

# Medium Modification of the Inclusive Gluon Spectrum at an EIC

**Matthew D. Sievert**  
with Ivan Vitev



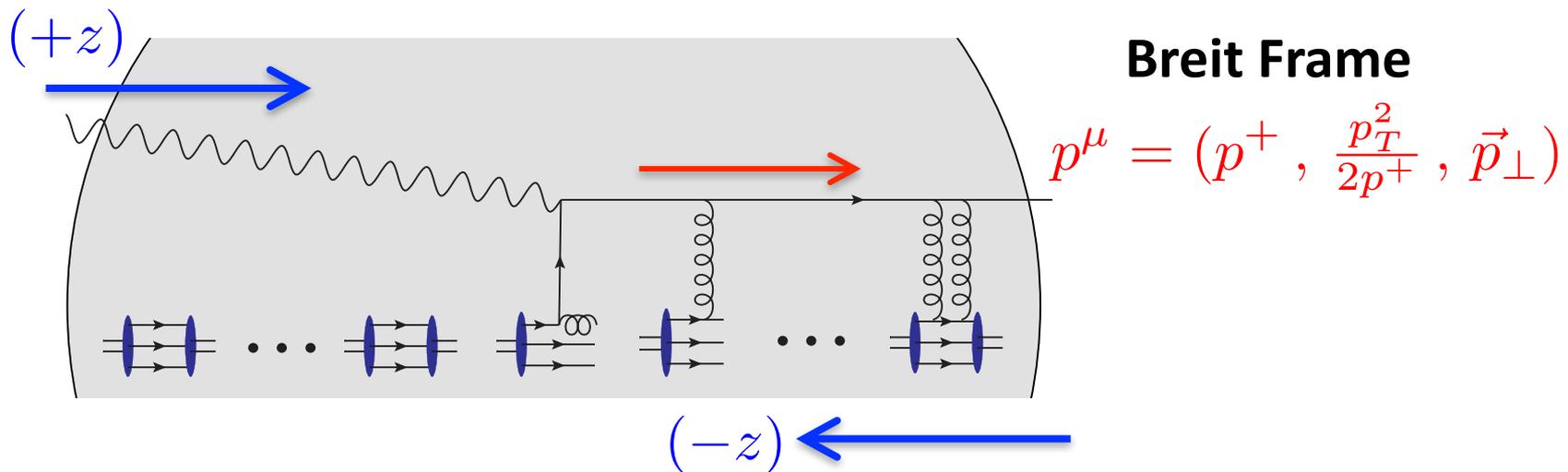
**Santa Fe Jets and  
Heavy Flavor 2018**

Wed. Jan. 31, 2018

# Outline

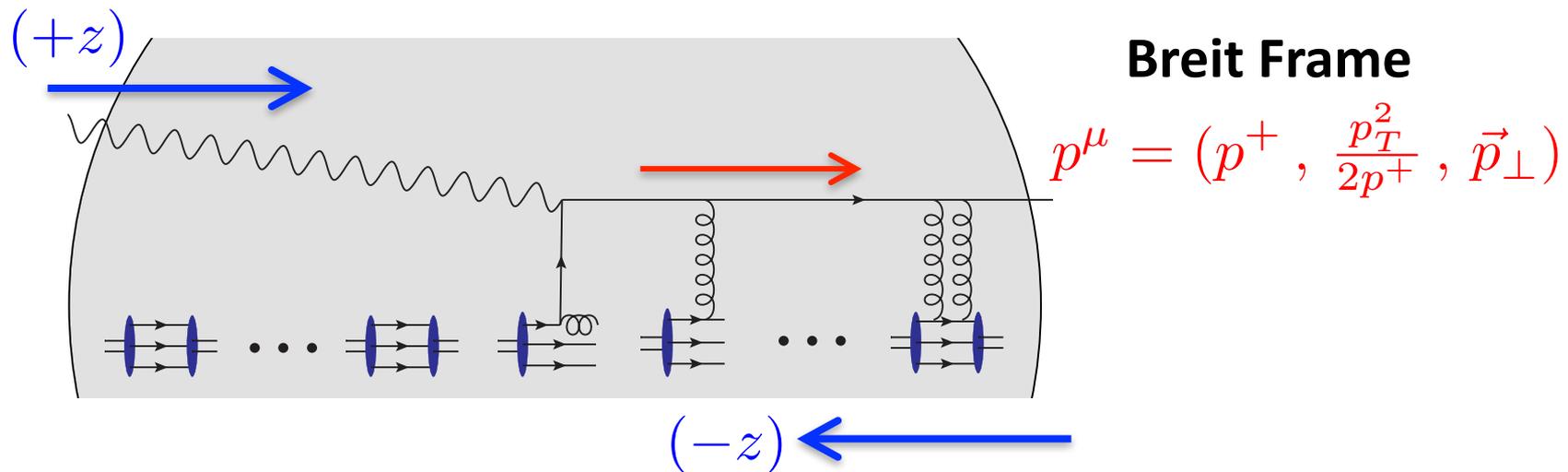
- I. Introduction: Radiative Energy Loss at an EIC
- II. Parton Splitting in Vacuum and in Medium
- III. General Structure: Recursion Relations
- IV. The Coordinate-Space Approach
- V. Preliminary Results and Outlook

# Deep Inelastic Scattering at an EIC



- (SI)DIS at an EIC produces quark jets at leading order
- Modified by multiple scattering in cold nuclear matter

# Deep Inelastic Scattering at an EIC



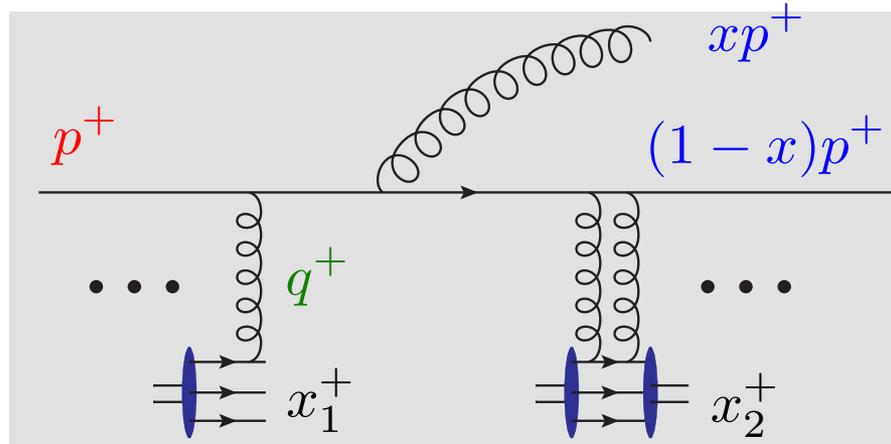
- (SI)DIS at an EIC produces quark jets at leading order
  - Modified by multiple scattering in cold nuclear matter
- Characteristic scales: **formation length**, **mean free path**, **opacity**

$$\ell_f^+ \sim \frac{1}{p^-} \sim \frac{2p^+}{p_T^2}$$

$$\lambda_{MFP} = \frac{1}{\rho \sigma^{el}}$$

$$\chi = \frac{L}{\lambda_{MFP}} = \langle N \rangle$$

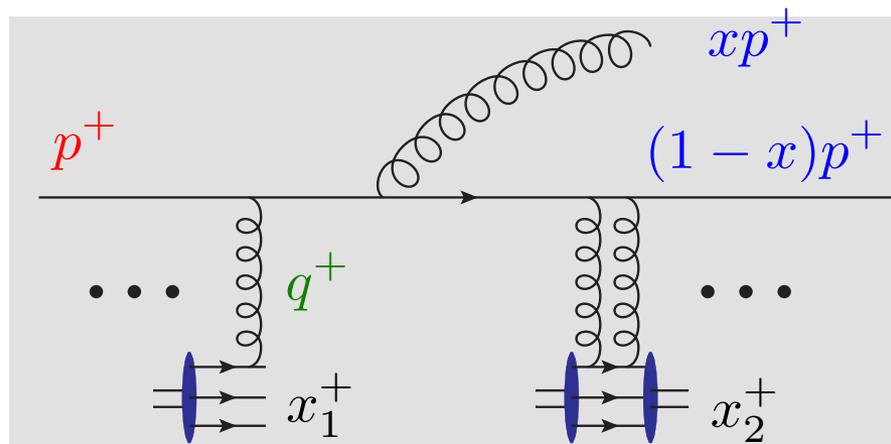
# Collisional vs. Radiative Energy Loss



- **Collisional Energy Loss:**  
direct energy transfer to medium

$$\left( \frac{\delta p^+}{p^+} \right)_{Coll} \sim \frac{q_T^2}{s}$$

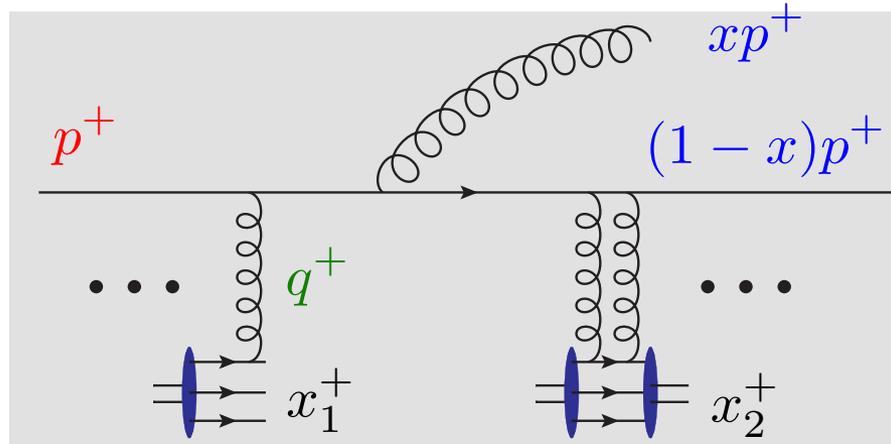
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- Radiative Energy Loss:**  
 stimulated radiation due to scattering
 
$$\left(\frac{\delta p^+}{p^+}\right)_{Rad} = x$$

$$|\mathcal{M}|^2 \propto \left( e^{-i\Delta E^- x_2^+} - e^{-i\Delta E^- x_1^+} \right) \sim \frac{k_T^2}{s} \frac{A^{1/3}}{\chi}$$

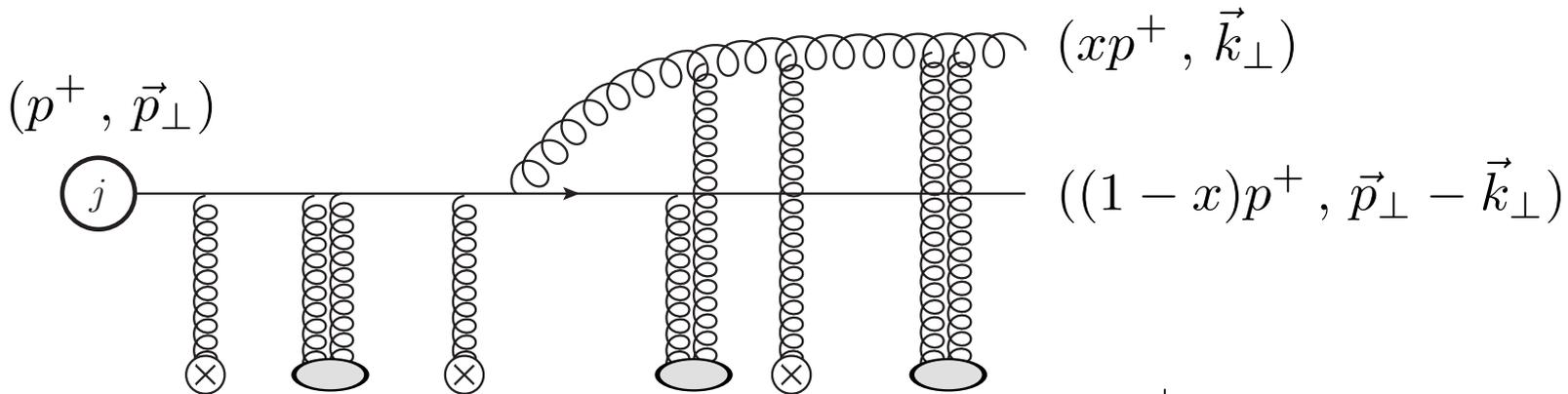
# Collisional vs. Radiative Energy Loss



- Collisional Energy Loss:**  
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$$|\mathcal{M}|^2 \propto \left( e^{-i\Delta E^- x_2^+} - e^{-i\Delta E^- x_1^+} \right) \sim \frac{k_T^2}{s} \frac{A^{1/3}}{\chi}$$
- **Radiative losses dominate** except at very large opacities.

# Jet Coupling to a Generic Medium

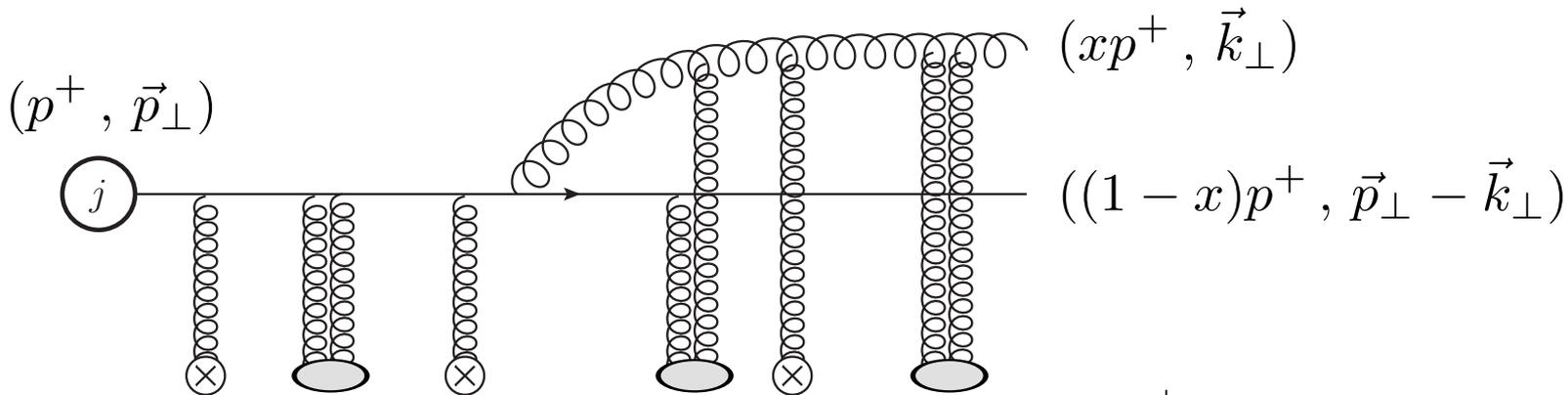


$A^+ = 0$  Light-Cone Gauge

- Compute the gluon spectrum of a quark jet with **any number of scatterings** in the medium

$$xp^+ \frac{dN}{d^2k dx dp^+}$$

# Jet Coupling to a Generic Medium



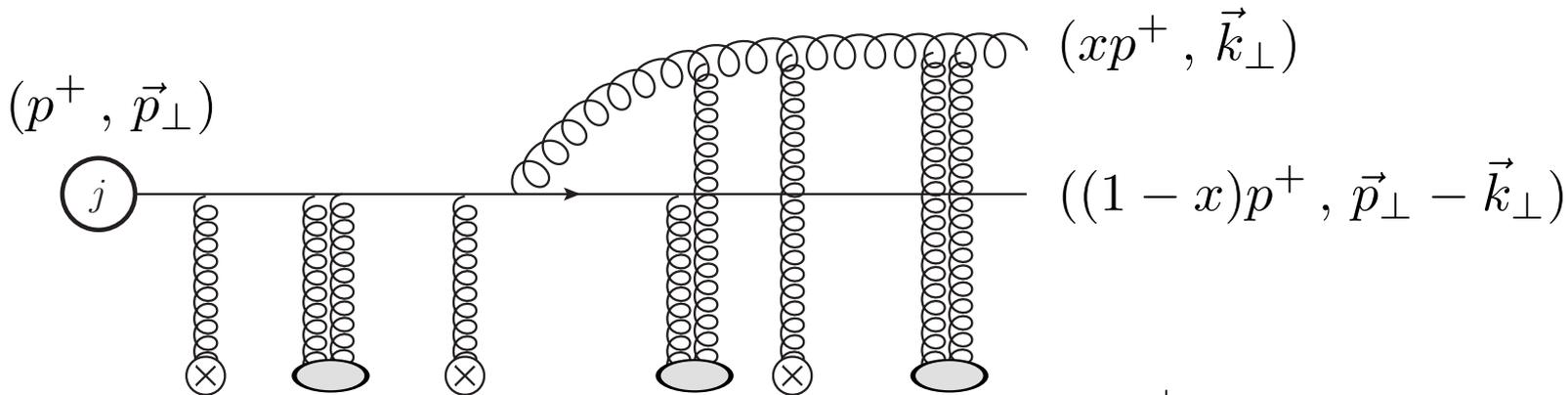
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- Semi-classical scattering: 2 gluons / scattering center

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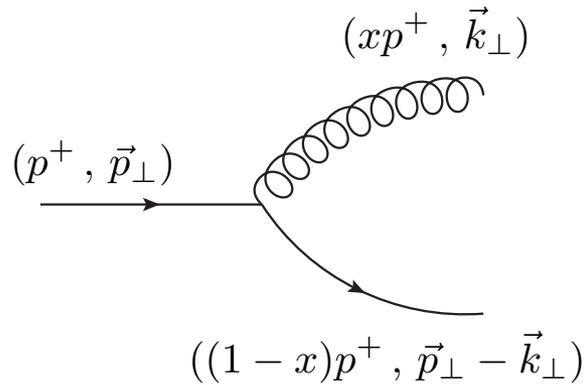
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- Power counting: one factor of opacity per scattering  $\mathcal{O}(\chi^N)$

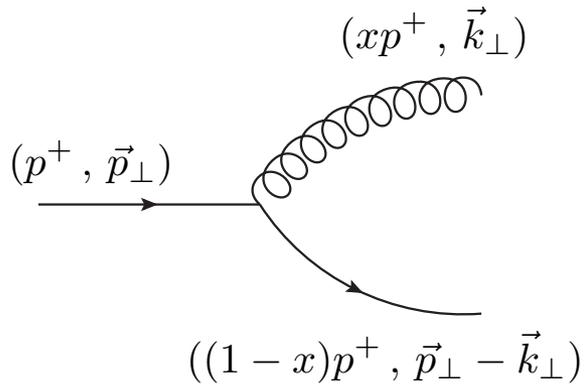
# Parton Splitting in Vacuum



$$\begin{aligned}\psi(k, p) &= \frac{-g [\bar{u}(p-k)(\gamma \cdot \epsilon^*)u(p)]}{2p^+(p^- - k^- - (p-k)^-)} \\ &= \psi(x, \vec{k}_\perp - x\vec{p}_\perp)\end{aligned}$$

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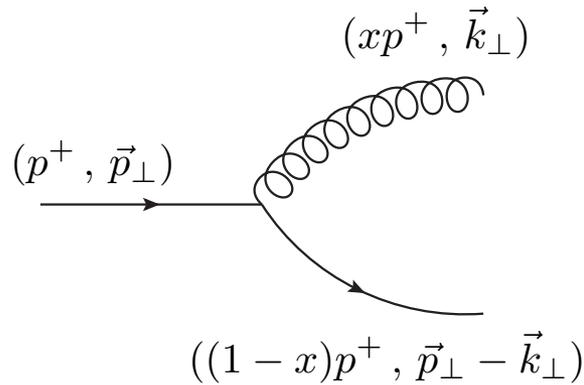


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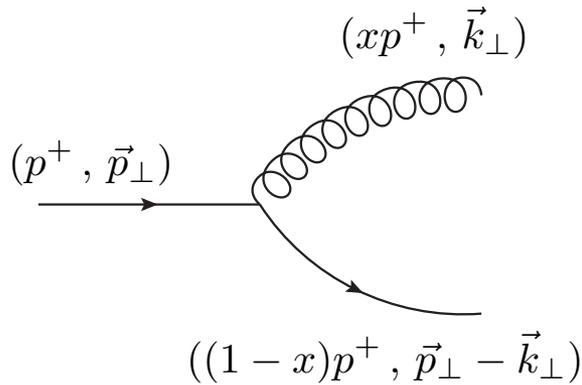
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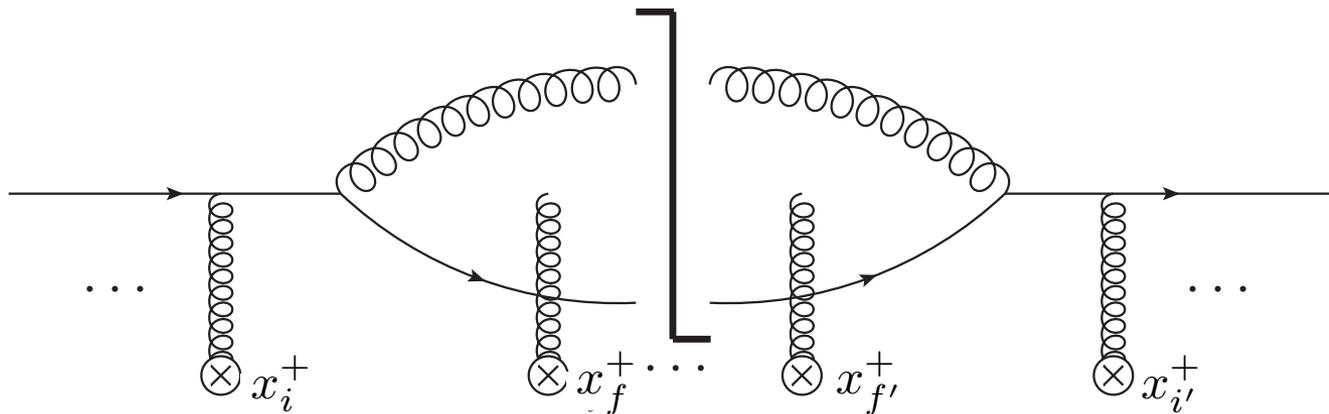
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- Intrinsic transverse momentum:**  $\vec{\kappa}_\perp = \vec{k}_\perp - x\vec{p}_\perp$
- Energy denominator / virtuality:**

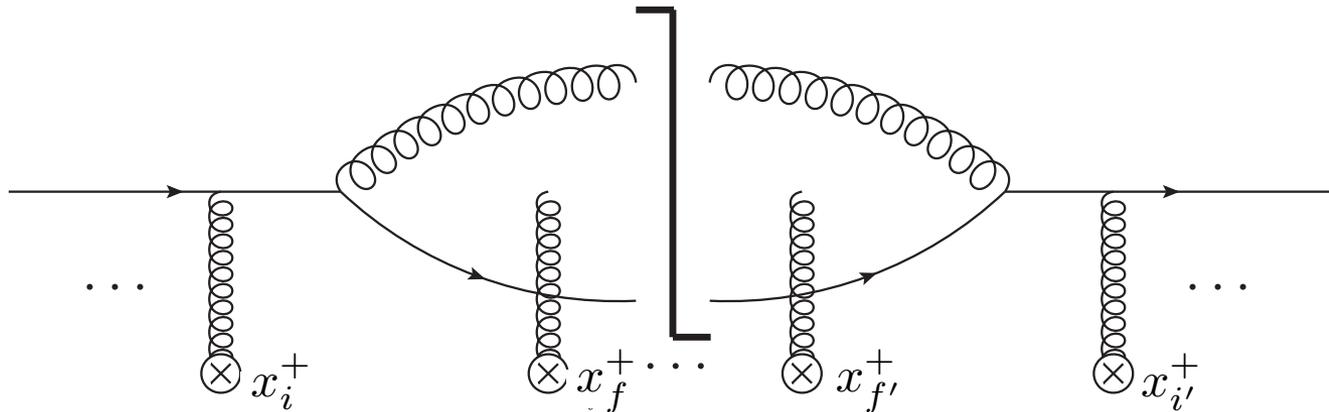
$$\Delta E^-(\vec{\kappa}_\perp) = \frac{1}{2p^+(p^- - k^- - (p-k)^-)} = \frac{-\kappa_T^2}{2x(1-x)p^+}$$

# Stimulated Emission in Medium



- Scatterings **stimulate radiation** by generating **phase factors**:

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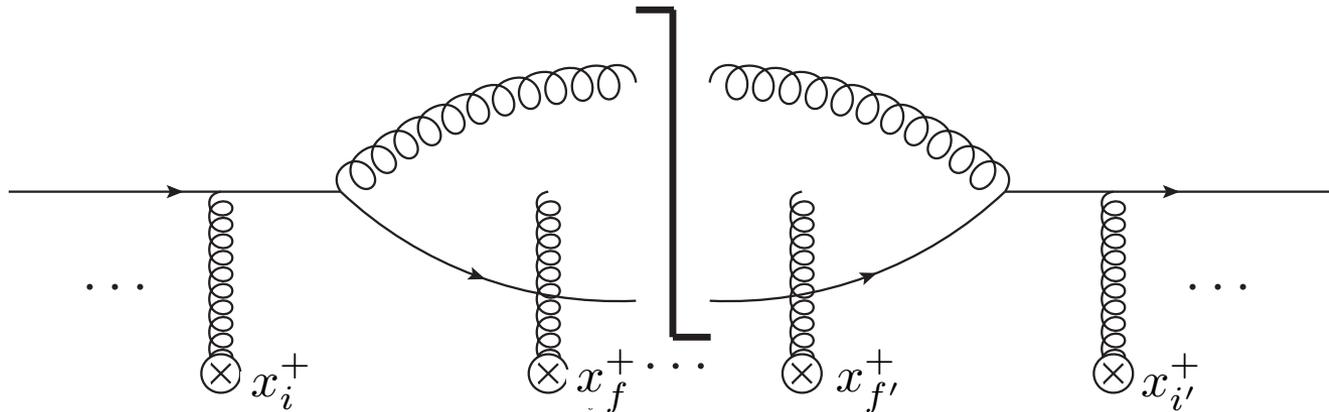
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➤ **Emission phases** due to bounds on gluon emission time:

$$\int_{x_i^+}^{x_f^+} dt_{LF} e^{-i\Delta E^- t_{LF}} [-g \bar{u}(\gamma \cdot \epsilon^*) u] =$$

$$= \psi(x, \vec{k}_\perp - x \vec{p}_\perp) \left[ e^{-i\Delta E^- x_f^+} - e^{-i\Delta E^- x_i^+} \right]$$

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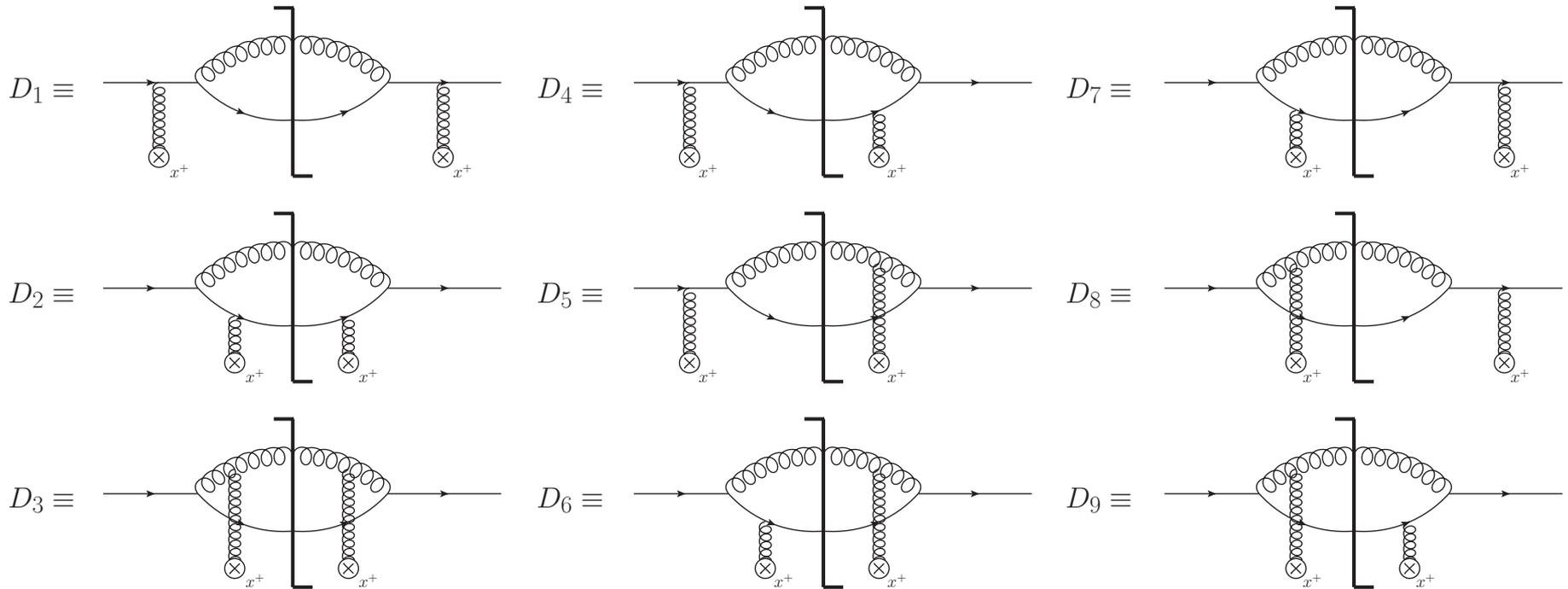
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- **Impulse phases**: virtuality changes due to scattering

$$e^{-i[\Delta E^-(p_f) - \Delta E^-(p_i)]x^+}$$

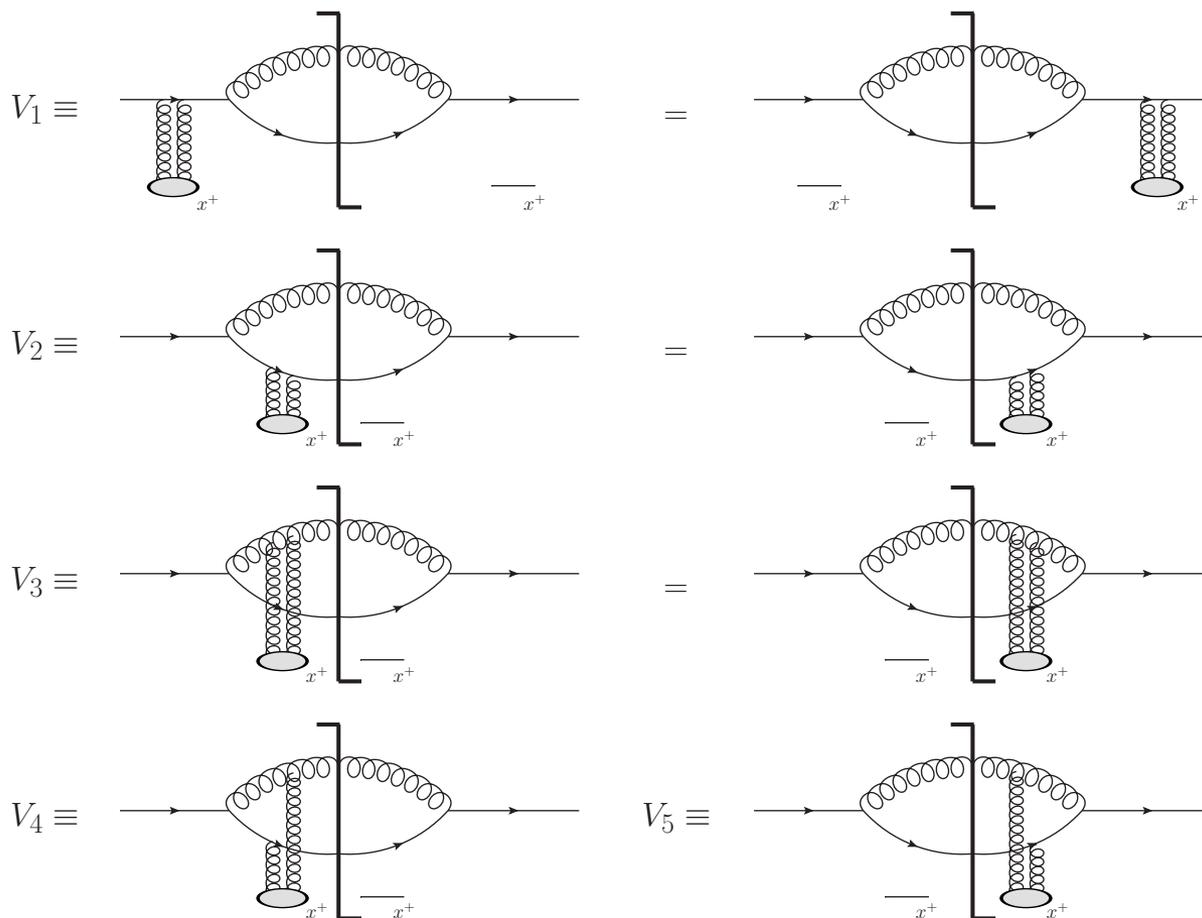
# Diagrammatic Building Blocks: "Direct"



- **“Direct scattering:”**

single-gluon exchange (Born) on a scattering center, squared

# Diagrammatic Building Blocks: “Virtual”



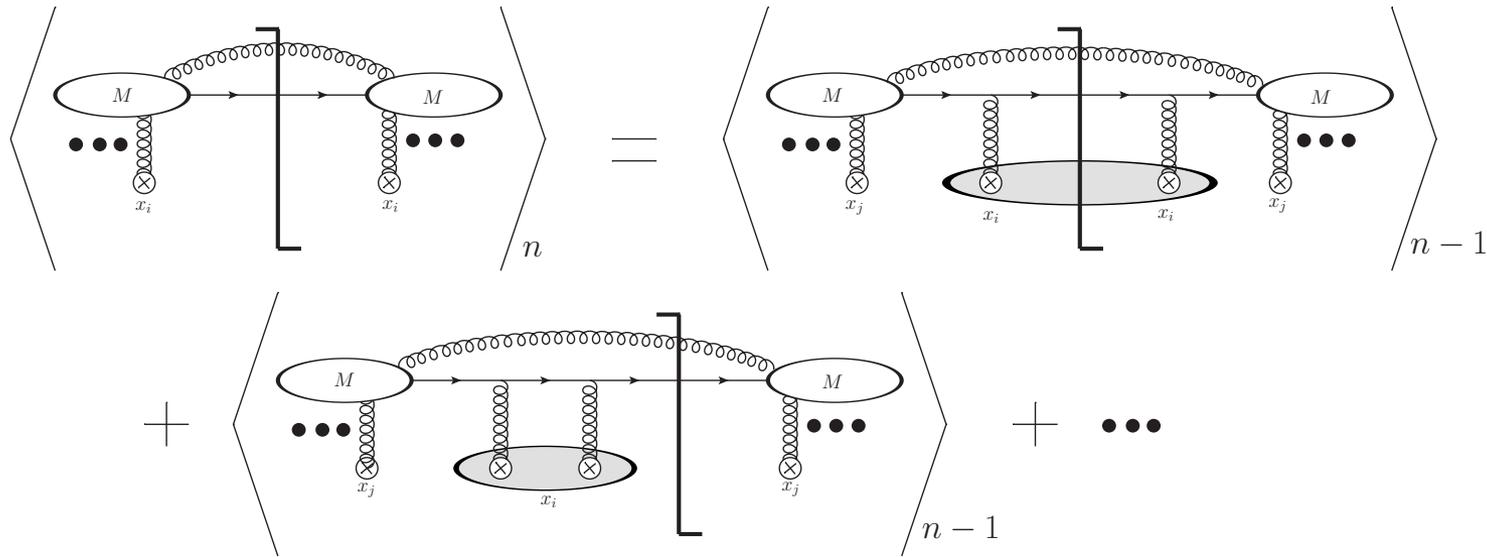
- “Virtual scattering:”

double-gluon exchange (double-Born) on a scattering center

# Results at First Order in Opacity

$$\begin{aligned}
 x p^+ \frac{dN}{d^2k dx dp^+} \Big|_{N=1} &= \frac{C_F}{4\pi(1-x)} \int_{0^+}^{L^+} \frac{dz_1^+}{\lambda^+} \int \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \left( \frac{(2\pi)^2 d\sigma^{el}}{\sigma_{el} d^2q} \right) \left( p^+ \frac{dN_0}{d^2p dp^+} \right) \left\{ \right. \\
 &\times |\psi(\underline{k} - x\underline{p} - x\underline{q})|^2 \\
 &+ |\psi(\underline{k} - x\underline{p})|^2 \left[ 2 \left( 1 - \cos [\Delta E^-(\underline{k} - x\underline{p}) z_1^+] \right) - 1 - \frac{N_c}{C_F} \left( 1 - \cos [\Delta E^-(\underline{k} - x\underline{p}) z_1^+] \right) \right] \\
 &+ |\psi(\underline{k} - x\underline{p} - \underline{q})|^2 \left[ 2 \frac{N_c}{C_F} \left( 1 - \cos [\Delta E^-(\underline{k} - x\underline{p} - \underline{q}) z_1^+] \right) \right] \\
 &+ \psi(\underline{k} - x\underline{p} - x\underline{q}) \psi^*(\underline{k} - x\underline{p}) \left[ \frac{1}{N_c C_F} \left( 1 - \cos [\Delta E^-(\underline{k} - x\underline{p}) z_1^+] \right) \right] \\
 &+ \psi(\underline{k} - x\underline{p} - x\underline{q}) \psi^*(\underline{k} - x\underline{p} - \underline{q}) \left[ - \frac{N_c}{C_F} \left( 1 - \cos [\Delta E^-(\underline{k} - x\underline{p} - \underline{q}) z_1^+] \right) \right] \\
 &\left. + \psi(\underline{k} - x\underline{p}) \psi^*(\underline{k} - x\underline{p} - \underline{q}) \left[ - \frac{N_c}{C_F} \left( 1 - \cos [\Delta E^-(\underline{k} - x\underline{p} - \underline{q}) z_1^+] \right) \right] \right\}
 \end{aligned}$$

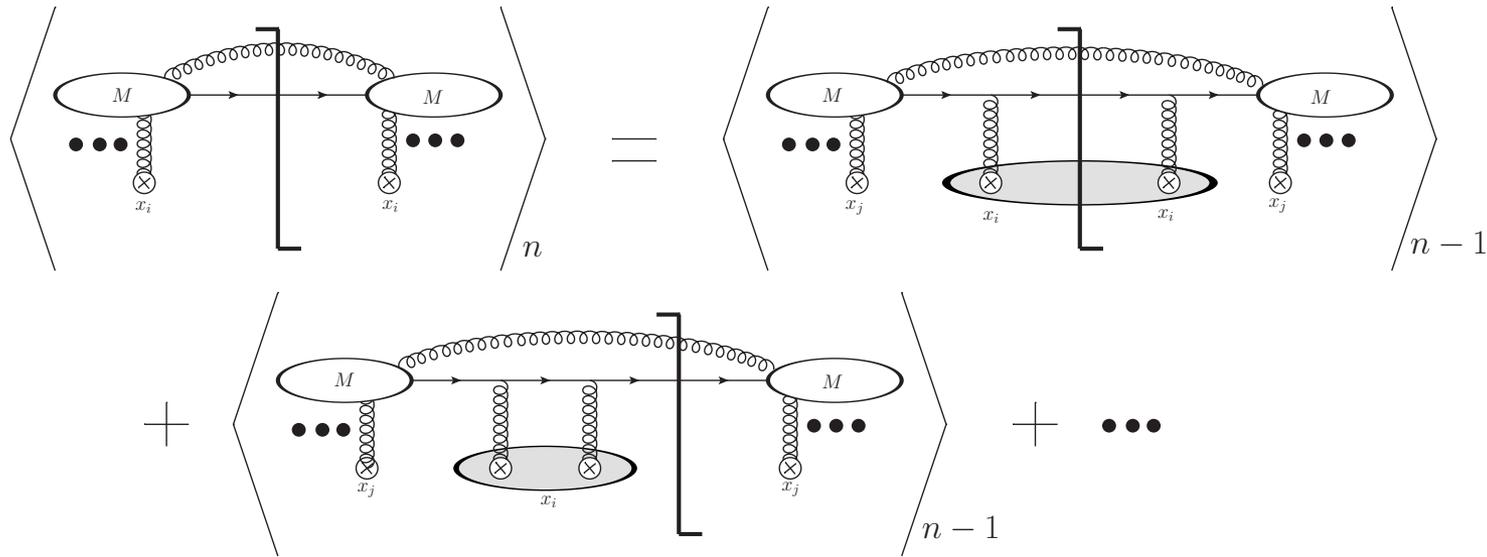
# Reaction Operators: Recursion Relations



$$\langle |M|^2 \rangle_n = (D \otimes D^\dagger + V \otimes 1 + 1 \otimes V^\dagger) \langle |M|^2 \rangle_{n-1}$$

- Formulate all-orders result in terms of a **recursion relation**:

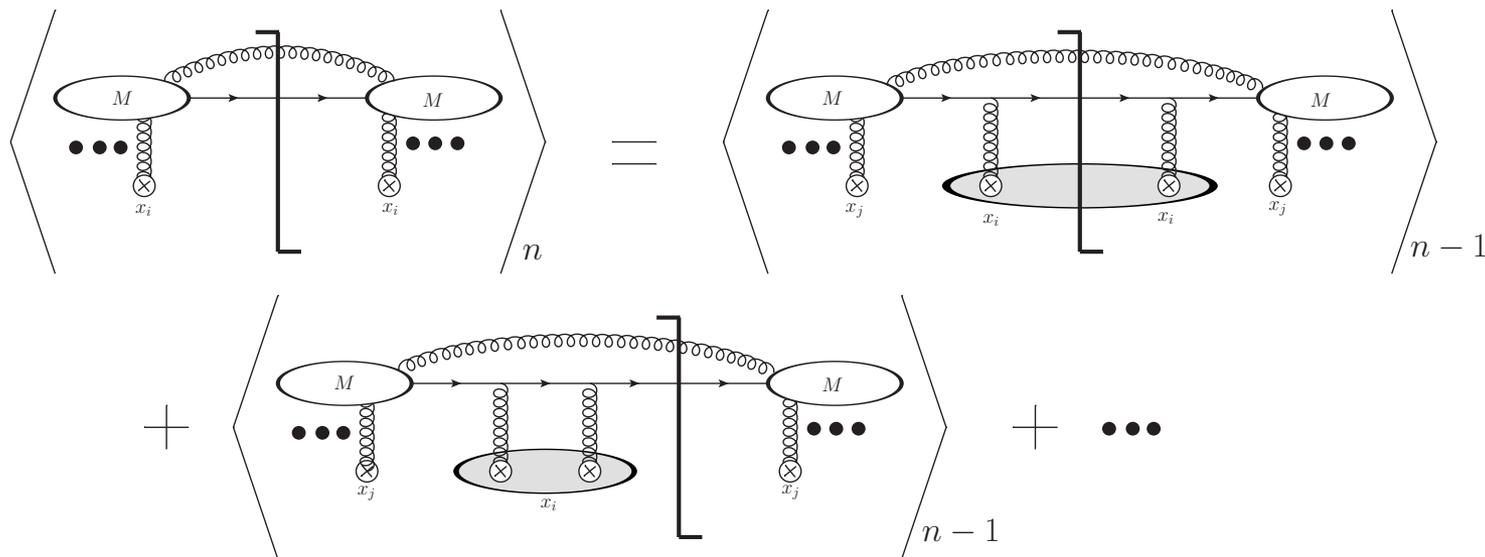
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  - Proliferation of **phase structures**
  - Proliferation of **convolution integrals**

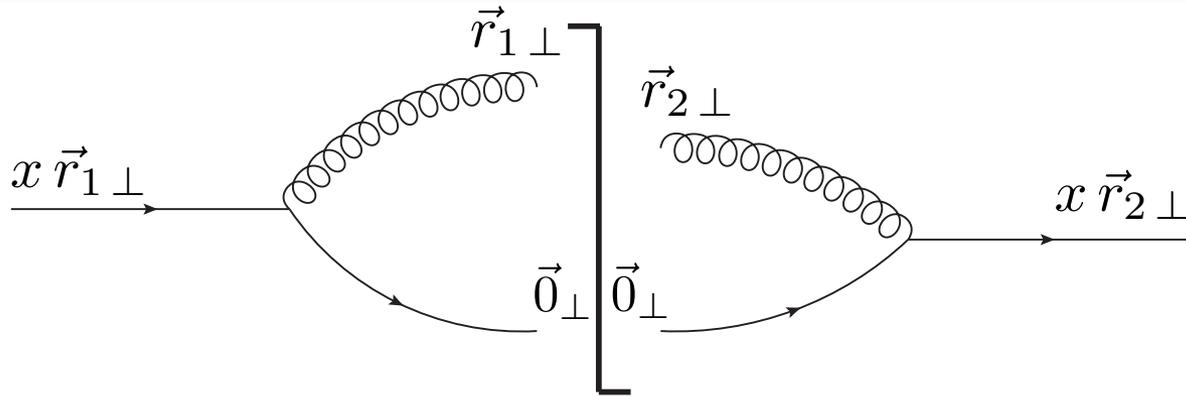
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- Formulate all-orders result in terms of a **recursion relation**:
  - Proliferation of **phase structures**
  - Proliferation of **convolution integrals**
- Can be solved analytically in soft-gluon approximation  $x \ll 1$

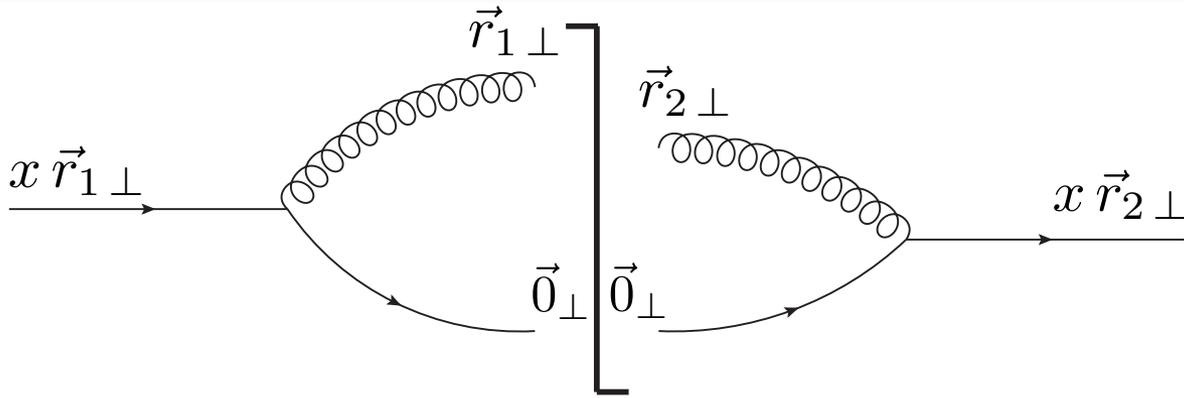
# The Coordinate Space Picture



- The strategy: transform to **transverse coordinate space**

$$xp^+ \frac{dN}{d^2k dx dp^+} = \frac{C_F}{4\pi x(1-x)} \int d^2r_1 d^2r_2 e^{-i\vec{k}_{\perp} \cdot (\vec{r}_{1\perp} - \vec{r}_{2\perp})} j(x|\vec{r}_{1\perp} - \vec{r}_{2\perp}|) \langle M(\vec{r}_{1\perp}) M^*(\vec{r}_{2\perp}) \rangle$$

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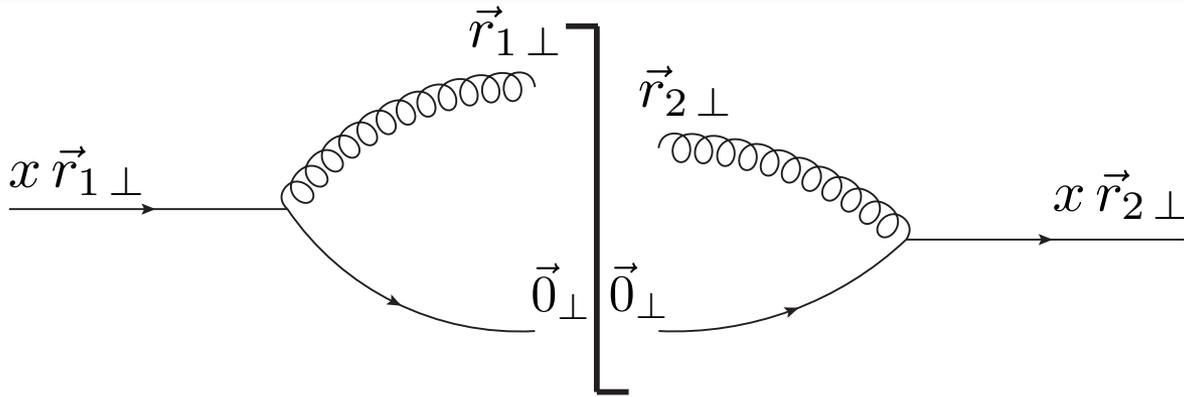
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- Absorb emission phases into the wave functions

$$\tilde{\psi}(x, \vec{r}_\perp; x^+) = \int \frac{d^2\kappa}{(2\pi)^2} e^{i\vec{\kappa}_\perp \cdot \vec{r}_\perp} \psi(x, \vec{\kappa}_\perp) e^{-i\Delta E^-(\vec{\kappa}_\perp)x^+}$$

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- Momentum convolutions  $\rightarrow$  **simple products**

$$\langle M(\vec{r}_{1\perp}) M^*(\vec{r}_{2\perp}) \rangle = [\tilde{\psi}(x, \vec{r}_{1\perp}; x_f^+) - \tilde{\psi}(x, \vec{r}_{1\perp}; x_i^+)] \\ \times [\text{Scattering Factors}] \times [\tilde{\psi}^*(x, \vec{r}_{2\perp}; x_{f'}^+) - \tilde{\psi}^*(x, \vec{r}_{2\perp}; x_{i'}^+)]$$



# Building Blocks in Coordinate Space

$$\begin{aligned}
 D_1 &\equiv \text{Diagram 1} = \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el} \left( x |\vec{r}_{1\perp} - \vec{r}_{2\perp}| \right) \\
 D_2 &\equiv \text{Diagram 2} = 1
 \end{aligned}$$

The diagrams illustrate two building blocks,  $D_1$  and  $D_2$ , in coordinate space.   
 Diagram 1 ( $D_1$ ) shows a central vertical line with a semi-circular arc above it. The arc is composed of two parts, each with a series of small circles. A vertical vector  $\vec{0}_\perp$  points downwards from the center of the arc. Two horizontal vectors,  $x \vec{r}_{1\perp}$  and  $x \vec{r}_{2\perp}$ , point from the center of the arc to the left and right respectively. Two vertical lines with circles and crosses at the bottom, labeled  $x^+$ , are attached to the ends of the arc.   
 Diagram 2 ( $D_2$ ) shows a similar central vertical line with a semi-circular arc above it, but the arc is composed of a single series of small circles. Two vertical lines with circles and crosses at the bottom, labeled  $x^+$ , are attached to the ends of the arc.

# Building Blocks in Coordinate Space

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 D_2 &\equiv \text{Diagram 2} = 1 \\
 D_3 &\equiv \text{Diagram 3} = \frac{N_c}{C_F} \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el} \left( |\vec{r}_{1\perp} - \vec{r}_{2\perp}| \right)
 \end{aligned}$$

The diagrams illustrate the building blocks in coordinate space. Each diagram shows a central vertical line representing a Wilson line, with a semi-circular arc above it representing a gluon exchange. 
   
 - **Diagram 1:** Shows a semi-circular arc above the Wilson line. Two external lines, labeled  $x \vec{r}_{1\perp}$  and  $x \vec{r}_{2\perp}$ , enter from the left and right respectively. Two vertical lines with a cross in a circle and  $x^+$  label extend downwards from the arc. A central vertical line is labeled  $\vec{0}_\perp$ .
   
 - **Diagram 2:** Shows a semi-circular arc below the Wilson line. Two external lines enter from the left and right. Two vertical lines with a cross in a circle and  $x^+$  label extend downwards from the arc.
   
 - **Diagram 3:** Shows a semi-circular arc above the Wilson line. Two external lines enter from the left and right. Two vertical lines with a cross in a circle and  $x^+$  label extend downwards from the arc.

# Building Blocks in Coordinate Space

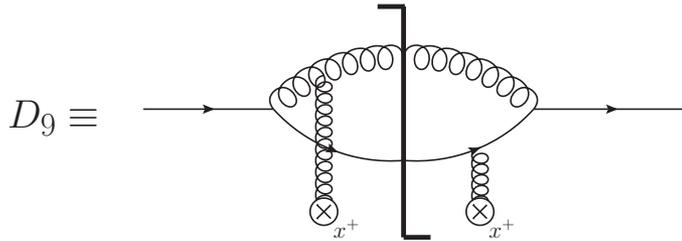
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 V_3 &\equiv \text{Diagram 4} = -\frac{N_c}{2C_F}
 \end{aligned}$$

The diagrams illustrate four building blocks in coordinate space, each represented by a diagram and an equation. The diagrams show a central vertical line with a semi-circular arc above it, containing a chain of circles. Various external lines and labels are present:
 

- Diagram 1:** Shows a semi-circular arc with a chain of circles. A vertical line passes through the center. Labels include  $\vec{r}_{1\perp}$ ,  $\vec{r}_{2\perp}$ ,  $\vec{0}_\perp$ , and  $x^+$ . External lines are labeled  $x \vec{r}_{1\perp}$  and  $x \vec{r}_{2\perp}$ .
- Diagram 2:** Shows a semi-circular arc with a chain of circles. A vertical line passes through the center. Labels include  $x^+$ . External lines are shown as simple arrows.
- Diagram 3:** Shows a semi-circular arc with a chain of circles. A vertical line passes through the center. Labels include  $x^+$ . External lines are shown as simple arrows.
- Diagram 4:** Shows a semi-circular arc with a chain of circles. A vertical line passes through the center. Labels include  $x^+$ . External lines are shown as simple arrows.

# Impulse Phases: Derivative Phase Shifts

- Impulse phases can be cast as **derivative phase shifts**:

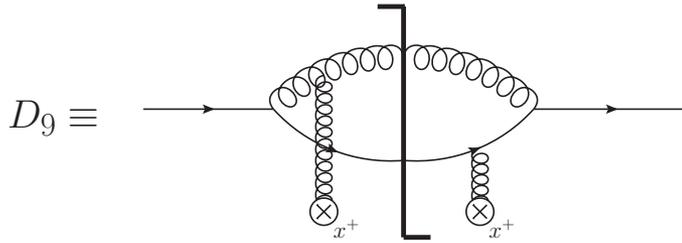


$$D_g \equiv \rightarrow \left[ \text{diagram} \right] \rightarrow$$

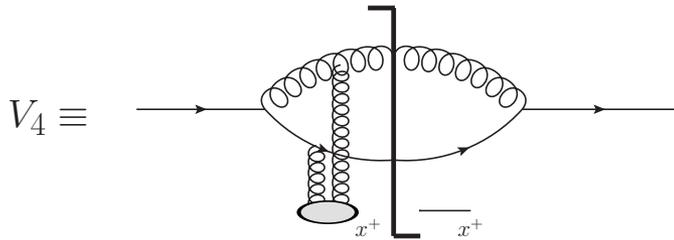
$$= e^{i\Delta E^- \overleftarrow{(i\nabla_{r_1})}x^+} \left[ -\frac{N_c}{2C_F} \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(r_{1T}) \right] e^{-i\Delta E^- \overrightarrow{(-i\nabla_{r_2})}x^+}$$

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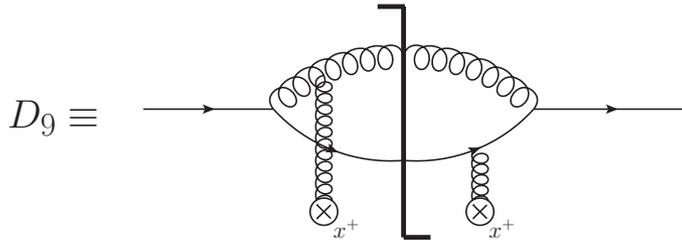
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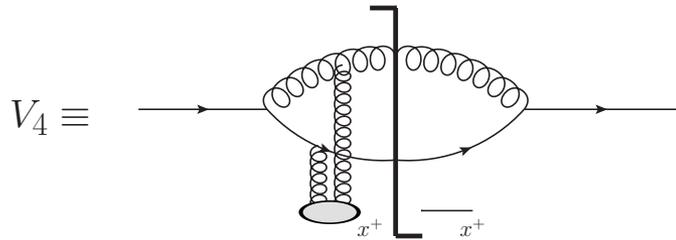
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- Impulse phases can be cast as **derivative phase shifts**:



$$= e^{i\Delta E^- (i\nabla_{r_1})x^+} \left[ -\frac{N_c}{2C_F} \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(r_{1T}) \right] e^{-i\Delta E^- (-i\nabla_{r_2})x^+}$$



$$= e^{i\Delta E^- (i\nabla_{r_1})x^+} \left[ \frac{N_c}{2C_F} \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(r_{1T}) \right] e^{-i\Delta E^- (-i\nabla_{r_2})x^+}$$

$$\tilde{\psi}(\vec{r}_{1\perp}; x^+) e^{i\Delta E^- (i\nabla_{r_1})y^+} = \tilde{\psi}(\vec{r}_{1\perp}; x^+ - y^+)$$

# Jet Broadening in the Initial and Final States

- Initial-state jet broadening:

$$= \left[ \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(x|\vec{r}_{1\perp} - \vec{r}_{2\perp}|) - 1 \right]$$

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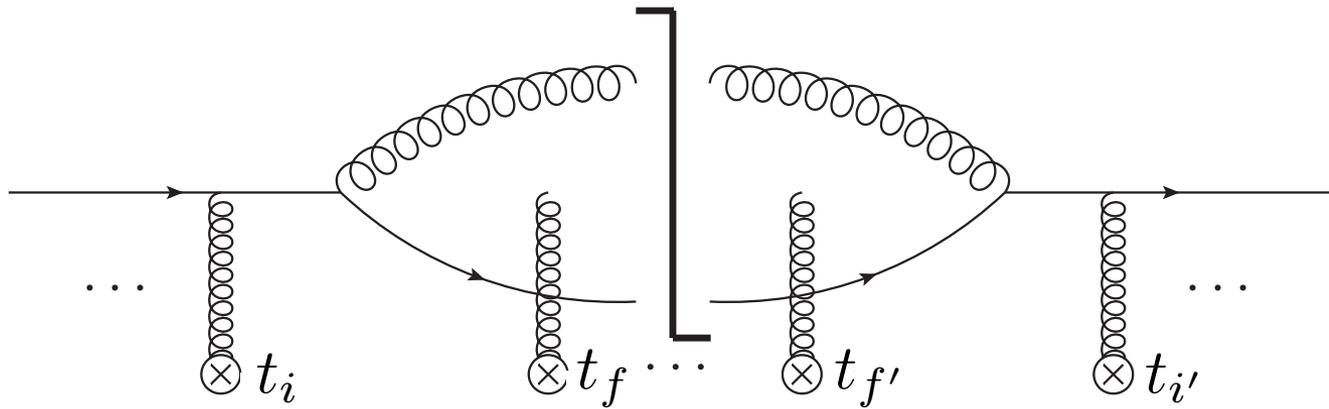
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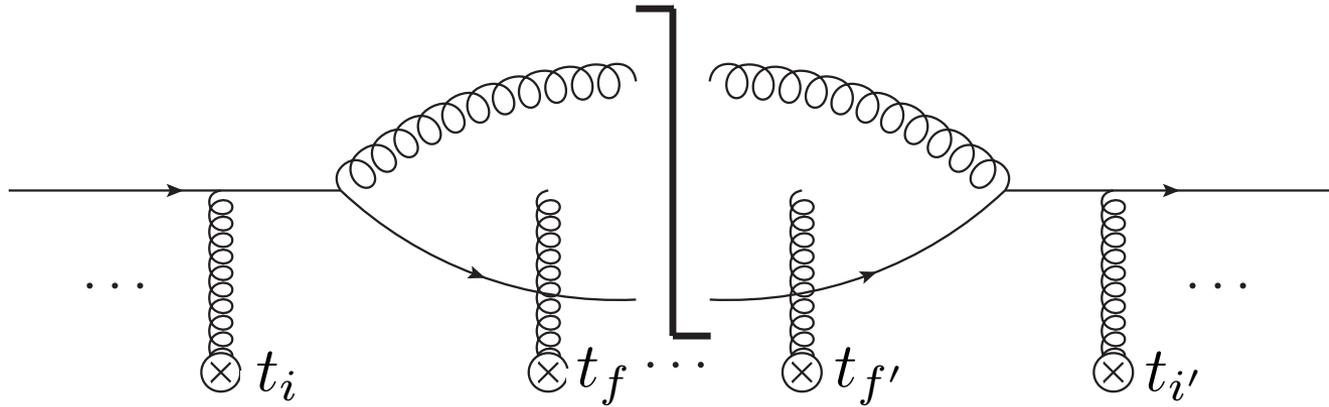
$$= \frac{N_c}{C_F} \left[ \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el} (|\vec{r}_{1\perp} - \vec{r}_{2\perp}|) - 1 \right]$$

# Schwinger-Keldysh Organization Scheme



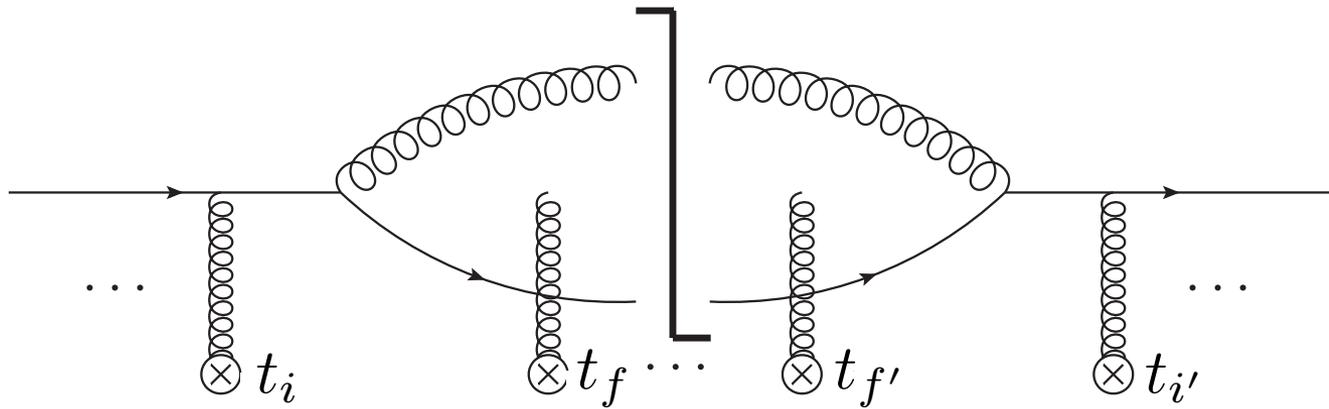
- Scatterings which don't affect the emission phases enter in a simple way with helpful cancellations.

# Schwinger-Keldysh Organization Scheme



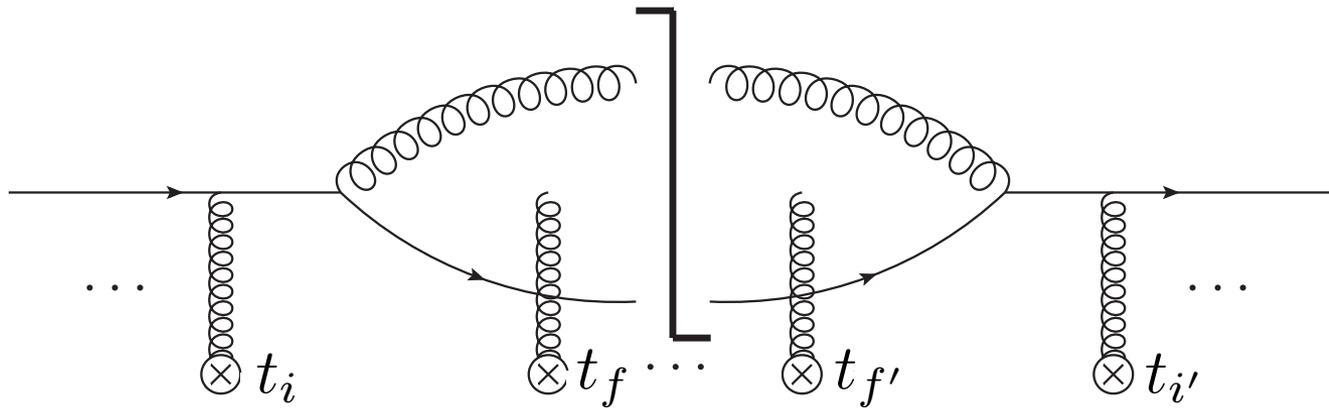
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Initial State Broadening:  $t < t_i, t_{i'} : \left[ \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(x | \vec{r}_{1\perp} - \vec{r}_{2\perp}) \right]^n$

Final State Broadening:  $t_f, t_{f'} < t : \left[ \frac{N_c}{C_F} \left( \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(x | \vec{r}_{1\perp} - \vec{r}_{2\perp}) \right) \right]^n$

# “Region” Scattering Factors

- Possible regions, using constraints:  $t_i < t_f$        $t_{i'} < t_{f'}$ 
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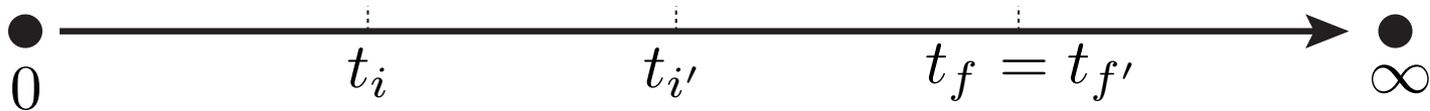
<i>ISI/ISI*</i>	$t < t_i, t_{i'}$	$R_1 = D_1 + 2V_1$
<i>ISI/FSI*</i>	$t_{f'} < t < t_i$	$R_2(x^+) = D_4(x^+) + D_5(x^+) + V_1 + V_2 + V_3 + V_5(x^+)$
<i>FSI/ISI*</i>	$t_f < t < t_{i'}$	$R_3(x^+) = D_7(x^+) + D_8(x^+) + V_1 + V_2 + V_3 + V_4(x^+)$
<i>FSI/FSI*</i>	$t_f, t_{f'} < t$	$R_4 = D_3 + 2V_3$
<i>ISI</i> only	$t_{i'} < t < t_i, t_{f'}$	$R_5 = V_1$
<i>FSI</i> only	$t_{i'}, t_f < t < t_{f'}$	$R_6(x^+) = V_2 + V_3 + V_4(x^+)$
<i>ISI*</i> only	$t_i < t < t_{i'}, t_f$	$R_7 = V_1$
<i>FSI*</i> only	$t_i, t_{f'} < t < t_f$	$R_8(x^+) = V_2 + V_3 + V_5(x^+)$
No scattering	$t_i, t_{i'} < t < t_f, t_{f'}$	1.

# “Boundary Point” Scattering Factors

$ISI/ISI^*$	$t = t_i = t_{i'}$	$B_1 = D_1$
$ISI/FSI^*$	$t = t_i = t_{f'}$	$B_2(x^+) = D_4(x^+) + D_5(x^+)$
$FSI/ISI^*$	$t = t_f = t_{i'}$	$B_3(x^+) = D_7(x^+) + D_8(x^+)$
$FSI/FSI^*$	$t = t_f = t_{f'}$	$B_4(x^+) = D_2 + D_3 + D_6(x^+) + D_9(x^+)$
$ISI$ only	$t_{i'} < t = t_i < t_{f'}$	$B_5 = V_1$
$FSI$ only	$t_{i'} < t = t_f < t_{f'}$	$B_6(x^+) = V_2 + V_3 + V_4(x^+)$
$ISI^*$ only	$t_i < t = t_{i'} < t_f$	$B_7 = V_1$
$FSI^*$ only	$t_i < t = t_{f'} < t_f$	$B_8(x^+) = V_2 + V_3 + V_5(x^+)$
$ISI (+ ISI^*)$	$t = t_i < t_{i'}$	$B_9 = D_1 + V_1$
$ISI (+ FSI^*)$	$t_{f'} < t = t_i$	$B_{10}(x^+) = D_4(x^+) + D_5(x^+) + V_1$
$FSI (+ ISI^*)$	$t = t_f < t_{i'}$	$B_{11}(x^+) = D_7(x^+) + D_8(x^+) + V_2 + V_3 + V_4(x^+)$
$FSI (+ FSI^*)$	$t_{f'} < t = t_f$	$B_{12}(x^+) = D_2 + D_3 + D_6(x^+) + D_9(x^+) + V_2 + V_3 + V_4(x^+)$
$(ISI +) ISI^*$	$t = t_{i'} < t_i$	$B_{13} = D_1 + V_1$
$(FSI +) ISI^*$	$t_f < t = t_{i'}$	$B_{14}(x^+) = D_7(x^+) + D_8(x^+) + V_1$
$(ISI +) FSI^*$	$t = t_{f'} < t_i$	$B_{15}(x^+) = D_4(x^+) + D_5(x^+) + V_2 + V_3 + V_5(x^+)$
$(FSI +) FSI^*$	$t_f < t = t_{f'}$	$B_{16}(x^+) = D_2 + D_3 + D_6(x^+) + D_9(x^+) + V_2 + V_3 + V_5(x^+).$

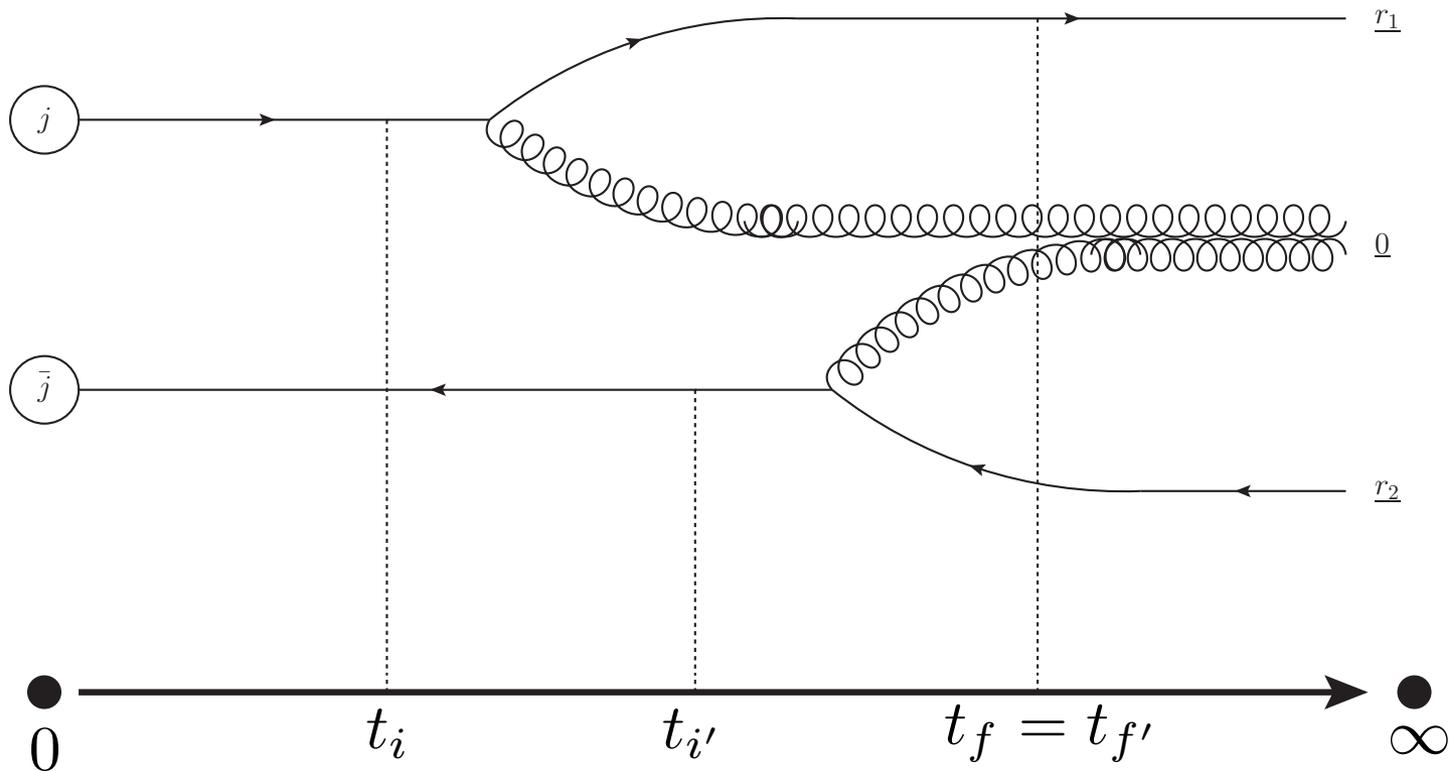
# An Example at Arbitrary Order

$$0 < t_i < t_{i'} < t_f = t_{f'} < \infty$$



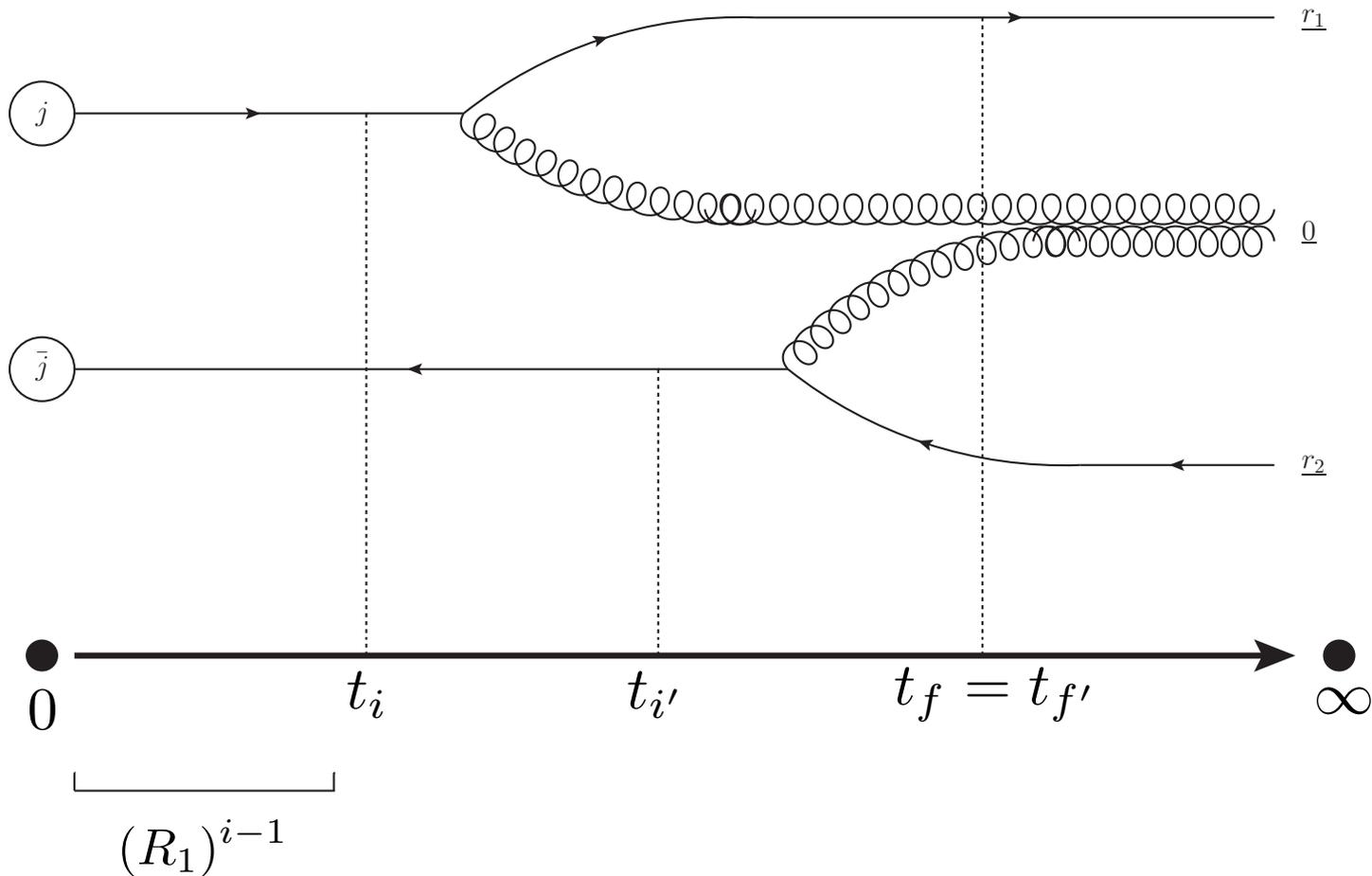
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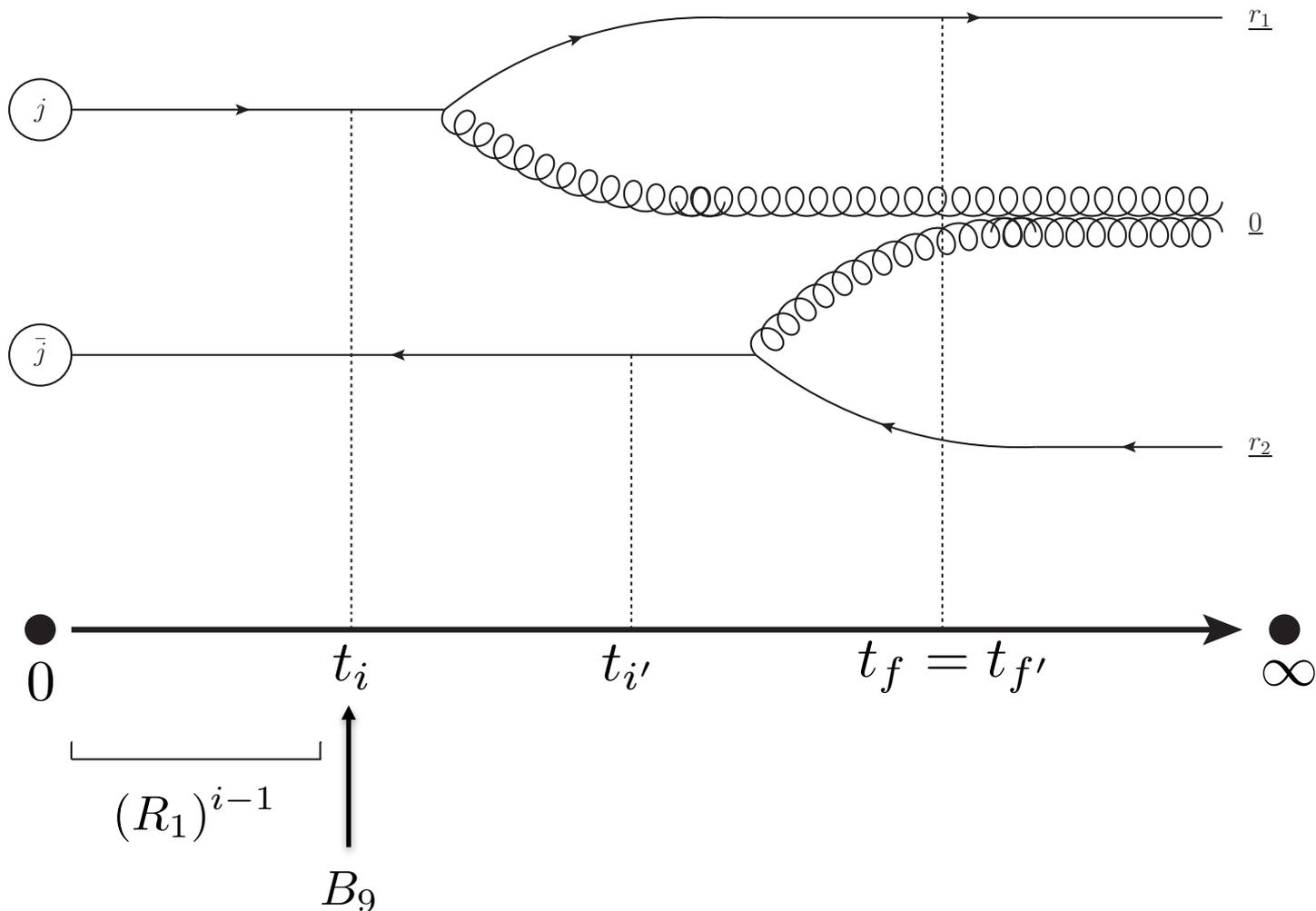
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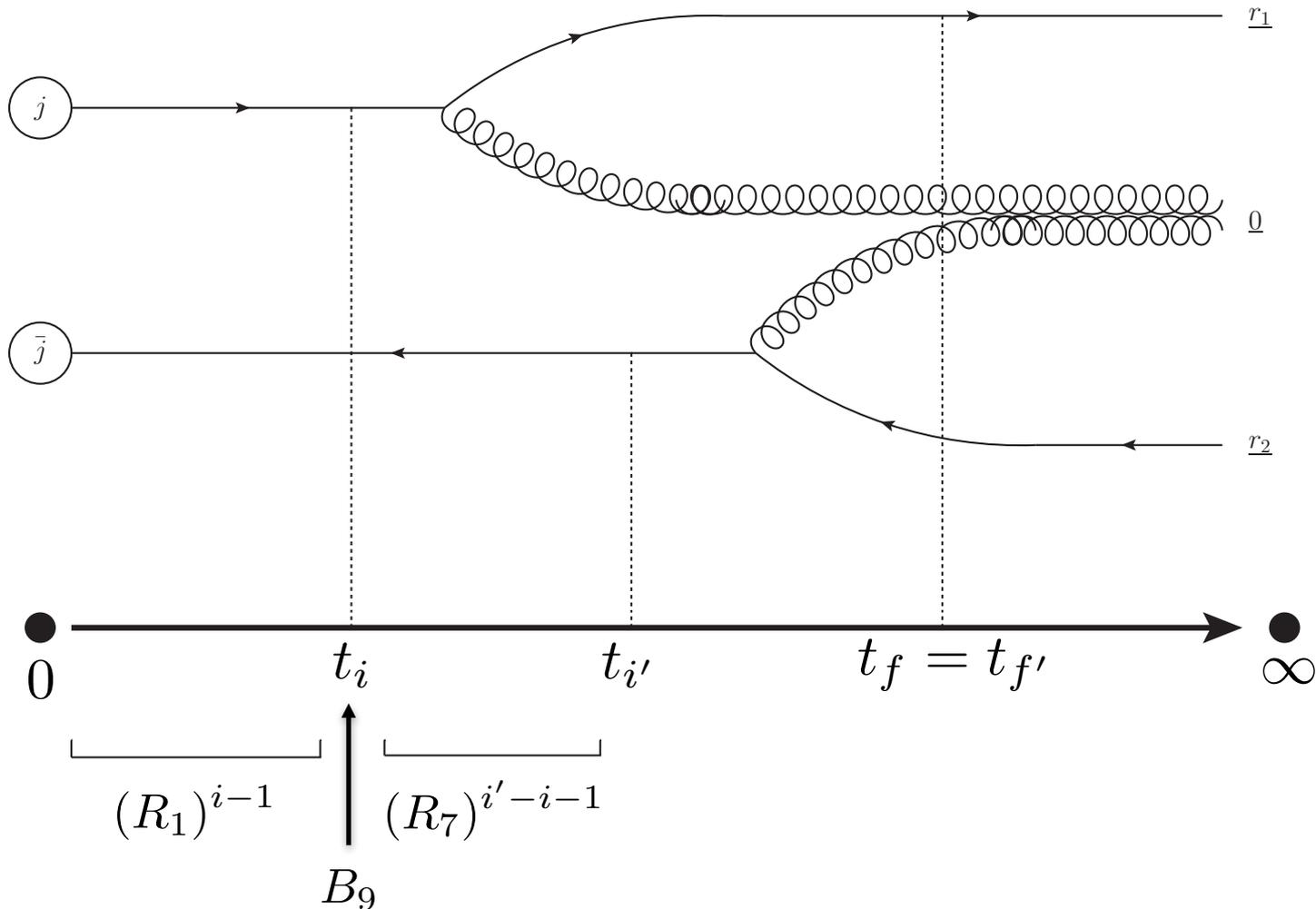
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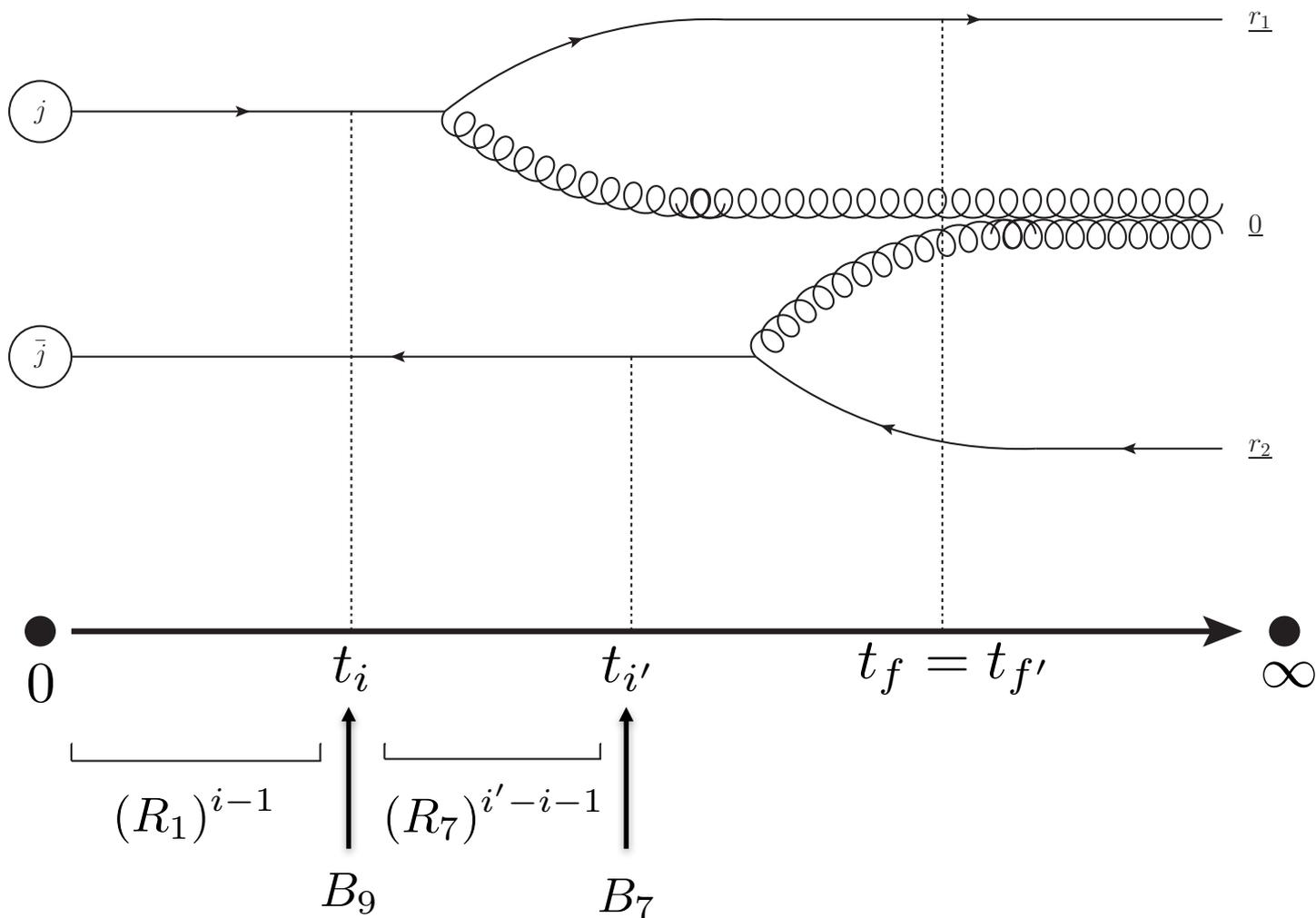
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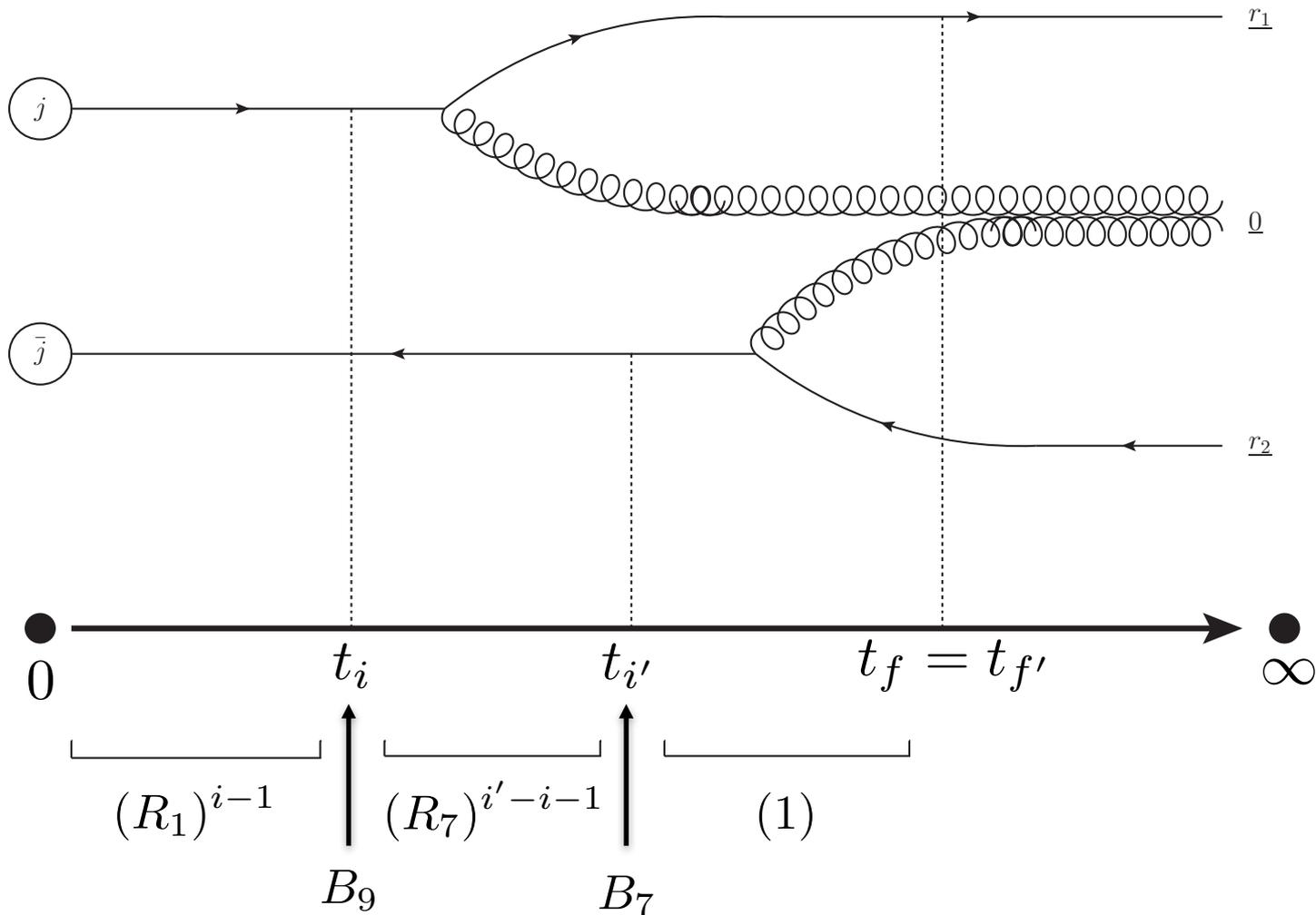
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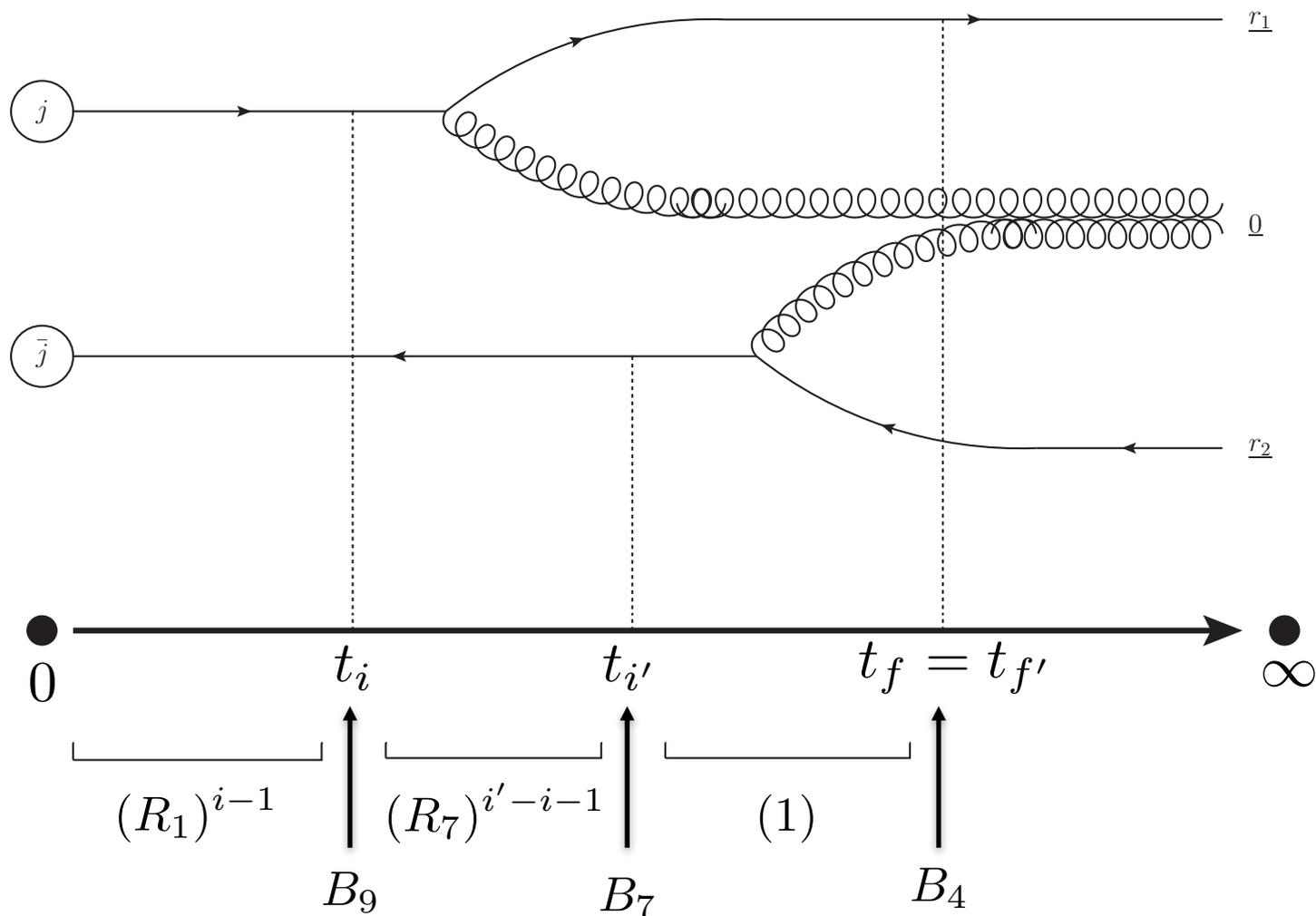
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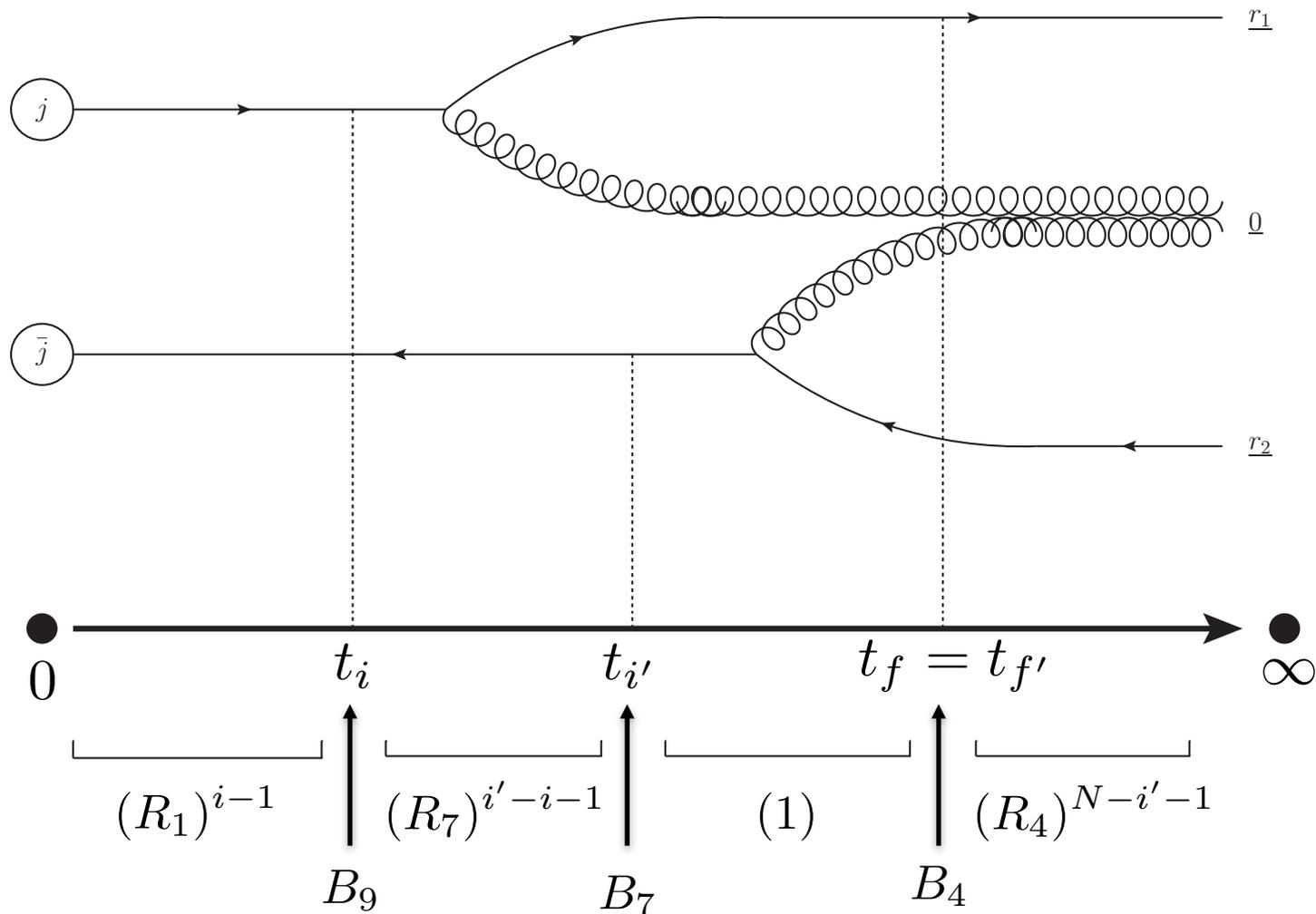
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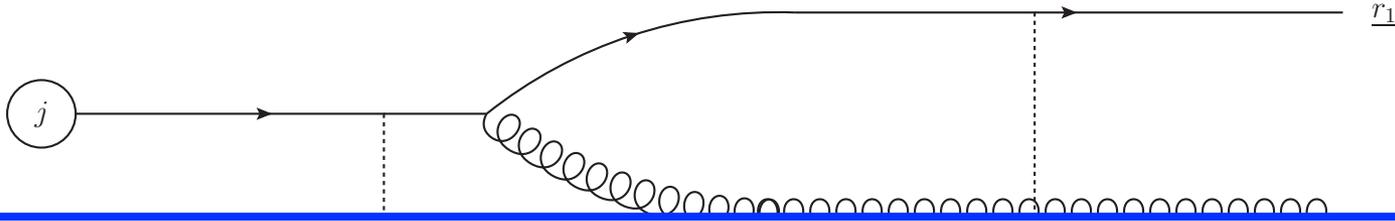
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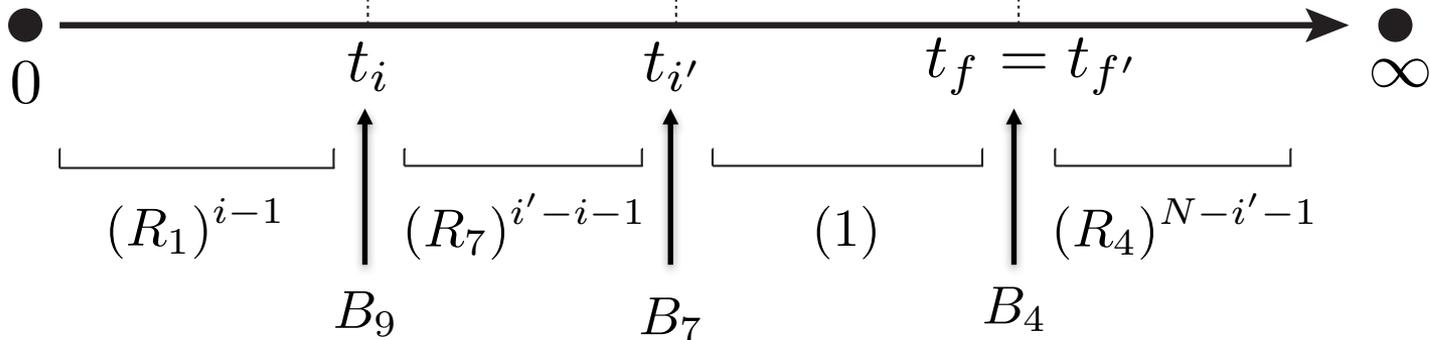


# An Example at Arbitrary Order

$$0 < t_i < t_{i'} < t_f = t_{f'} < \infty$$



$$\begin{aligned}
 &= \sum_{i=1}^{N-2} \sum_{i'=i+1}^{N-1} \left[ \tilde{\psi}(x, \underline{r}_1; x_{i'+1}^+) - \tilde{\psi}(x, \underline{r}_1; x_i^+) \right] \left\{ \left( \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(x|r_1 - r_2|_T) - 1 \right)^{i-1} \left( \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(x|r_1 - r_2|_T) - \frac{1}{2} \right) \left( -\frac{1}{2} \right)^{i'-i-1} \right. \\
 &\quad \times \left( -\frac{1}{2} \right) \left( 1 + \frac{N_c}{C_F} \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(|r_1 - r_2|_T) - \frac{N_c}{2C_F} \frac{(2\pi)^2}{\sigma_{el}} e^{i\Delta E^- x_{i'+1}^+} \left[ \sigma^{el}(r_{1T}) + \sigma^{el}(r_{2T}) \right] e^{-i\Delta E^- x_{i'+1}^+} \right) \\
 &\quad \left. \times \left( \frac{N_c}{C_F} \left[ \frac{(2\pi)^2}{\sigma_{el}} \sigma^{el}(|r_1 - r_2|_T) - 1 \right] \right)^{N-i'-1} \right\} \left[ \tilde{\psi}^*(x, \underline{r}_2; x_{i'+1}^+) - \tilde{\psi}^*(x, \underline{r}_2; x_i^+) \right] \quad (21)
 \end{aligned}$$



# Explicit Results: N = 2

- Straightforward construction at second order:

1.)	$\langle 0, \infty   0, 1 \rangle$	$[\tilde{\psi}(x_1^+) - \tilde{\psi}(x_0^+)] [B_6(x_1^+) R_6(x_2^+)] [0 - \tilde{\psi}^*(x_0^+)]$	15.)	$\sum_{i=1}^2 \langle i, \infty   0, i \rangle$	$\sum_{i=1}^2 [\tilde{\psi}(x_i^+) - \tilde{\psi}(x_0^+)] [(R_7)^{i-1} B_3(x_i^+) \prod_{j=i+1}^2 R_6(x_j^+)] [0 - \tilde{\psi}^*(x_i^+)]$
2.)	$\langle 0, 1   0, \infty \rangle$	$[0 - \tilde{\psi}(x_0^+)] [B_8(x_1^+) R_8(x_2^+)] [\tilde{\psi}^*(x_1^+) - \tilde{\psi}^*(x_0^+)]$	16.)	$\langle 2, \infty   0, 1 \rangle$	$[\tilde{\psi}(x_1^+) - \tilde{\psi}(x_0^+)] [B_{11}(x_1^+) B_{14}(x_2^+)] [0 - \tilde{\psi}^*(x_2^+)]$
3.)	$\langle 0, 1   0, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_0^+)] [B_8(x_1^+) B_{12}(x_2^+)] [\tilde{\psi}^*(x_1^+) - \tilde{\psi}^*(x_0^+)]$	17.)	$\langle 1, 2   0, \infty \rangle$	$[0 - \tilde{\psi}(x_0^+)] [B_7 B_8(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_1^+)]$
4.)	$\langle 0, 1   0, 1 \rangle$	$[\tilde{\psi}(x_1^+) - \tilde{\psi}(x_0^+)] [B_4(x_1^+) R_4] [\tilde{\psi}^*(x_1^+) - \tilde{\psi}^*(x_0^+)]$	18.)	$\langle 1, 2   0, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_0^+)] [B_7 B_4(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_1^+)]$
5.)	$\langle 0, 2   0, 1 \rangle$	$[\tilde{\psi}(x_1^+) - \tilde{\psi}(x_0^+)] [B_6(x_1^+) B_{16}(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_0^+)]$	19.)	$\langle 1, 2   0, 1 \rangle$	$[\tilde{\psi}(x_1^+) - \tilde{\psi}(x_0^+)] [B_3(x_1^+) B_{16}(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_1^+)]$
6.)	$\langle 0, \infty   2, \infty \rangle$	$[0 - \tilde{\psi}(x_2^+)] [R_5 B_5] [0 - \tilde{\psi}^*(x_0^+)]$	20.)	$\langle 2, \infty   1, \infty \rangle$	$[0 - \tilde{\psi}(x_1^+)] [B_9 B_7] [0 - \tilde{\psi}^*(x_2^+)]$
7.)	$\langle 0, \infty   1, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_1^+)] [B_5 B_6(x_2^+)] [0 - \tilde{\psi}^*(x_0^+)]$	21.)	$\langle 2, \infty   1, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_1^+)] [B_9 B_3(x_2^+)] [0 - \tilde{\psi}^*(x_2^+)]$
8.)	$\langle 0, 2   1, \infty \rangle$	$[0 - \tilde{\psi}(x_1^+)] [B_5 B_8(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_0^+)]$	22.)	$\langle 2, \infty   2, \infty \rangle$	$[0 - \tilde{\psi}(x_2^+)] [R_1 B_1] [0 - \tilde{\psi}^*(x_2^+)]$
9.)	$\langle 0, 2   1, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_1^+)] [B_5 B_4(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_0^+)]$	23.)	$\langle 1, \infty   1, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_1^+)] [B_1 B_6(x_2^+)] [0 - \tilde{\psi}^*(x_1^+)]$
10.)	$\sum_{i=1}^2 \langle 0, i   i, \infty \rangle$	$\sum_{i=1}^2 [0 - \tilde{\psi}(x_i^+)] [(R_5)^{i-1} B_2(x_i^+) \prod_{j=i+1}^2 R_8(x_j^+)] [\tilde{\psi}^*(x_i^+) - \tilde{\psi}^*(x_0^+)]$	24.)	$\langle 1, 2   1, \infty \rangle$	$[0 - \tilde{\psi}(x_1^+)] [B_1 B_8(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_1^+)]$
11.)	$\langle 0, 1   1, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_1^+)] [B_2(x_1^+) B_{12}(x_2^+)] [\tilde{\psi}^*(x_1^+) - \tilde{\psi}^*(x_0^+)]$	25.)	$\langle 1, 2   1, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_1^+)] [B_1 B_4(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_1^+)]$
12.)	$\langle 0, 1   2, \infty \rangle$	$[0 - \tilde{\psi}(x_2^+)] [B_{15}(x_1^+) B_{10}(x_2^+)] [\tilde{\psi}^*(x_1^+) - \tilde{\psi}^*(x_0^+)]$	26.)	$\langle 1, \infty   2, \infty \rangle$	$[0 - \tilde{\psi}(x_2^+)] [B_{13} B_5] [0 - \tilde{\psi}^*(x_1^+)]$
13.)	$\langle 2, \infty   0, \infty \rangle$	$[0 - \tilde{\psi}(x_0^+)] [R_7 B_7] [0 - \tilde{\psi}^*(x_2^+)]$	27.)	$\langle 1, 2   2, \infty \rangle$	$[0 - \tilde{\psi}(x_2^+)] [B_{13} B_2(x_2^+)] [\tilde{\psi}^*(x_2^+) - \tilde{\psi}^*(x_1^+)]$
14.)	$\langle 1, \infty   0, 2 \rangle$	$[\tilde{\psi}(x_2^+) - \tilde{\psi}(x_0^+)] [B_7 B_6(x_2^+)] [0 - \tilde{\psi}^*(x_1^+)]$			

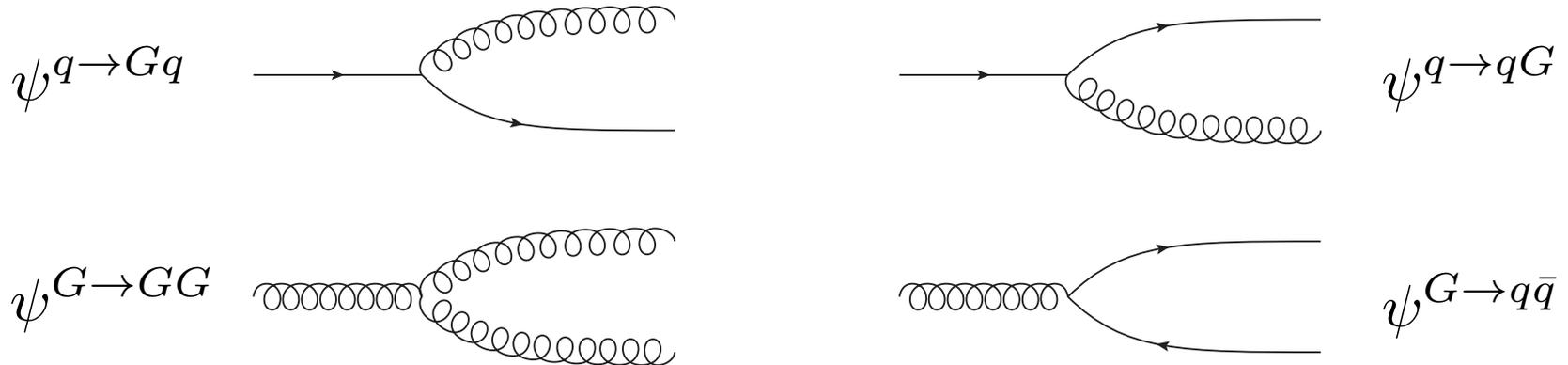
➤ **27 time orderings = 221 diagrams**

# Extending the Results to any Order

- The same architecture applies to **higher orders**:
  - Enumerate time orderings (up to 51 at  $N \geq 4$ )
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  - Adjust color factors for gluon vs. quark scattering
  - Replace splitting wave functions

# Summary and Outlook

- Medium-modified splitting function formulated in **coordinate space**
  - Multiple scattering enters with multiplicative factors
  - Organize solution based on Schwinger-Keldysh time orderings
  - Explicit, automatic solution at any order in opacity

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- Next steps:
  - Detailed **phenomenology at  $N = 2$**
  - Straightforward extension to jet substructure in **other systems**