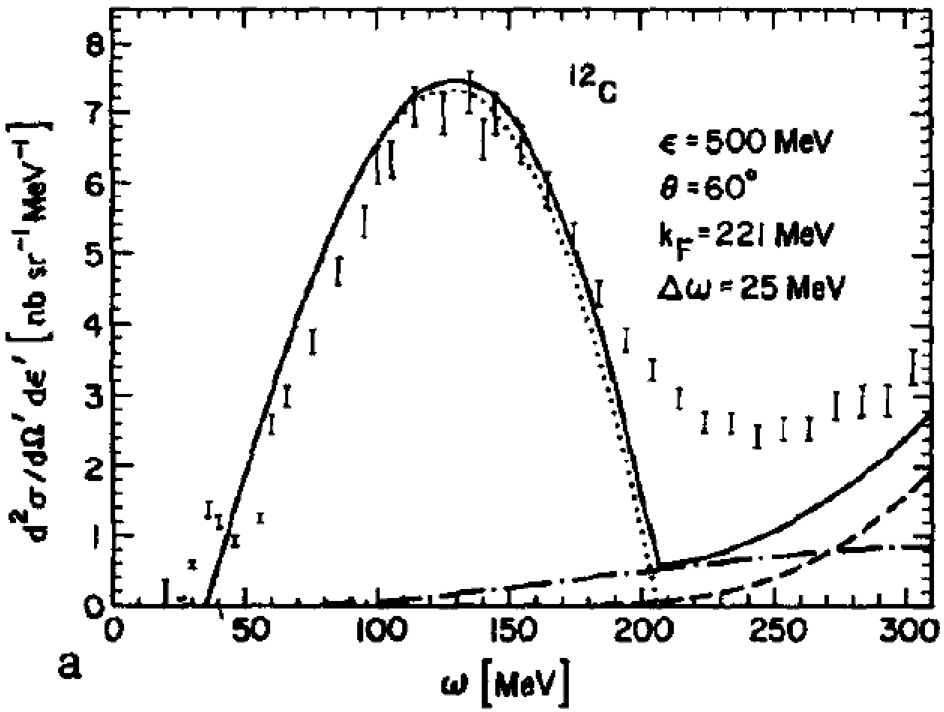


Excitattion Energy (MeV)
35
30
25
20
15
10
5
0
^{16}O
Neutron Separation Energy
Proton Separation Energy
Ground State

The Relativistic Fermi Gas Model

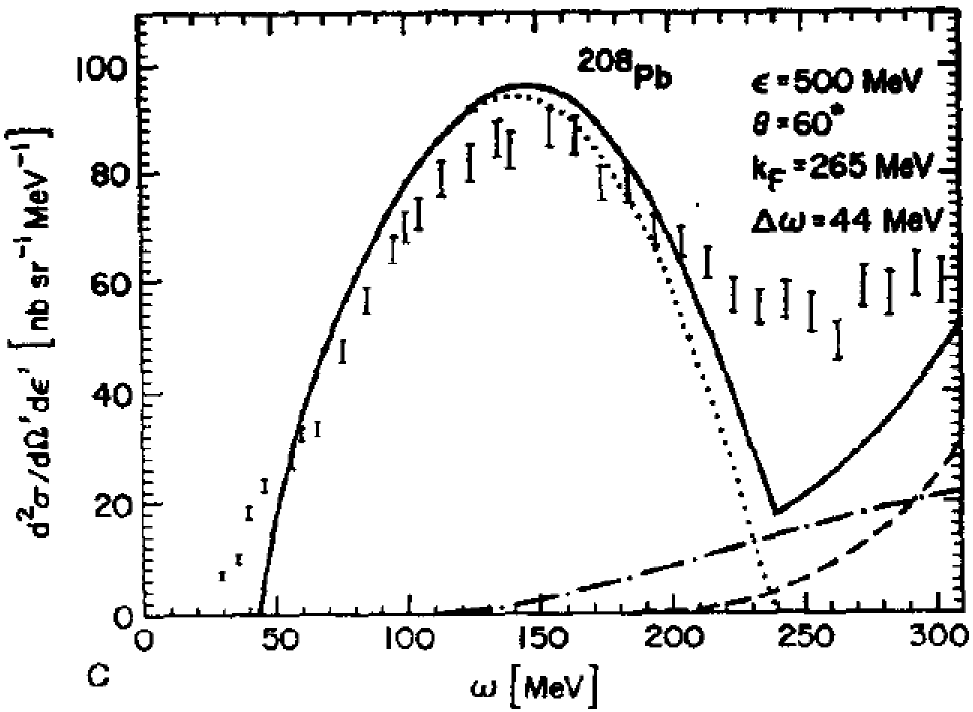


^{16}O
$k = 2$ GeV
$\theta_l = 30^\circ$
$\frac{d^2\sigma}{dk' d\Omega_{k'}}$ (nb GeV^{-1}sr^{-1})
ω (GeV)
0
10
20
30
40
50
60
70
80
0.6
0.8
1.0
1.2
1.4



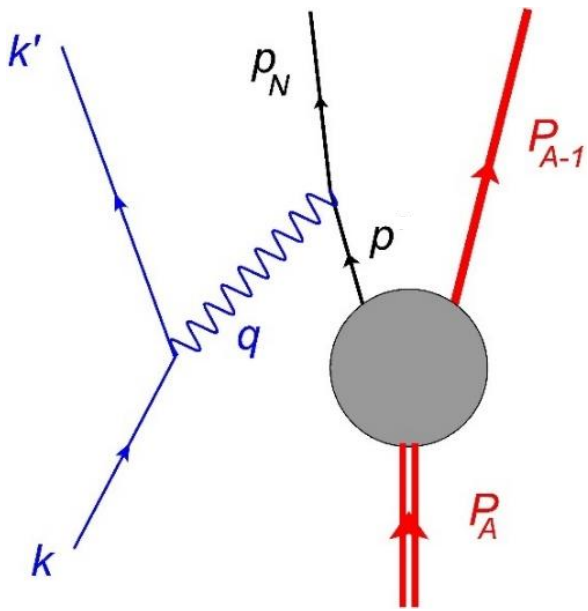
^{12}C
$\epsilon = 500$ MeV
$\theta = 60°$
$k_F = 221$ MeV
$\Delta\omega = 25$ MeV
$d^2\sigma/d\Omega' de'$ [nb sr^{-1} MeV^{-1}]
ω [MeV]
0
50
100
150
200
250
300
1
2
3
4
5
6
7
8

a

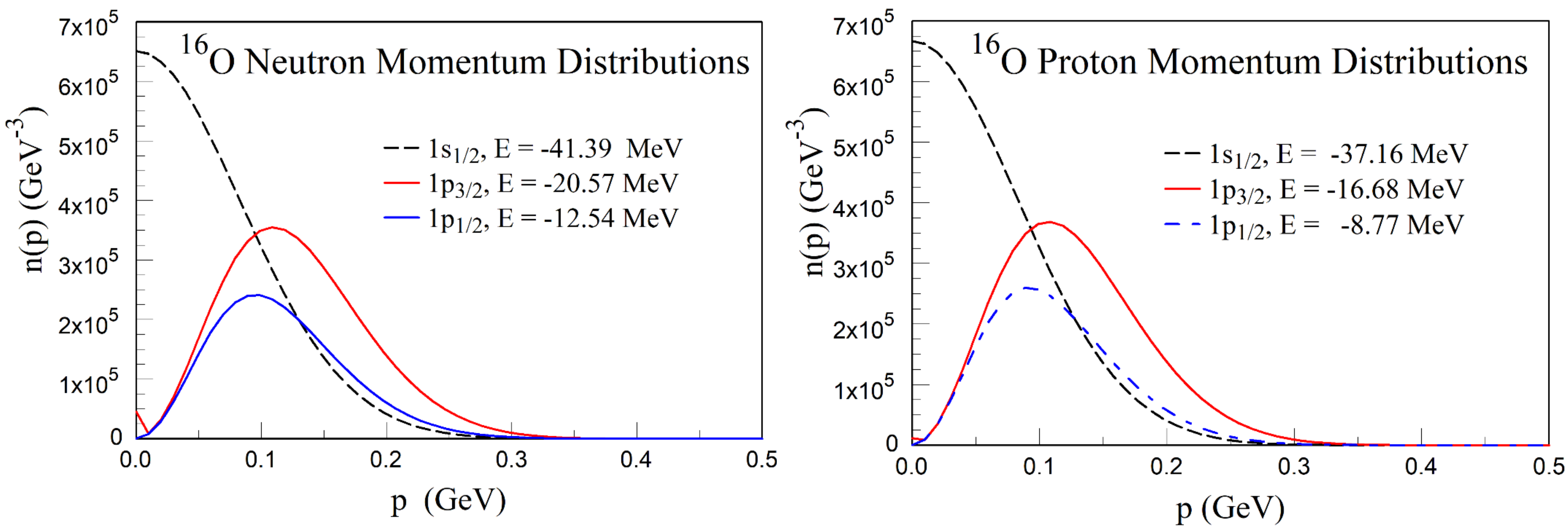


208Pb
ε = 500 MeV
θ = 60°
k_F = 265 MeV
Δω = 44 MeV
d²σ/dΩ′dε′ [nb sr⁻¹ MeV⁻¹]
ω [MeV]
0
20
40
60
80
100
0
50
100
150
200
250
300
C

The Plane Wave Impulse Approximation



The Relativistic Mean Field Model



^{16}O Neutron Momentum Distributions
$1s_{1/2}$, E = -41.39 MeV
$1p_{3/2}$, E = -20.57 MeV
$1p_{1/2}$, E = -12.54 MeV
n(p) (GeV^{-3})
p (GeV)
^{16}O Proton Momentum Distributions
$1s_{1/2}$, E = -37.16 MeV
$1p_{3/2}$, E = -16.68 MeV
$1p_{1/2}$, E = -8.77 MeV
n(p) (GeV^{-3})
p (GeV)

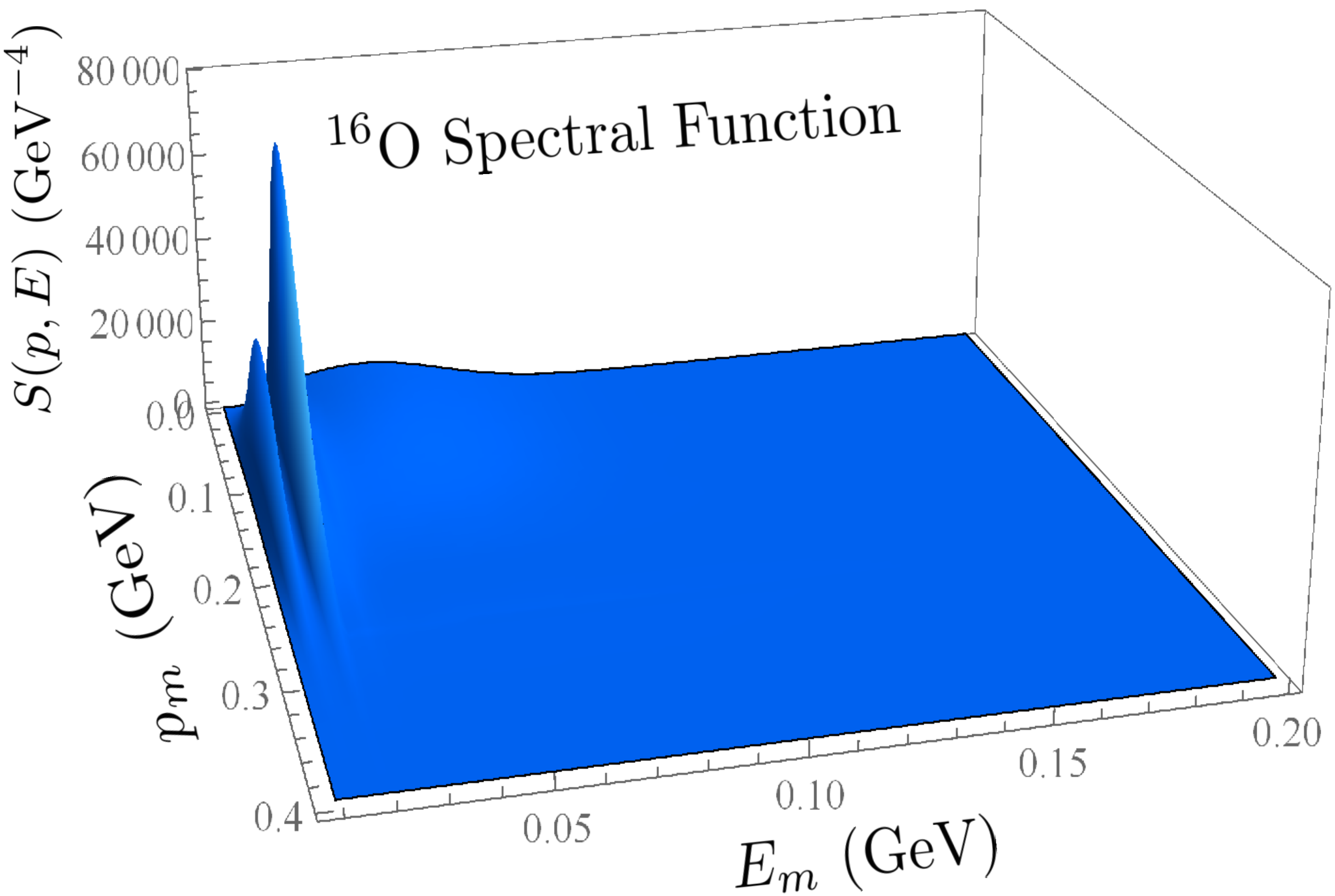
The Spectral Function

$$
\begin{aligned}
W^{\mu\nu} =& \sum_{s_N}\sum_{s_A}\sum_{s_{A-1}}\sum_{s_m} \bar{u}(\boldsymbol{p}_N, s_N)_a J^\nu(q)_{ab} \Psi(p_m, s_m; P_{A-1}, s_{A-1}; P_A, s_A)_{bc} \\
& \times \bar{\Psi}(p_m, s_m; P_{A-1}, s_{A-1}; P_A, s_A)_{cd} J^\mu(-q)_{de} u(\boldsymbol{p}_N, s_N)_e \\
=& \sum_{s_N} \bar{u}(\boldsymbol{p}_N, s_N)_a J^\nu(q)_{ab} \frac{1}{8\pi} \Lambda^+(\boldsymbol{p}_m)_{bd} S(p_m, E_m) J^\mu(-q)_{de} u(\boldsymbol{p}_N, s_N)_e \\
=& \frac{1}{8\pi} \mathrm{Tr}\left[J^\mu(-q)\Lambda^+(\boldsymbol{p}_N) J^\nu(q) \Lambda^+(\boldsymbol{p}_m)\right] S(p_m, E_m) \\
=& \frac{1}{8\pi} w^{\mu\nu}(P_A - P_{A-1}, Q) S(p_m, E_m)
\end{aligned}
$$

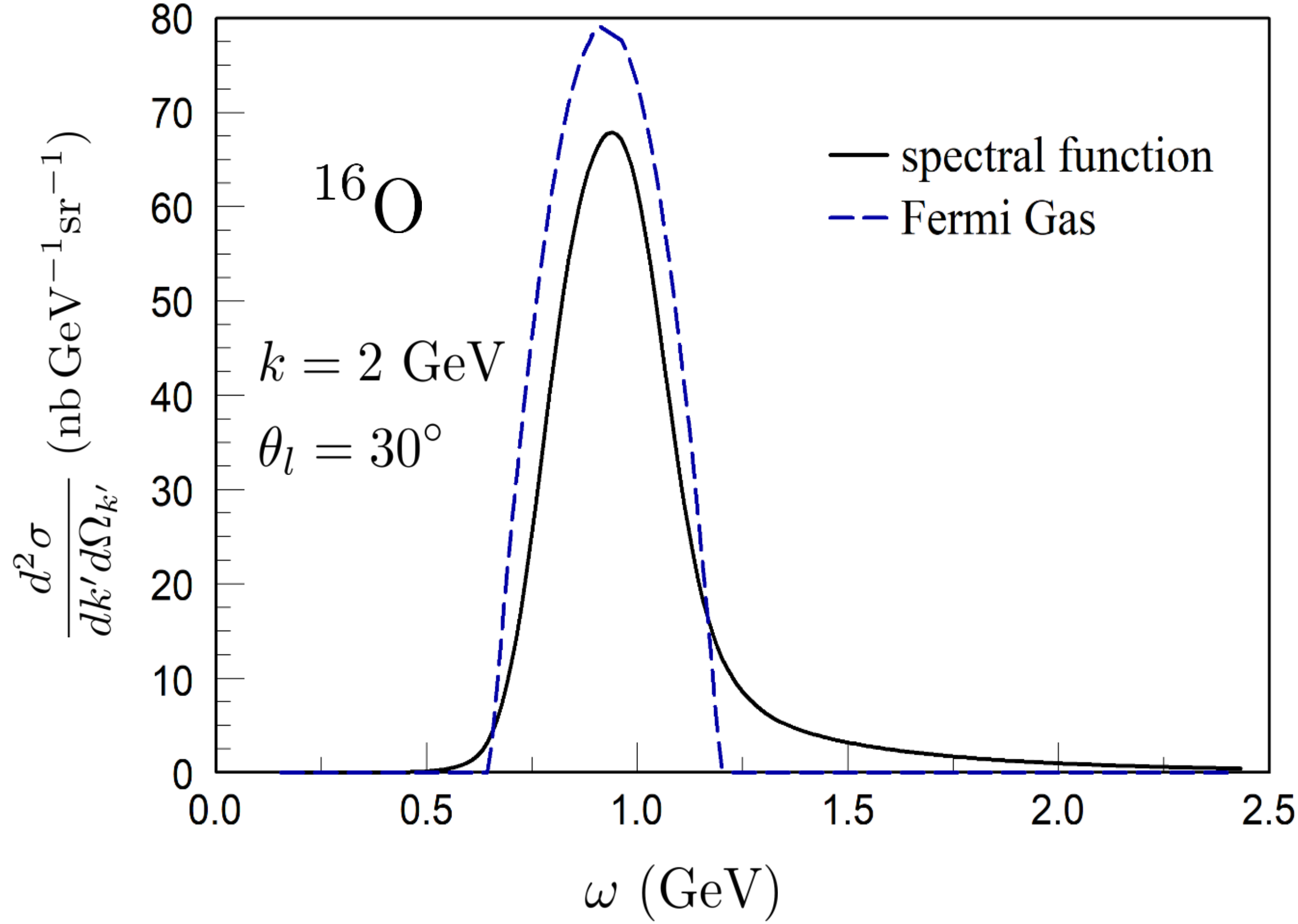
where $E_m = E_s + \mathcal{E}$ and $\mathcal{E} = \sqrt{p_m^2 + W_{A-1}^2} - \sqrt{p_m^2 + M_{A-1}^2}$

The Spectral Function is normalized such that:

$$\int_0^\infty dE_m S(p_m, E_m) = n(p_m) \qquad \frac{1}{(2\pi)^3}\int_0^\infty dp_m\, p_m^2 n(p_m) = N$$

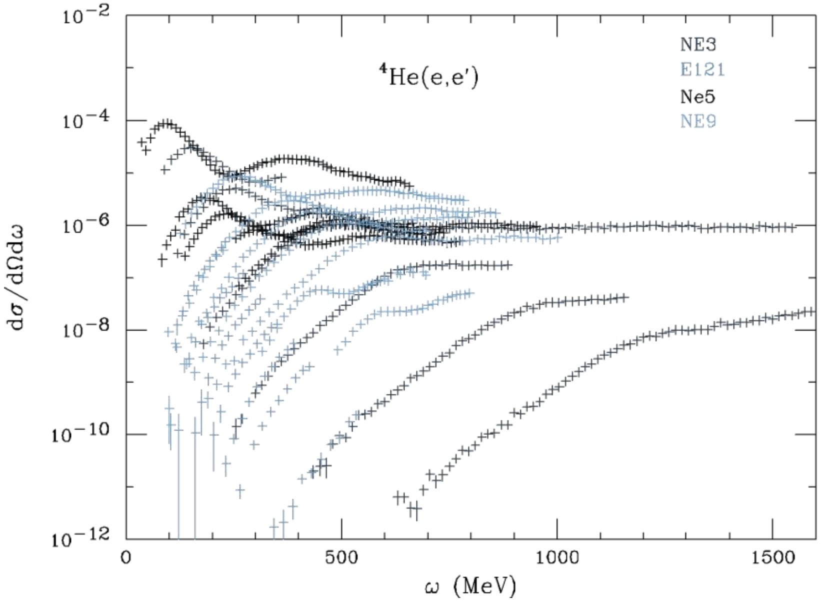


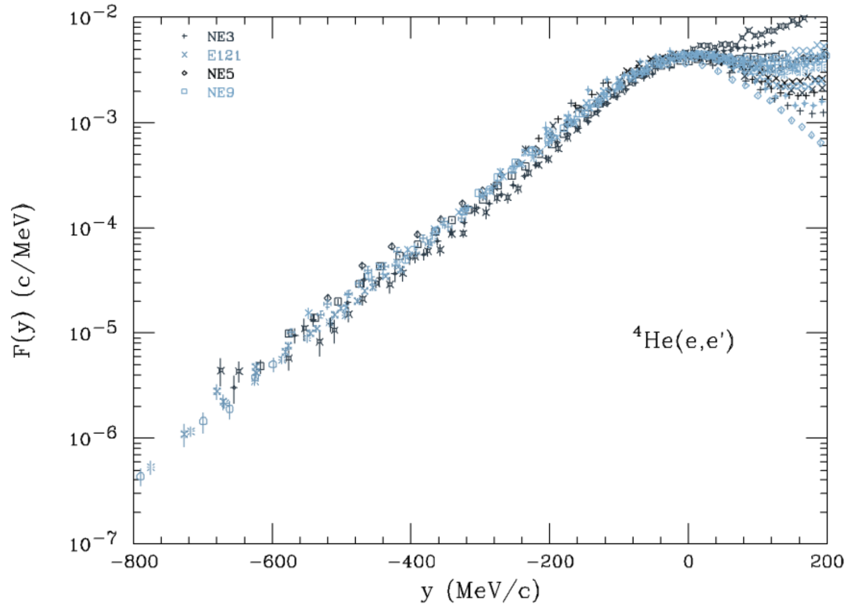
^{16}O Spectral Function
$S(p,E)$ (GeV^{-4})
80 000
60 000
40 000
20 000
0
0.0
0.1
0.2
0.3
0.4
p_m (GeV)
0.05
0.10
0.15
0.20
E_m (GeV)

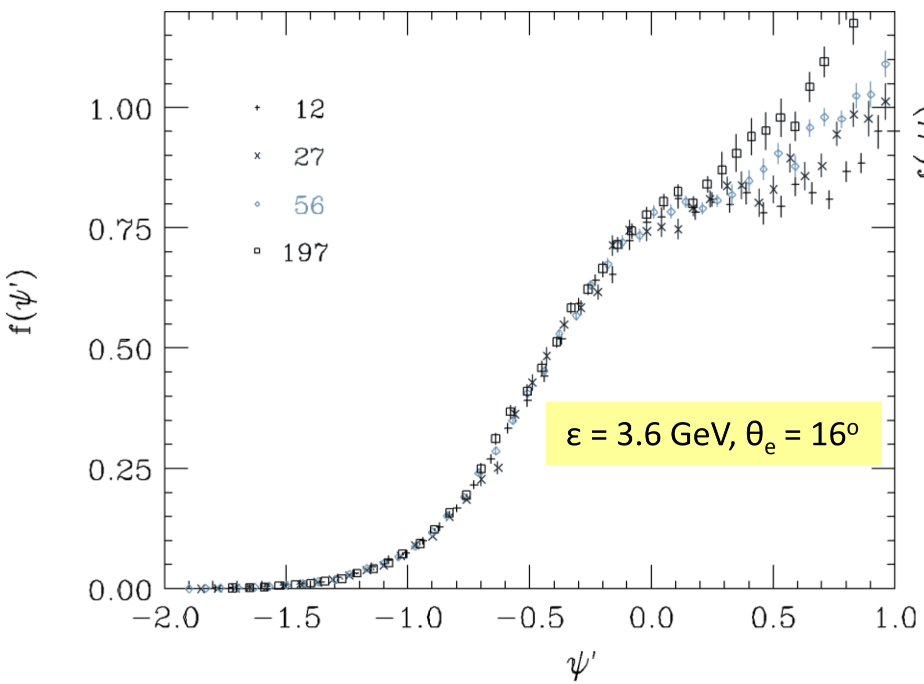


80
70
60
50
40
30
20
10
0
0.0
0.5
1.0
1.5
2.0
2.5
^{16}O
$k = 2$ GeV
$\theta_l = 30^\circ$
spectral function
Fermi Gas
$\frac{d^2\sigma}{dk' d\Omega_{k'}}$ (nb GeV^{-1}sr^{-1})
ω (GeV)

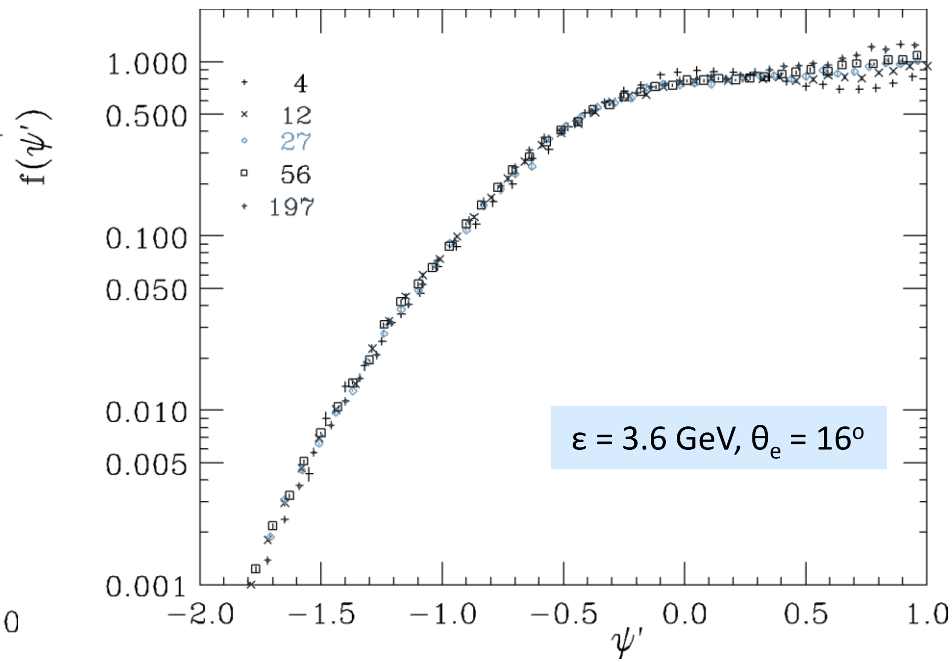
Scaling and Super Scaling





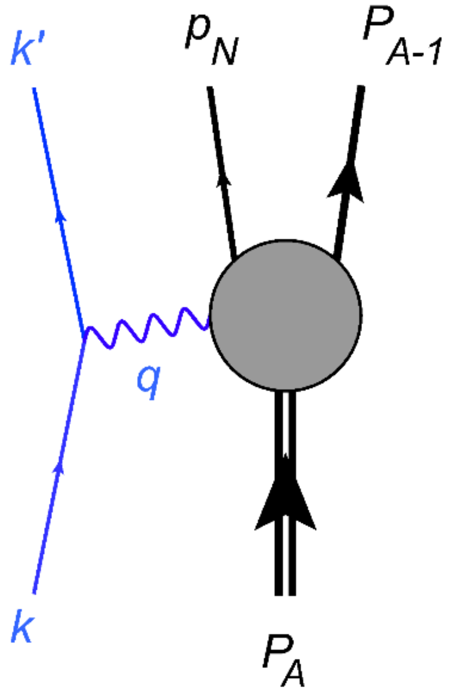


1.00
0.75
0.50
0.25
0.00
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
f(ψ')
ψ'
12
27
56
197
ε = 3.6 GeV, θe = 16o



1.000
0.500
0.100
0.050
0.010
0.005
0.001
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
f(ψ')
ψ'
4
12
27
56
197
ε = 3.6 GeV, θe = 16o

Semi-Inclusive Scattering



$$\left(\frac{d\sigma^4}{dk' d\Omega_{k'} dp_N d\Omega_N}\right)_h = \frac{m_N\, p_N^2}{(2\pi)^3\, E_N}\, \sigma_{Mott} \Big[v_L R_L^{(I)} + v_T R_T^{(I)} + v_{TT} R_{TT}^{(I)} \cos 2\phi_N + v_{LT} R_{LT}^{(I)} \cos\phi_N + h\, v_{LT'} R_{LT'}^{(II)} \sin\phi_N \Big]$$

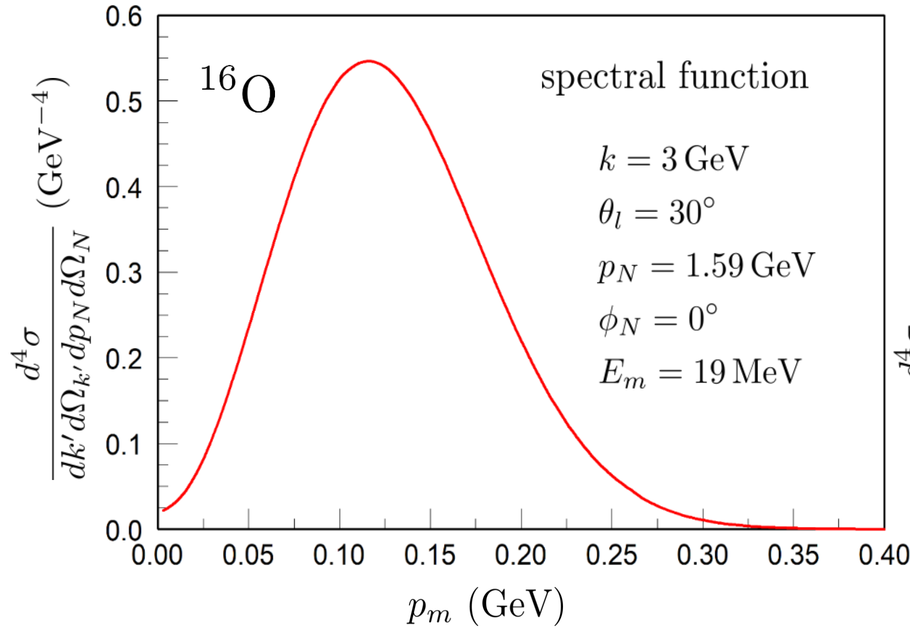
$$v_L = \frac{Q^4}{q^4}$$

$$v_T = \frac{Q^2}{2q^2} + \tan^2\frac{\theta_l}{2}$$

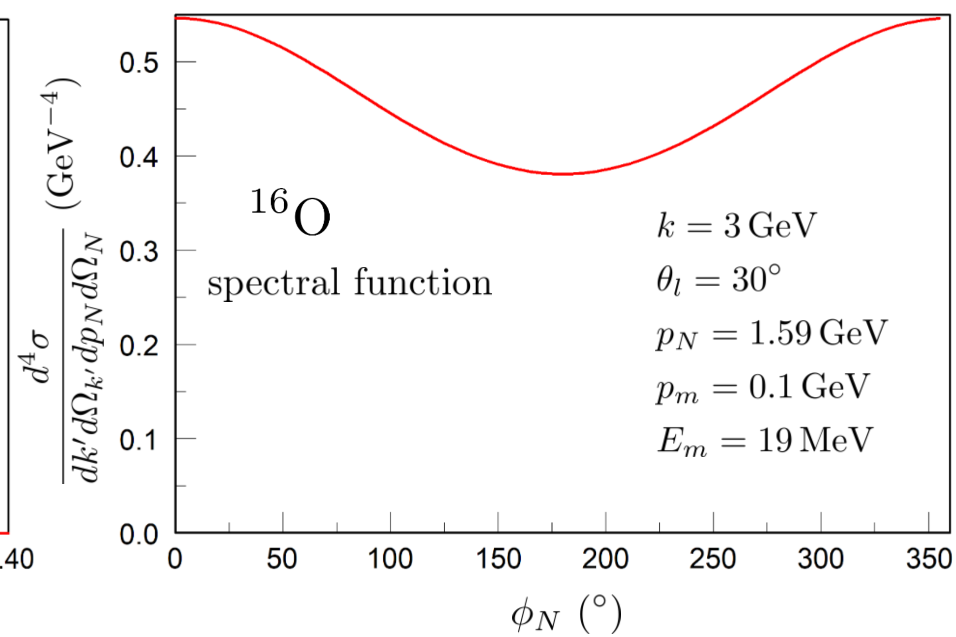
$$v_{TT} = -\frac{Q^2}{2q^2}$$

$$v_{LT} = -\frac{Q^2}{\sqrt{2}q^2}\sqrt{\frac{Q^2}{q^2} + \tan^2\frac{\theta_l}{2}}$$

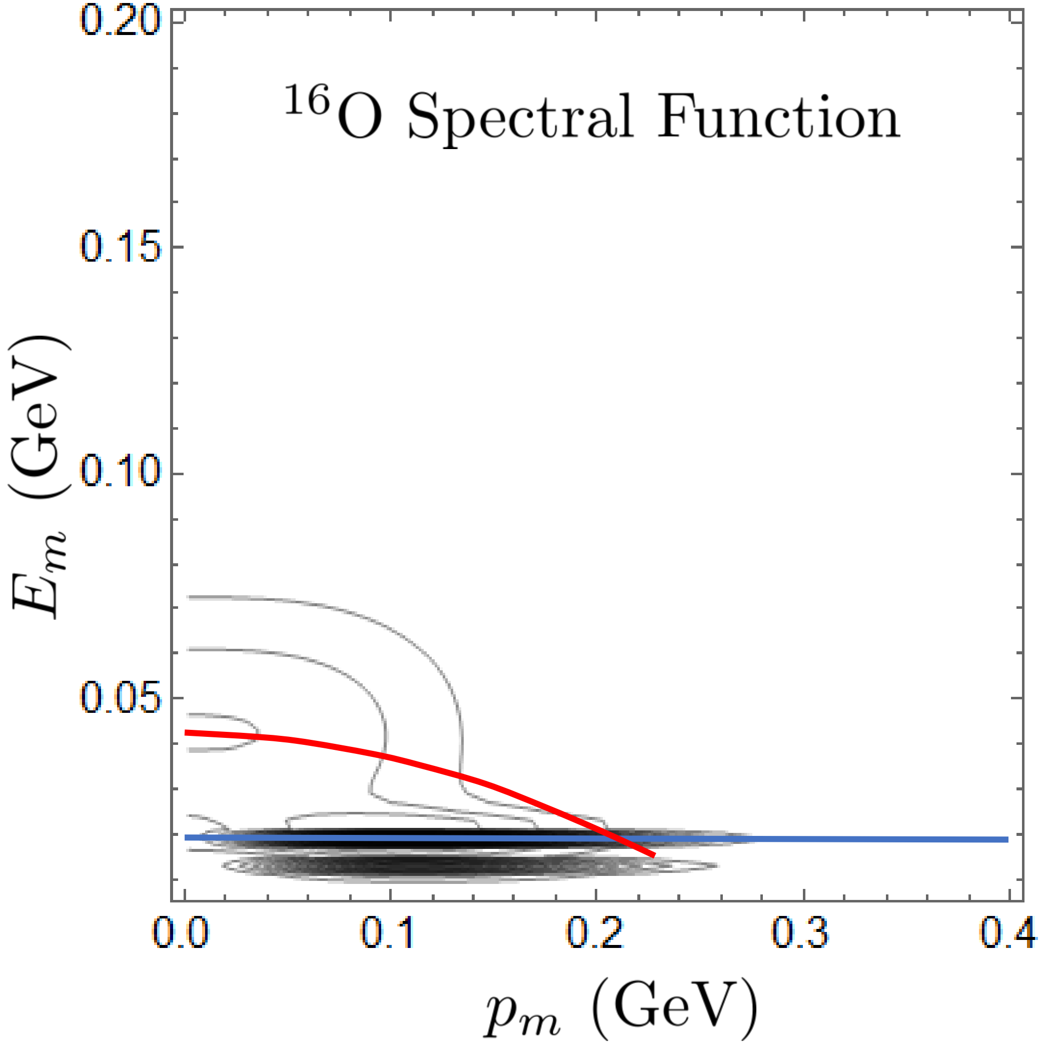
$$v_{LT'} = -\frac{Q^2}{\sqrt{2}q^2}\tan\frac{\theta_l}{2}\sqrt{\frac{Q^2}{q^2} + \tan^2\frac{\theta_l}{2}}$$



$\frac{d^4\sigma}{dk' d\Omega_{k'} dp_N d\Omega_N}$ (GeV^{-4})
0.6
0.5
0.4
0.3
0.2
0.1
0.0
^{16}O
spectral function
$k = 3\,\mathrm{GeV}$
$\theta_l = 30^\circ$
$p_N = 1.59\,\mathrm{GeV}$
$\phi_N = 0^\circ$
$E_m = 19\,\mathrm{MeV}$
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40
p_m (GeV)

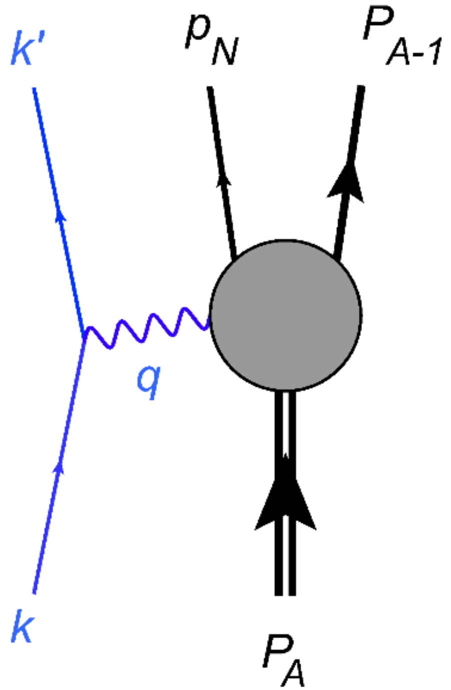


$\frac{d^4\sigma}{dk' d\Omega_{k'} dp_N d\Omega_N}$ (GeV^{-4})
0.5
0.4
0.3
0.2
0.1
0.0
^{16}O
spectral function
$k = 3\,\mathrm{GeV}$
$\theta_l = 30^\circ$
$p_N = 1.59\,\mathrm{GeV}$
$p_m = 0.1\,\mathrm{GeV}$
$E_m = 19\,\mathrm{MeV}$
0 50 100 150 200 250 300 350
ϕ_N ($^\circ$)



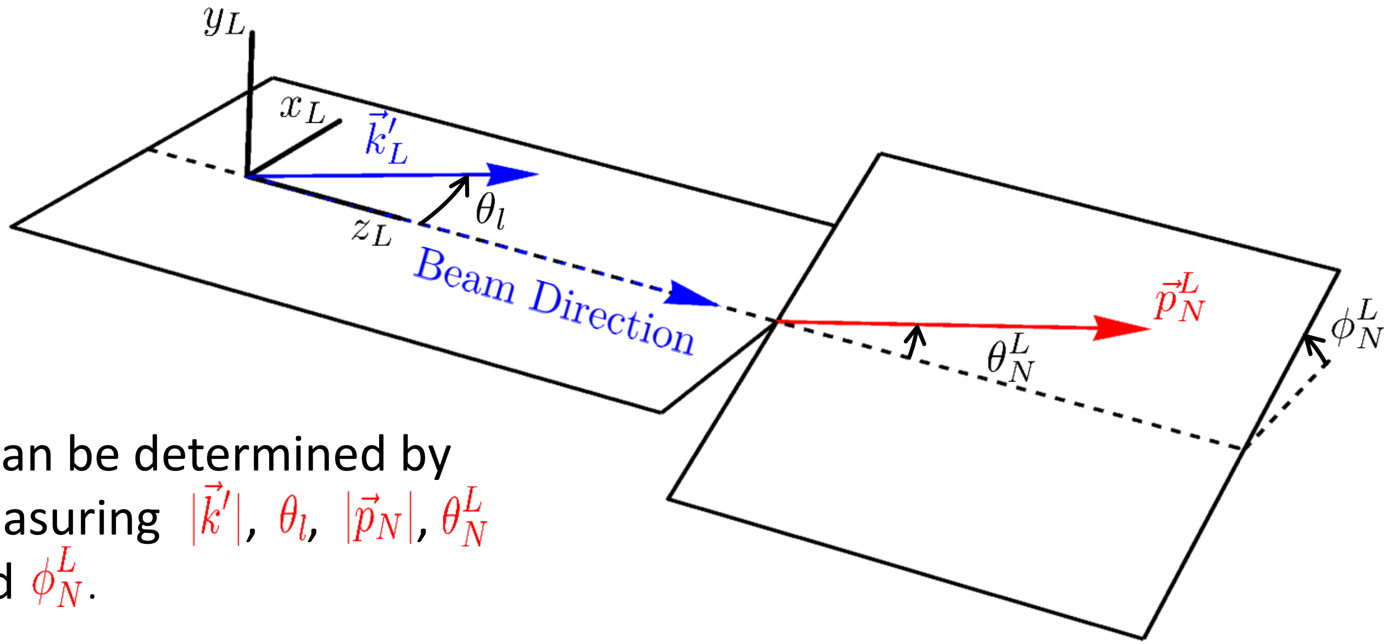
$$S_{RFG}(p_m, E_s + \mathcal{E}) = \frac{3(2\pi)^3 N}{k_F^3} \delta(\mathcal{E} - \sqrt{k_F^3 + m_n^2} + \sqrt{p_m^2 + m_n^2}) \theta(k_f - p_m)$$

Charge-Changing Neutrino Scattering (CCν)



Kinematic Variables in the "Lab" Frame

Since the objective is to determine the incident neutrino energy to study neutrino oscillations and given that the beam direction is known but not the incident momentum, it is best to consider the frame.



$|\vec{k}|$ can be determined by measuring $|\vec{k}'|$, θ_l, $|\vec{p}_N|$, θ^L_N and ϕ^L_N.

Inclusive Cross Section

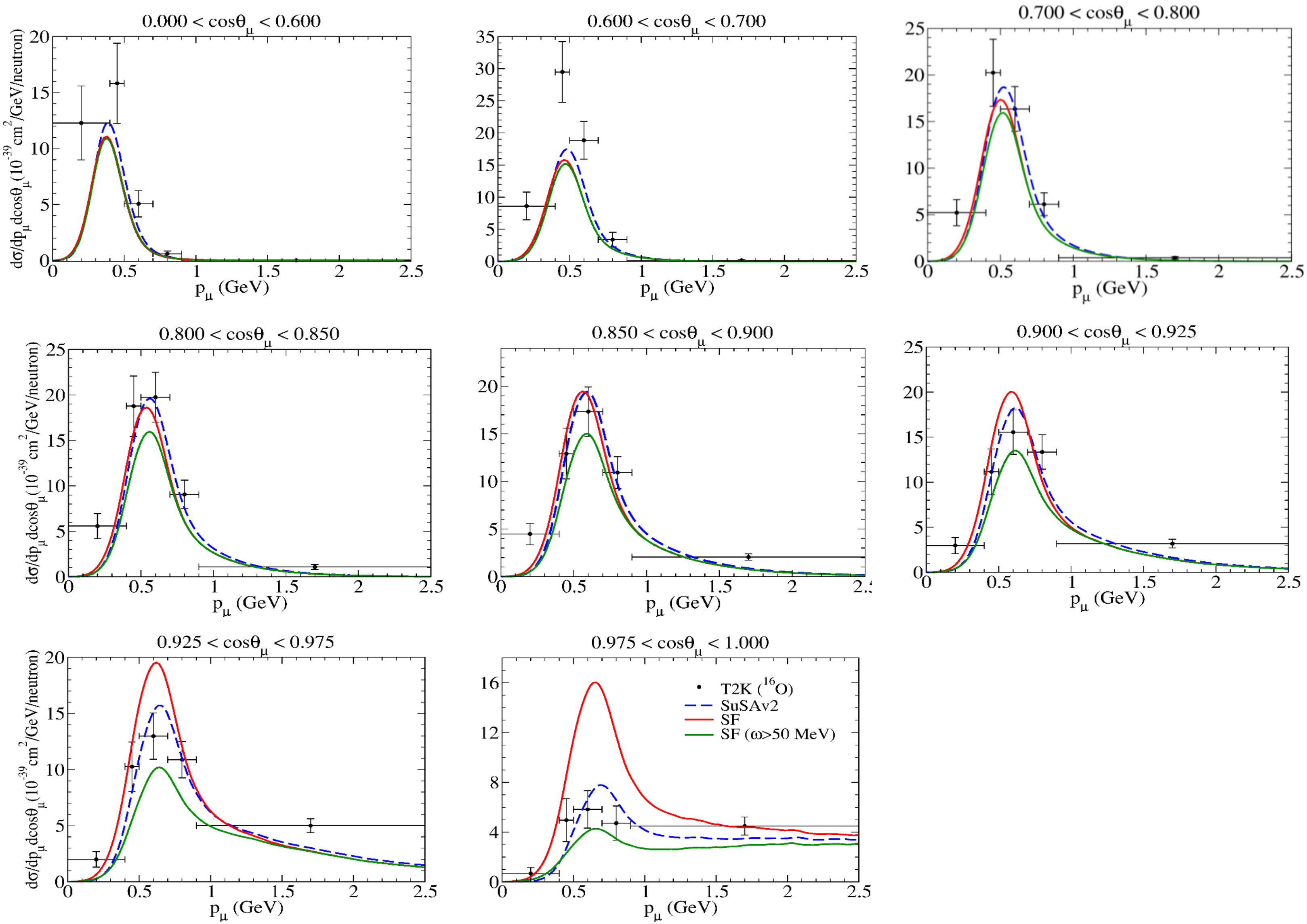
$$\left\langle \frac{d^4\sigma}{dk' d\Omega_{k'}} \right\rangle = \frac{G_F^2 \cos\theta_c m_N {k'}^2 v_0}{2(2\pi)^5 \varepsilon'} \int_{E_0}^{\infty} \frac{dk}{k} P(k) \Big[\hat{V}_{CC} (w_{CC}^{VV(I)} + w_{CC}^{AA(I)})$$
$$+ 2\hat{V}_{CL} (w_{CL}^{VV(I)} + w_{CL}^{AA(I)}) + \hat{V}_{LL} (w_{LL}^{VV(I)} + w_{LL}^{AA(I)}) + \chi \hat{V}_{T'} w_{T'}^{VA(I)} \Big]$$

$$E_0 = \varepsilon' + M_{A-1} + m_N - M_A$$

$$\chi = \begin{cases} -1 & \text{for neutrinos} \\ 1 & \text{for antineutrinos} \end{cases}$$

$$v_0 \equiv (\varepsilon + \varepsilon')^2 - q^2$$

Comparison to Recent Data From T2K



Semi-Inclusive Cross Section

$$\left\langle \frac{d^4\sigma}{dk' d\Omega_{k'} dp_N d\Omega_N^L} \right\rangle = \int_{M_{A-1}}^{\infty} dW_{A-1} \int_0^{\infty} dk \frac{G^2 \cos^2\theta_c m_N k'^2 \varepsilon\, p_N^2 W_{A-1}}{2(2\pi)^5 k\varepsilon' E_N \sqrt{X_B^2 + m^2 a_B}} v_0 \mathcal{F}_\chi^2 \delta(k-k_0) P(k)$$

$$= \int_{M_{A-1}}^{\infty} dW_{A-1} \frac{G^2 \cos^2\theta_c m_N k'^2 \varepsilon_0\, p_N^2 W_{A-1} v_0}{2(2\pi)^5 k_0 \varepsilon' E_N \sqrt{X_B^2 + m^2 a_B}} \mathcal{F}_\chi^2 P(k_0)$$

$$E_B = \varepsilon' + E_N - M_A$$

$$\boldsymbol{p}_B = \boldsymbol{k}' + \boldsymbol{p}_N$$

$$X_B = \frac{1}{2}\left(p_B^2 - E_B^2 + W_{A-1}^2 - m^2\right)$$

$$a_B = p_B^2 \cos^2\theta_B - E_B^2$$

$$k_0 = \frac{1}{a_B}\left(X_B p_B \cos\theta_B + E_B \sqrt{X_B^2 + m^2 a_B}\right)$$

$$\varepsilon_0 = \frac{1}{a_B}\left(E_B X_B + p_B \cos\theta_B \sqrt{X_B^2 + m^2 a_B}\right)$$

O. Moreno, T. W. Donnelly, W. P. Ford and J. W. Van Orden, Phys. Rev. D **90**, 013014 (2014).

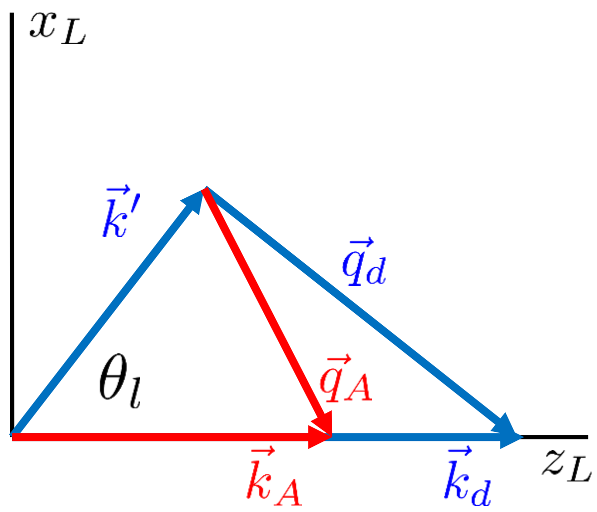
$$
\begin{aligned}
\mathcal{F}_\chi^2 =& \hat{V}_{CC}(w_{CC}^{VV(I)} + w_{CC}^{AA(I)}) + 2\hat{V}_{CL}(w_{CL}^{VV(I)} + w_{CL}^{AA(I)}) + \hat{V}_{LL}(w_{LL}^{VV(I)} + w_{LL}^{AA(I)}) \\
&+ \hat{V}_T(w_T^{VV(I)} + w_T^{AA(I)}) \\
&+ \hat{V}_{TT}\left[(w_{TT}^{VV(I)} + w_{TT}^{AA(I)})\cos 2\phi_N + (w_{TT}^{VV(II)} + w_{TT}^{AA(II)})\sin 2\phi_N\right] \\
&+ \hat{V}_{TC}\left[(w_{TC}^{VV(I)} + w_{TC}^{AA(I)})\cos \phi_N + (w_{TC}^{VV(II)} + w_{TC}^{AA(II)})\sin \phi_N)\right] \\
&+ \hat{V}_{TL}\left[(w_{TL}^{VV(I)} + w_{TL}^{AA(I)})\cos \phi_N + (w_{TL}^{VV(II)} + w_{TL}^{AA(II)})\sin \phi_N\right] \\
&+ \chi\left[\hat{V}_{T'}w_{T'}^{VA(I)} + \hat{V}_{TC'}(w_{TC'}^{VA(I)}\sin \phi_N + w_{TC'}^{VA(II)}\cos \phi_N)\right. \\
&\left. + \hat{V}_{TL'}(w_{TL'}^{VA(I)}\sin \phi_N + w_{TL'}^{VA(II)}\cos \phi_N)\right]
\end{aligned}
$$

$$
\begin{aligned}
\cos\theta_N &= \cos\theta_N^L \cos\theta_{kq} - \cos\phi_N^L \sin\theta_N^L \sin\theta_{kq} \\
\sin\theta_N &= \sqrt{1 - \cos^2\theta_N} \\
\cos\phi_N &= \frac{\cos\phi_N^L \sin\theta_N^L \cos\theta_{kq} + \cos\theta_N^L \sin\theta_{kq}}{\sin\theta_N} \\
\sin\phi_N &= \frac{\sin\phi_N^L \sin\theta_N^L}{\sin\theta_N}
\end{aligned}
$$

Mathematica

Heavy Water

$^2H_2{}^{16}O$
Kinematics



Optimize kinematics for the deuteron

$$s_d = (M_d + \omega)^2 - \boldsymbol{q}^2$$

$$y = \frac{(M_d + \omega)\sqrt{s(s - 4m_N^2)}}{2s} - \frac{|\boldsymbol{q}|}{2}$$

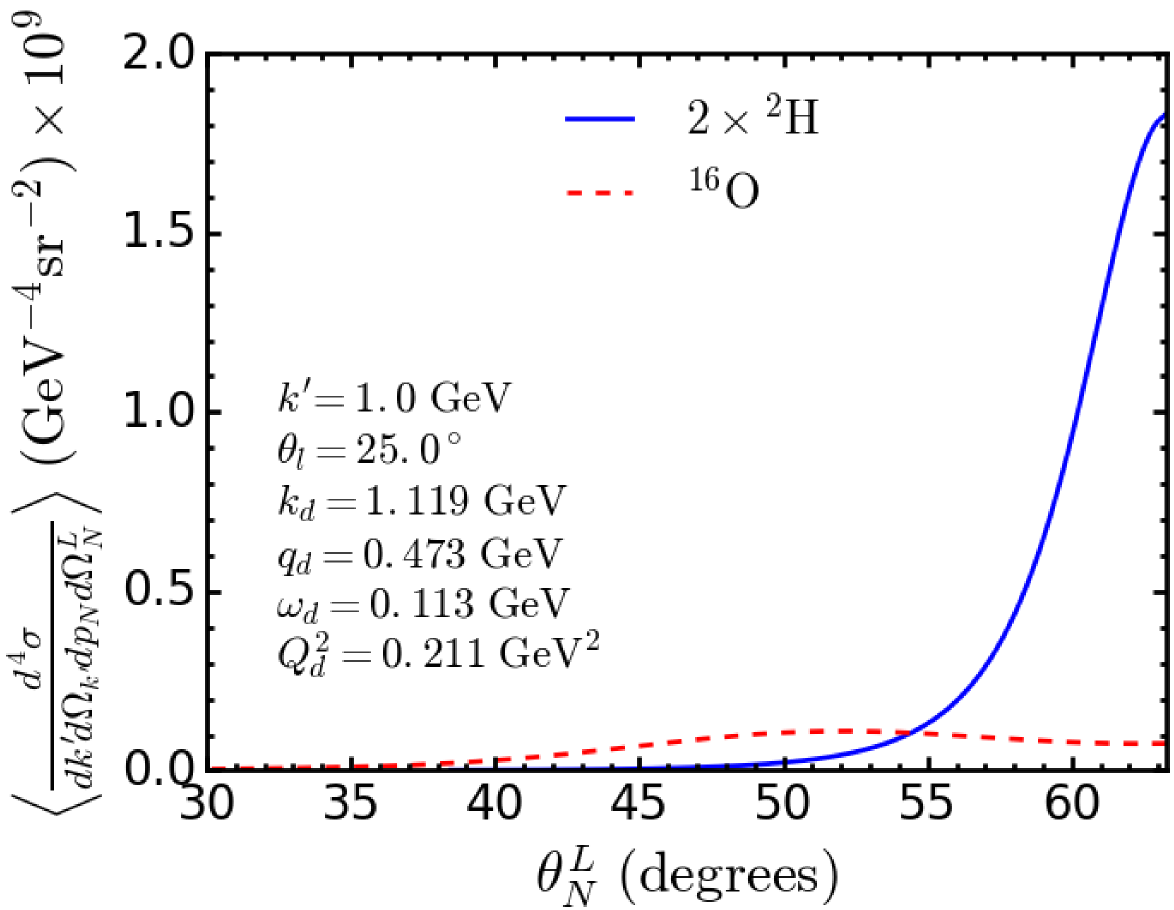
$$Y = y + |\boldsymbol{q}|$$

$$|y| \leq p \leq Y$$

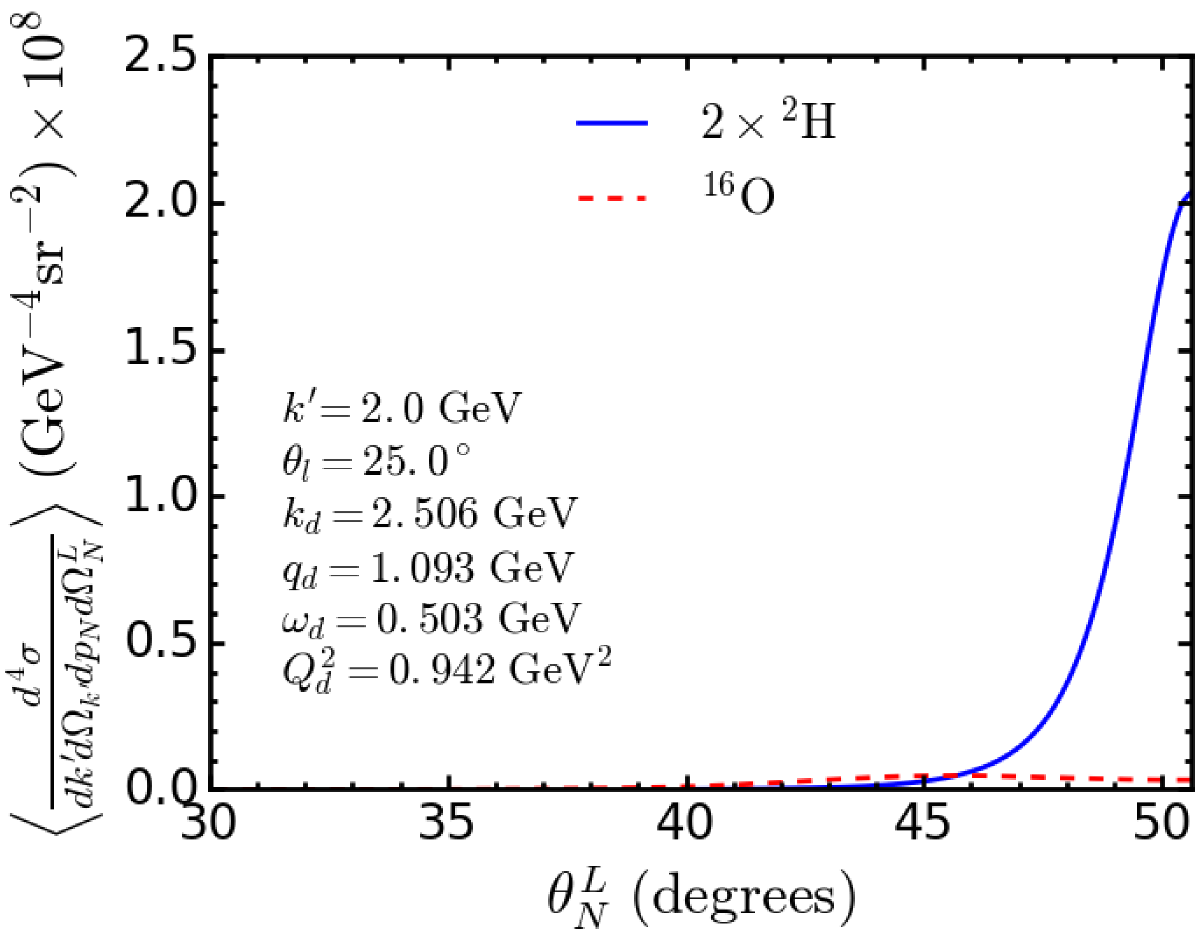
Given $|\boldsymbol{k}'|$ and θ_l, choose $|\boldsymbol{k}_d|$ such that $y = 0$.

Then $|\boldsymbol{p}_N|$, θ_N^L and ϕ_N^L can be determined as functions of $|\boldsymbol{p}_N - \boldsymbol{q}_d|$

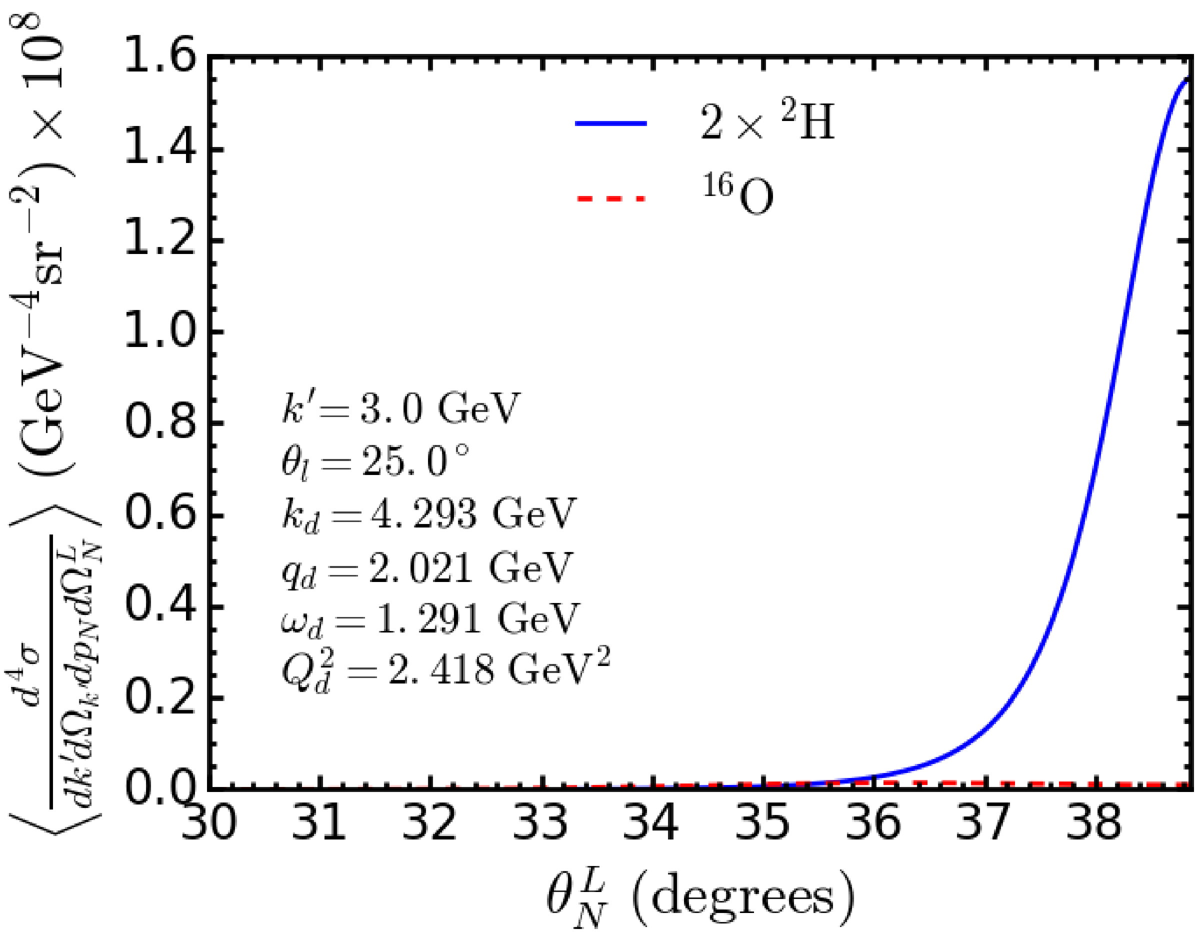
The cross section for semi-inclusive scattering from $^{16}\mathrm{O}$ can then be determined as a function of $|\boldsymbol{k}_A|$ and $|\boldsymbol{p}_N - \boldsymbol{q}_d|$.



$\left\langle \frac{d^4\sigma}{dk' d\Omega_k dp_N d\Omega_N^L} \right\rangle$ (GeV^{-4}sr^{-2}) $\times 10^9$
2.0
1.5
1.0
0.5
0.0
30
35
40
45
50
55
60
θ_N^L (degrees)
$2 \times {}^2$H
^{16}O
$k' = 1.0$ GeV
$\theta_l = 25.0^\circ$
$k_d = 1.119$ GeV
$q_d = 0.473$ GeV
$\omega_d = 0.113$ GeV
$Q_d^2 = 0.211$ GeV^2

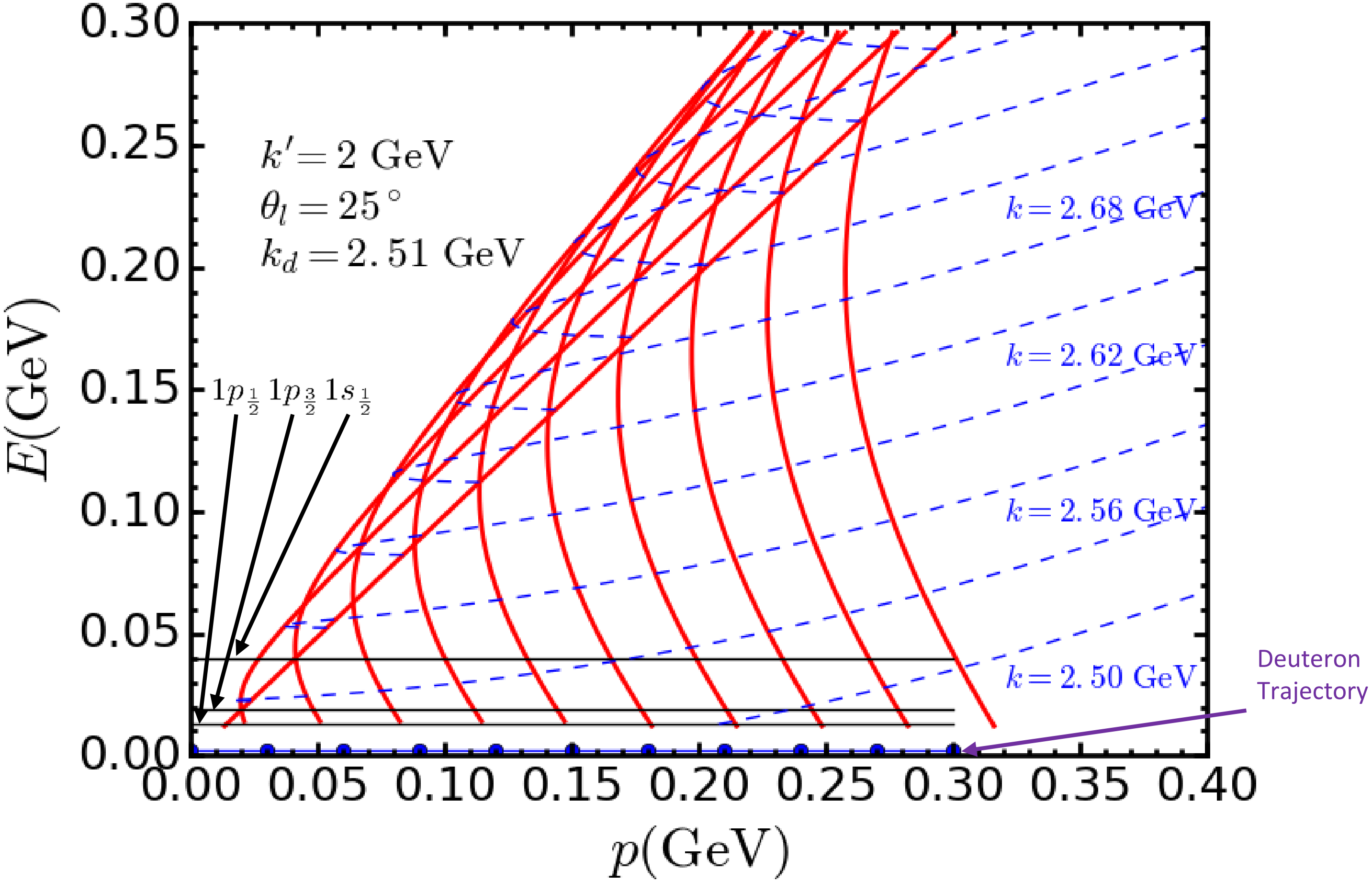


$\left\langle \frac{d^4\sigma}{dk' d\Omega_{k'} dp_N d\Omega_N^L} \right\rangle$ (GeV^{-4}sr^{-2}) $\times 10^8$
θ_N^L (degrees)
$2 \times {}^2$H
^{16}O
$k' = 2.0$ GeV
$\theta_l = 25.0^\circ$
$k_d = 2.506$ GeV
$q_d = 1.093$ GeV
$\omega_d = 0.503$ GeV
$Q_d^2 = 0.942$ GeV^2
2.5
2.0
1.5
1.0
0.5
0.0
30
35
40
45
50



$\left\langle \frac{d^4\sigma}{dk' d\Omega_k dp_N d\Omega_N^L} \right\rangle$ (GeV^{-4}sr^{-2}) $\times 10^8$
θ_N^L (degrees)
$2 \times {}^2$H
^{16}O
$k' = 3.0$ GeV
$\theta_l = 25.0^\circ$
$k_d = 4.293$ GeV
$q_d = 2.021$ GeV
$\omega_d = 1.291$ GeV
$Q_d^2 = 2.418$ GeV^2

Trajectories in *p* and *E*



$$
\begin{aligned}
\Delta_1 =& m^2 + {m'}^2 \\
\Delta_2 =& 2\varepsilon\varepsilon' - 2|\boldsymbol{k}||\boldsymbol{k}'| \\
\Delta_3 =& 4\boldsymbol{k}^2\boldsymbol{k}'^2 - 4\varepsilon^2\varepsilon'^2 \\
\Delta_4 =& m'^2 - m^2 \\
v_0 =& (\varepsilon + \varepsilon')^2 - \boldsymbol{q}^2 \\
=& \Delta_2 + \Delta_1 + 4|\boldsymbol{k}||\boldsymbol{k}'|\cos^2\frac{\theta_l}{2}
\end{aligned}
$$

$$
\kappa = \varepsilon + \varepsilon'
$$

$$
\begin{aligned}
\hat{V}_{CC} =& \left(1 + \frac{\Delta_1}{v_0}\right) \\
\hat{V}_{CL} =& -\frac{1}{|\boldsymbol{q}|}\left(\omega + \frac{\Delta_4\kappa}{v_0}\right) \\
\hat{V}_{LL} =& \left(\frac{\omega^2}{\boldsymbol{q}^2} - \frac{\Delta_1}{v_0} + \frac{\Delta_4^2}{\boldsymbol{q}^2 v_0} + \frac{2\Delta_4\kappa\omega}{\boldsymbol{q}^2 v_0}\right) \\
\hat{V}_{T} =& \left[Q^2\left(\frac{1}{2\boldsymbol{q}^2} + \frac{1}{v_0}\right) + \Delta_1\left(\frac{1}{2\boldsymbol{q}^2} - \frac{1}{v_0}\right) - \frac{\Delta_1^2 - \Delta_3 + \Delta_1 Q^2}{2\boldsymbol{q}^2 v_0}\right] \\
\hat{V}_{TT} =& -\left[\frac{\Delta_1 + Q^2}{2\boldsymbol{q}^2}\left(1 - \frac{\Delta_1}{v_0}\right) + \frac{\Delta_3}{2\boldsymbol{q}^2 v_0}\right] \\
\hat{V}_{TC} =& -\frac{1}{\sqrt{2}v_0}\sqrt{1 + \frac{v_0}{\boldsymbol{q}^2}}\sqrt{\Delta_3 + (\Delta_1 + Q^2)(v_0 - \Delta_1)} \\
\hat{V}_{TL} =& \frac{1}{\sqrt{2}\boldsymbol{q}^2 v_0}\sqrt{\Delta_3 + (\Delta_1 + Q^2)(v_0 - \Delta_1)}\,(\Delta_4 + \omega\kappa) \\
\hat{V}_{T'} =& \frac{1}{v_0}\left(Q^2\sqrt{1 + \frac{v_0}{\boldsymbol{q}^2}} - \frac{\Delta_4\omega}{|\boldsymbol{q}|}\right) \\
\hat{V}_{TC'} =& -\frac{1}{\sqrt{2}v_0}\sqrt{\Delta_3 + (\Delta_1 + Q^2)(v_0 - \Delta_1)} \\
\hat{V}_{TL'} =& \frac{1\omega}{\sqrt{2}|\boldsymbol{q}|v_0}\sqrt{\Delta_3 + (\Delta_1 + Q^2)(v_0 - \Delta_1)}
\end{aligned}
$$